Advances and Applications in Mathematical Sciences **ISSN 0974-6803**
Volume 22, Issue 8, June 2023, Pages 1765-1770

SOME NEW RESULTS ON PALEY GRAPHS

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Abstract

$P_q(A, B)$ graph is a Paley graph with vertices $a \in A$ are the elements of finite field F_q and $b \in B$ are edges between elements $x, y \in A$ if and only if $x - y$ is a non-zero square in F_q, where $q \equiv 1(\operatorname{mod} 4) = p^n$ (p is a prime number, n is any positive integer) is a prime power. This paper aims to prove some new results on Paley graph. The main new results are closure, planarity and the edge pebbling number of a Paley graph.

1. Introduction

Paley graph is a very good example which shows how graph theory and algebra interplace with each other. Paley graphs are named after Raymond Paley. They are closely related to the Paley construction for constructing Hadamard matrices from quadratic residues. They were introduced as graphs independently by Sachs and Erdos and Renyi. Sachs was interested in them for their self-complementarity properties, while Erdos and Renyi studied their symmetries. Paley graphs are dense undirected graphs constructed from the members of a suitable finite field by connecting pairs of elements that differ by a quadratic residue. The Paley graphs form an infinite family of conference graphs, which yield an infinite family of symmetric conference matrices. Anyone who seriously studies algebraic graph theory will, sooner or later come across the Paley graphs.

2020 Mathematics Subject Classification: 05Cxx.
Keywords: Paley graph, finite field, closure, planarity, edge pebbling number.
Received March 31, 2022; Accepted April 19, 2022

2. Preliminaries

In this section, for convenience of the reader and also for later use, we recall some known supporting results.

2.1 Theorem [2]. *For every prime 'p' and $n \in N$ there is a field with p^n elements.*

2.2 Theorem [2]. *The number of elements of a finite field F is equal to p^n, where p is prime and $n \in N$.*

2.3 Definition. Paley Graph [2]. Let p be a prime number and n be a positive integer such that $q = p^n \equiv 1 (\text{mod } 4)$. The graph $P = P_q = (V, E)$ with $v(P) = F_{p^n}$ and $E(P) = \{\{(x, y)\} : x, y \in F_{p^n}, x - y \in (F^*_{p^n})^2\}$ is called the Paley graph of order p^n. Here $F^*_{p^n} = F_{p^n} - \{0\}$ and $(F^*_{p^n})^2 = \{a^2 : a \in F^*_{p^n}\}$.

2.4 Illustration. Paley graphs exist for orders 5, 9, 13, 17, 25, 29, 37, 41, 49, 53, 61, 73, ... (1) Let $q = 5$, $Z_5 = \{0, 1, 2, 3, 4\}$ can be taken as F_q, $(F^*_q)^2 = \{1, 4\}$ $\therefore$ $E(q) = \{(0, 1), (1, 2), (2, 3), (3, 4), (4, 0)\}$

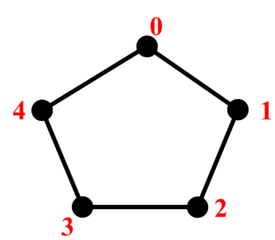

Figure 2.1. Paley graph of P_5.

Here we have illustrated the Paley graph of order 5, successive Paley graphs are illustrated in [2].

3. Closure of a Paley Graph

In this section, the closure of a Paley graph is determined.

3.1 Definition. Closure [5]**.** The closure of a graph G with n vertices, denoted by $C(G)$, is the graph obtained from G by repeatedly adding edges between pair of non-adjacent vertices whose degree sum is at least n, until this can no longer be done.

3.2 Theorem. *The closure of a Paley graph is itself.*

Proof. Since Paley graphs are regular. The degree of each vertex is $\frac{q-1}{2}$, where 'q' is the number of vertices. Now, consider any two non-adjacent vertices, say v_i and v_j having a degree $\frac{q-1}{2}$. By adding the degree of vertices, we get

$$\frac{q-1}{2}+\frac{q-1}{2}=\frac{2q-2}{2}=\frac{2(q-1)}{2}=q-1<q$$

Since v_i and v_j are arbitrary, it is true for every pair of non-adjacent vertices. So, by the definition of closure, no more edges can be added. Therefore, the graph P_q is its own closure.

Hence the proof.

4. Planarity of a Paley Graph

4.1 Definition[1]**.** A graph G is called "planar graph" if G can be drawn in the plane so that no two of its edges cross each other. Therefore, a graph that is not planar is called "non-planar".

4.2 Theorem[1]**.** *If G is a planar graph of order $n \geq 3$ and size m, then $m \leq 3n-6$. Here $n =$ number of vertices and $m =$ number of edges.*

4.3 Theorem. *Paley graphs are non-planar whenever $q \neq 5$.*

Proof. Paley graphs exist for order 5, 9, 13, 25, 17, 29,….

Claim. Paley graphs are non-planar except for $q = 5$

WKT, if G is a planar graph of order $n \geq 3$ and size m, then

$$m \leq 3n-6 \tag{1}$$

When $q = 5 = n$, number of edges, $m = \frac{q(q-1)}{4} = \frac{5(4)}{4} = 5$

Substituting in (1), we get $5 \leq 3(5) - 6 \Rightarrow 5 \leq 15 - 6 \Rightarrow 5 \leq 9$

$\therefore$ This inequality holds

Also, it is obvious from the diagram of P_5, that no two edges intersect each other

$\therefore P_5$ is a planar graph.

Let $q = 9 = n$.

We cannot draw a graph P_9 without the edge crossing $\exists$ at least any two edges say e_i and e_j cross with each other

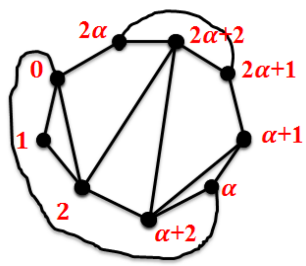

Figure 4.1. $P_9 - \{(2^\alpha, \alpha), (2^\alpha + 1, 1), (1, \alpha + 1)\}$.

In figure 4.1, if we draw any one of the edges in $\{(2^\alpha, \alpha), (2^\alpha + 1, 1), (1, \alpha + 1)\}$, then these edges will definitely intersect. P_9 is non-planar.

Now,

$$\frac{q(q-1)}{4} \leq 3q - 6 \tag{2}$$

Suppose, $q \geq 13$ then

$$\frac{q(q-1)}{4} = \frac{q(13-1)}{4} = 3q$$

$\therefore$ (2) becomes, $3q \leq 3q - 6$ which doesn't hold

$\therefore q$ violates the inequality when $q \geq 13$

$\therefore P_q$ is non-planar whenever $q \neq 5$.

Hence the proof.

5. Edge Pebbling Number of a Paley Graph

5.1 Definition. Edge pebbling [3]**.** An edge pebbling move on a graph G is defined to be the removal of two pebbles from one edge and the addition of one pebble to an adjacent edge.

Edge pebbling number [3]**.** An edge pebbling number $P_E(G)$ is defined to be least number of pebbles such that any distribution of $P_E(G)$ pebbles on the edges of G allows one pebble to be any specific, but arbitrary edge.

5.2 Theorem. *The edge pebbling number of a Paley graph is* $\frac{q(q-1)}{4}$.

Proof.

Case (i). The pebbles are distributed in a way that all edges with exactly one pebble except the target edge, then the edge pebble move is not possible, so we should place one more pebble in any of the $\left\{\frac{q(q-1)}{4}\right\} - 1$ edges. Then the target edge can be reached. In this case, we need $\frac{q(q-1)}{4}$ pebbles.

Case (ii). All the pebbles are placed on a single edge, say e_1.

Subcase (i). If this target edge is adjacent to e_1, then only 2 pebbles are enough to reach the target edge because of adjacency.

Subcase (ii). If the target edge is not adjacent to e_1 [i.e. Immediate non-adjacent to e_1], then we need 4 pebbles to attain the target edge.

Subcase (iii). Consider another way of distributing the pebbles on any edge then we need 8 pebbles to reach the target edge. Here the target edge is arbitrary non-adjacent to e_1. This is true for all possible distributions.

But the maximum number of pebbles needed to reach the target edge is the edge pebbling number, so the edge pebbling number for a Paley graph is $\frac{q(q-1)}{4}$.

6. Conclusion

In this paper, we have determined the closure, planarity and the edge pebbling number of the Paley graph. There are many more interesting results pertaining to Paley graphs which have wider applications in other fields.

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