Advances and Applications in Mathematical Sciences ISSN 0974-6803
Volume 22, Issue 8, June 2023, Pages 1905-1914

MORE RESULTS ON DEGREE PARTITION NUMBER

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Abstract

The whole world is currently dealing with a major problem caused by Covid 19, which necessitates social separation in many aspects. In certain circumstances, a need may emerge in which a certain group of individuals or components must be divided into multiple groups in order to meet certain requirements. We define a vertex partition $\pi_k = \{V_1, V_2, \ldots, V_k\}$ on the vertex set V of a graph G which is said to be a similar degree partition if the sum of degrees of vertices in each class $V_i, 1 \leq i \leq k,$ differs from that of other by at most 1. The degree partition number of G, $\psi_D(G) = \max \{k / \pi_k$ is a similar degree partition of $G\}$. In this paper we present the degree partition number of some graphs and we establish some bounds for this parameter.

1. Introduction

Only finite, simple, undirected graphs are considered in this study. For basic notations and terminology that are not included here, [1, 2] can be used to look up. The degree set of a graph is indicated by $D(G)$, while the degree of a vertex v is denoted by $\deg v$ or $d(v)$. In a graph, the minimum and maximum degree of vertices are represented by δ and Δ respectively.

2020 Mathematics Subject Classification: 05Cxx.
Keywords: graph partition, partitioning, degree partition number, degree partitioning.
Received June 25, 2022; Accepted August 20, 2022

If every vertex of a graph G has degree r, the graph is said to be r-regular. A graph with n vertices is complete graph if it is $(n-1)$-regular. The graph is known to be $(r, r+1)$-biregular if any vertex of G is of degree either r or $r+1$. A graph $G(V, E)$ is connected if there exists a path connecting every two vertices of G. Path and cycle on n vertices are denoted by P_n and C_n respectively.

A graph $G(V, E)$ is called a bipartite graph with bipartition (V', V'') if any edge $uv \in E$ has its one of its ends in V' and other in V''. If every vertex in V' is adjacent to every other vertex in V'', such bipartite graph is called complete bipartite graph denoted by $K_{m,n}$ where $|V_1| = m$ and $|V_2| = n$.

The Cartesian product graph $G = G_1 \square G_2$ of two graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge set E_1 and E_2 is the graph with vertex set $V_1 \times V_2$ and the vertex $u = (u_1, u_2)$ is adjacent to the vertex $v = (v_1, v_2)$ if $u_1 = v_1$ and u_2 is adjacent to v_2 or $u_2 = v_2$ and u_1 is adjacent to v_1. Grid graph $G(m, n)$ is a cartesian product of two paths P_m and P_n. The ladder graph can be obtained as the Cartesian product of two path graphs P_2 and P_n. The friendship graph F_n can be created by linking n copies of the cycle graph C_3 with a common vertex.

In many cases, a situation may arise where a single group of people or components must be separated into various groups in order to meet special needs. Graph models are one of the techniques to depict any system. To investigate the nature and properties of a network, our mathematicians devise a variety of partitioning methods.

Graph partition is the process of reducing a graph to smaller graphs by partitioning its vertex set into mutually incompatible groups. There are numerous research concepts in the literature that are based on partitioning the vertex and edge sets of a graph.

The general chromatic partition, bilinear partitions, trilinear partitions are some examples of graph partitions that can be referred from [3, 4].

This study was prepared during the Corona virus pandemic, which necessitates social separation in all aspects. Every system, however, must be dynamic for the country's economic and educational well-being. To meet the need of the hour, the system must be subdivided into smaller groups with more or less identical capacity. This serves as the foundation for the investigation of degree partition number [5, 6], which is presented in this work.

Let $\pi_k = \{V_1, V_2, \ldots, V_k\}, (k \geq 2)$ be a partition of the vertex set $V(G)$. π_k is called a similar degree partition if the sum of degrees of vertices in any class of π_k differs from that of other by at most 1. i.e., if $|\sum_{v \in V_i} d(v) - \sum_{v \in V_j} d(v)| \leq 1$ for $1 \leq i, j \leq k$. When this difference equals zero for any two classes of a partition π_k, then it is called perfect similar degree partition. The degree partition number of a graph $\psi_D(G)$ is defined as $\max\{k/\pi_k$ is a similar degree partition of $G\}$ and such π_k is called the maximal similar degree partition.

For example, consider the graph given in Figure 1.

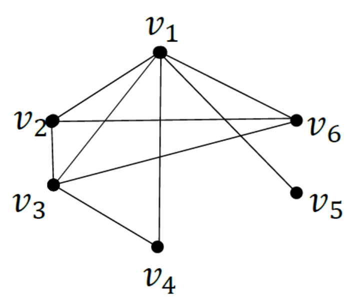

Figure 1.

Here $\pi_3 = \{V_1, V_2, V_3\}$ where $V_1 = \{v_1\}, V_2 = \{v_2, v_3\}, V_3 = \{v_4, v_5, v_6\}$

$\pi_2 = \{V_1, V_2\}$ where $V_1 = \{v_1, v_5, v_6\}, V_2 = \{v_2, v_3, v_4\}$

$\pi'_3 = \{V_1, V_2, V_3\}$ where $V_1 = \{v_1, v_5\}, V_2 = \{v_2, v_6\}, V_3 = \{v_3, v_4\}$.

One can confirm that π_3 is not a similar degree partition. π_2 and π'_3 are similar degree partitions. As no other similar degree partition $\pi_k, k \geq 4$ exists for this graph, $\psi_D(G) = 3$ and π'_3 is a maximal similar degree partition of the graph that we considered above.

2. Main Results

In this section, we present some basic results and bounds on the degree partition number of a graph.

Fact 1. The degree partition number of any grid graph is $|V| - 2$.

Proof. Grid graph $G(m, n)$ is a graph with mn vertices. Let $V = \{v_{11}, v_{12}, \ldots, v_{1n}, v_{21}, v_{22}, \ldots, v_{2n}, \ldots, v_{m1}, v_{m2}, \ldots, v_{mn}\}$.

We may note that,

$$\deg v_{ij} = \begin{cases} 2 & \text{if } (i, j) = (1, 1), (1, n), (m, 1), (m, n) \\ & \text{if } (i, j) = (1, 2), (1, 3), \ldots, (1, n-1), \\ 3 & (m, 2), (m, 3), \ldots, (m, n-1), \\ & (2, 1), (3, 1), \ldots, (m-1, 1), \\ & (2, n), (3, n), \ldots, (m-1, n) \\ 4 & \text{otherwise} \end{cases}$$

By taking corner vertices in pair, and remaining vertices as individual classes we get the required similar degree partition. So, $\psi_D(G(m, n)) = mn - 2$. □

Fact 2. The degree partition number of any friendship graph is 3.

Proof. Let the central vertex be denoted by v and the set of remaining vertices be $\{v_1, v_2, \ldots, v_{2n}\}$.

Then $\deg v = 2n$ and $\deg v_i = 2$ for all $i = 1, 2, \ldots, 2n$. The partition $\pi_3 = \{V_1, V_2, V_3\}$ where $V_1 = \{v\}, V_2 = \{v_1, v_2, \ldots, v_n\}$ and $V_3 = \{v_{n+1}, v_{n+2}, \ldots, v_{2n}\}$ is the required similar degree partition. Hence $\psi_D(G) = 3$. □

Fact 3. The degree partition number of bipartite graph G is at least 2.

In fact, we can note that the bipartition of the vertex set itself forms a degree partition of G. □

Fact 4. The degree partition number of complete graph, path, cycle, Peterson graph, $(n, n+1)$-complete bipartite graph and ladder graph is $|V(G)|$. □

In fact, we can state the following theorem.

Theorem 5. $\psi_D(G) = |V(G)|$ *if and only if G is either a regular graph or* $(n, n+1)$*-biregular graph.* □

Theorem 6. $1 \leq \psi_D(G) \leq \left\lfloor \frac{\sum_{v_i \in V_1} \deg v_i - 1}{\Delta - 1} \right\rfloor$.

Proof. Let G be a graph with n vertices. Let $\pi_k = \{V_1, V_2, \ldots, V_k\}$ be a maximal similar degree partition of G. Then $\psi_D(G) = k$.

Clearly, there exists at least one partition say V_1 such that $\sum_{v_i \in V_1} \deg v_i \geq \Delta$.

Also, $\sum_{v_i \in V_j} \deg v_i \geq \Delta - 1$ for $j = 2, 3, \ldots, k$.

Adding the above k inequalities, we get

$$\sum_{v_i \in V(G)} \deg v_i \geq \Delta + (k-1)(\Delta - 1).$$

$$\therefore k - 1 \leq \frac{\sum_{v_i \in V_1} \deg v_i - \Delta}{\Delta - 1} \Rightarrow k \leq \frac{\sum_{v_i \in V_1} \deg v_i - \Delta}{\Delta - 1} + 1$$

$$\Rightarrow k \leq \frac{\sum_{v_i \in V_1} \deg v_i - 1}{\Delta - 1}$$

Hence, $k \leq \left\lfloor \frac{\sum_{v_i \in V_1} \deg v_i - 1}{\Delta - 1} \right\rfloor$ since k is an integer.

Always $k \geq 1$.

Thus $1 \leq \psi_D(G) \leq \left\lfloor \frac{\sum_{v_i \in V_1} \deg v_i - 1}{\Delta - 1} \right\rfloor$. □

Corollary 7. *If* $\psi_D(G) = \left\lfloor \frac{\sum_{v_i \in V_1} \deg v_i - 1}{\Delta - 1} \right\rfloor$, *then there exists at least one partition class say* V_i *in* ψ_D *such that* V_i *contains max-degree vertex alone.* □

Theorem 8. *If degree of each vertex of* G *is even, then* $1 \leq \psi_D(G) \leq \left\lfloor \frac{\sum_{v_i \in V_1} \deg v_i}{\Delta} \right\rfloor$.

Proof. Let G be a graph with n vertices and degree of each vertex be even.

Let $\pi_k = \{V_1, V_2, \ldots, V_k\}$ be a maximal similar degree partition of G. Then $\psi_D(G) = k$.

Since degree of each vertex is even, π_k should be a perfect similar degree partition of G.

Then, $\sum_{v_i \in V_j} \deg v_i \geq \Delta$ for all $j = 1, 2, 3, \ldots, k$.

Adding the above k inequalities, we get

$$\sum_{v_i \in V(G)} \deg v_i \geq k\Delta.$$

$\therefore k \leq \frac{\sum_{v_i \in V_1} \deg v_i}{\Delta}$. Hence, $k \leq \left\lfloor \frac{\sum_{v_i \in V_1} \deg v_i}{\Delta} \right\rfloor$ since k is an integer.

Always $k \geq 1$. Thus $1 \leq \psi_D(G) \leq \left\lfloor \frac{\sum_{v_i \in V_1} \deg v_i}{\Delta} \right\rfloor$. □

Theorem 9. *For* $n \geq 4$, $\psi_D(K_{2,n}) = \begin{cases} 4 & \text{if } n \text{ is even} \\ 3 & \text{if } n \equiv \pm 1 (\bmod 6) \\ 2 & \text{if } n \equiv 3 (\bmod 6). \end{cases}$

Proof. Let $K_{2,n}$ be a complete bipartite graph with bipartition (V', V'')

where $V' = \{u_1, u_2\}$, $V'' = \{v_1, v_2, \ldots, v_n\}$. Here $\deg u_i = n$ for $i = 1, 2$ and $\deg v_i = 2$ for $i = 1, 2, \ldots, n$.

Also, $\sum_{v_i \in V(K_{2,n})} \deg v_i = 2(2)(n) = 4n$.

By theorem 6, no matter whether n is odd or even, $\psi_D(K_{2,n}) \leq 4$.

Also, since it is bipartite, $\psi_D(K_{2,n}) \geq 2$. So, $2 \leq \psi_D(K_{2,n}) \leq 4$.

Let $\pi_4 = \{V_1, V_2, V_3, V_4\}$ be a similar degree partition of $K_{2,n}$, then $V_1 = \{u_1\}$, $V_2 = \{u_2\}$.

Now since $\deg v_i = 2$ for $1 \leq i \leq n$ and $\sum_{v_i \in V''} \deg v_i = 2n$, we need to partition V'' into V_3 and V_4 so that $\sum_{v_i \in V_j} \deg v_i = n$ for $j = 3, 4$.

This is possible only when n is even.

Hence, $\psi_D(K_{2,n}) = 4$ only when n is even, i.e., $n \equiv 0 (\mathrm{mod}\, 2)$ or $(n \equiv 0, 2, 4 (\mathrm{mod}\, 6))$.

For the remaining cases, if $\pi_3 = \{V_1, V_2, V_3\}$ forms a similar degree partition of $K_{2,n}$, then $V_1 = \{u_1, v_1, v_2, \ldots, v_k\}$, $V_2 = \{u_2, v_{k+1}, v_{k+2}, \ldots, v_{2k}\}$ and $V_3 = \{v_{2k+1}, v_{2k+2}, \ldots, v_n\}$ where n is odd.

Here, $\sum_{v_i \in V_j} \deg v_i = n + 2k$ for $j = 1, 2$, and $\sum_{v_i \in V_3} \deg v_i = 2(n - 2k)$

Note that $n + 2k$ and $2(n - 2k)$ are of different parity considering the fact that n is odd.

$$\therefore 2(n - 2k) = n + 2k \pm 1 \Rightarrow n = 6k \pm 1$$

Hence, $\psi_D(K_{2,n}) = 3$ if $n \equiv \pm 1 (\mathrm{mod}\, 6)$.

And $\pi_2 = \{V', V''\}$ forms a maximal similar degree partition in the remaining case $n \equiv 3 (\mathrm{mod}\, 6)$. □

As an illustration, $K_{2,5}$ is shown in Figure 2.

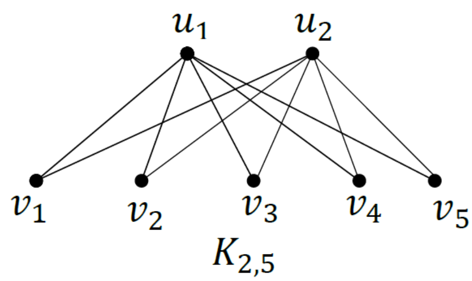

Figure 2.

Let $K_{2,5}$ have bipartition (V', V'') where $V' = \{u_1, u_2\}$ and $V'' = \{v_1, v_2, v_3, v_4, v_5\}$. The partition $\pi_3 = \{V_1, V_2, V_3\}$ where $V_1 = \{u_1, v_1\}, V_2 = \{u_2, v_2\}$ and $V_3 = \{v_3, v_4, v_5\}$ stands as the maximal similar degree partition of $K_{2,5}$. □

Theorem 10. $\psi_D(K_{3k+2,6k+5}) = 4k+3$ *for* $k \geq 1$.

Proof. Let $K_{3k+2,6k+5}$ be a complete bipartite graph with bipartition (V', V'') where $V' = \{u_1, u_2, \ldots, u_{3k+2}\}, V'' = \{v_1, v_2, \ldots, v_{6k+5}\}$.

Here $\deg u_i = 6k+5$ for $i = 1, 2, \ldots, 3k+2$ and $\deg v_i = 3k+2$ for $i = 1, 2, \ldots, 6k+5$.

$\pi_{4k+3} = \{V_1, V_2, \ldots, V_{4k+3}\}$ where $V_1 = \{u_1, v_1\}, V_2 = \{u_2, v_2\}, \ldots, V_{3k+2} = \{u_{3k+2}, v_{3k+2}\}, V_{3k+3} = \{v_{3k+3}, v_{3k+4}, v_{3k+5}\}, \ldots, V_{4k+3} = \{v_{6k+3}, v_{6k+4}, v_{6k+5}\}$ forms a maximal similar degree partition of $K_{3k+2,6k+5}$. It can be easily verified that the degree sum of the partition classes $V_1, V_2, \ldots, V_{3k+2}$ are $9k+7$ and the degree sum of the partition classes $V_{3k+3}, V_{3k+4}, \ldots, V_{4k+3}$ are $9k+6$.

$$\therefore \psi_D(K_{3k+2,6k+5}) = 4k+3. \qquad \square$$

For example, let the bipartition (V', V'') of $K_{5,11}$ be given by $V' = \{u_1, u_2, u_3, u_4, u_5\}$ and $V'' = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}\}$.

The partition $\pi_7 = \{V_1, V_2, V_3, V_4, V_5, V_6, V_7\}$ where $V_1 = \{u_1, v_1\}$, $V_2 = \{u_2, v_2\}$, $V_3 = \{u_3, v_3\}$, $V_4 = \{u_4, v_4\}$, $V_5 = \{u_5, v_5\}$, $V_6 = \{v_6, v_7, v_8\}$ and $V_7 = \{v_9, v_{10}, v_{11}\}$ serves as the maximal similar degree partition of $K_{5,11}$ with degree sum as 16 for the partition classes V_1, V_2, V_3, V_4, V_5 and as 15 for the partition classes V_6 and V_7. □

Theorem 11. $\psi_D(K_{3k+1,6k+1}) = 4k+1$ *for* $k \geq 1$.

Proof. Let $K_{3k+1,6k+1}$ be a complete bipartite graph with bipartition (V', V'') where $V' = \{u_1, u_2, \ldots, u_{3k+1}\}$, $V'' = \{v_1, v_2, \ldots, v_{6k+1}\}$.

Here $\deg u_i = 6k+1$ for $i = 1, 2, \ldots, 3k+1$ and $\deg v_i = 3k+1$ for $i = 1, 2, \ldots, 6k+1$.

$\pi_{4k+1} = \{V_1, V_2, \ldots, V_{4k+1}\}$ where $V_1 = \{u_1, v_1\}$, $V_2 = \{u_2, v_2\}, \ldots, V_{3k+1} = \{u_{3k+1}, v_{3k+1}\}$, $V_{3k+2} = \{v_{3k+2}, v_{3k+3}, v_{3k+4}\}, \ldots, V_{4k+1} = \{v_{6k-1}, v_{6k}, v_{6k+1}\}$ forms a maximal similar degree partition of $K_{3k+1,6k+1}$. One can verify that the degree sum of the partition classes $V_1, V_2, \ldots, V_{3k+1}$ are $9k+2$ and the degree sum of the partition classes $V_{3k+2}, V_{3k+3}, \ldots, V_{4k+1}$ are $9k+3$.

$$\therefore \psi_D(K_{3k+1,6k+1}) = 4k+1.$$ □

For instance, we consider $K_{7,13}$ with the bipartition (V', V'') where $V' = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ and $V'' = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}\}$.

The partition $\pi_9 = \{V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8, V_9\}$ where $V_1 = \{u_1, v_1\}$, $V_2 = \{u_2, v_2\}$, $V_3 = \{u_3, v_3\}$, $V_4 = \{u_4, v_4\}$, $V_5 = \{u_5, v_5\}$, $V_6 = \{u_6, v_6\}$, $V_7 = \{u_7, v_7\}$, $V_8 = \{v_8, v_9, v_{10}\}$ and $V_9 = \{v_{11}, v_{12}, v_{13}\}$ forms the maximal similar degree partition of $K_{7,13}$ with degree sum as 20 for the partition classes $V_1, V_2, V_3, V_4, V_5, V_6, V_7$ and as 21 for the partition classes V_8 and V_9. □

References

[1] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, The Macmillan Press Ltd, Britain, 1976.

[2] F. Harary, Graph Theory, Addison-Wesly, Reading Mass, 1972.

[3] E. Sampathkumar and C. V. Venkatachalam, Chromatic partitions of a graph, Discrete Mathematics, North-Holland 74(1-2) (1989), 227-239.

[4] E. Sampathkumar and L. Pushpa Latha, Bilinear and trilinear partitions of a graph, Indian J. Pure Appl. Math. 25(8) (1994), 843-850.

[5] N. Malathi, M. Bhuvaneshwari and Selvam Avadayappan, Degree Partition Number of Derived Graphs (Communicated).

[6] N. Malathi, M. Bhuvaneshwari and Selvam Avadayappan, A note on degree partition number of a graph, Journal of Emerging Technologies and Innovative Research 8(7) (2021).