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COMPLEMENTARY TRIPLE CONNECTED AT MOST TWIN DOMINATION NUMBER OF A GRAPH

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Abstract

In this article, we introduce the concept of complementary triple connected at most twin domination number of a graph. A set $S \subseteq V$ is called a complementary triple connected at most twin dominating set $(CTATD(G))$, if every vertex $v \in V - S; 1 \leq |N(v) \cap S| \leq 2$ and $\langle V - S \rangle$ is triple connected. The minimum cardinality taken over all the complementary triple connected at most twin dominating sets in G is called the complementary triple connected at most twin domination number of G and is denoted by $CTATD(G)$. In this article we investigate this parameter for some standard and special types of graphs.

1. Motivation

Mustapha Chellali et al. [4] first studied the concept of [1, 2] set. Xiaojing Yang and Baoyin-dureng Wu [2] extended the study of this parameter. G. Mahadevan et al. developed the theory of $\gamma_{[1,2]cc}$ [5] and the concept of at most twin outer perfect domination number of a graph [3]. Paulraj Joseph et al., [6] were introduced the triple connected graphs. Keeping all the above definitions as the motivation we keep the dominating set to be [1, 2] -

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dominating set and its complement to be triple connected, thereby we introduce a new domination parameter called CTATD-number of a graph.

2. Preliminary Definitions

For our further discussion, we mention the following definitions which are available in [1]. A Helm graph H_p is a graph obtained from the wheel $W_{1,n}$ by joining a pendent vertex to each vertex in the outer cycle of $W_{1,n}$. Subdivide every edge in the graph G, join the vertices that are adjacent in G, and join the subdivided vertices that are adjacent to a common vertex. The obtained graph is called total graph. The flower graph Fl_p is the graph obtained from the Helm H_p by joining each pendant vertex to the apex of the helm. A closed Helm CH_n is the graph obtained from a Helm H_n by joining each pendant vertex to form a cycle. The barbell graph $K_p \bigcup K_p + e$ is obtained by joining two copies of K_p by a bridge. The friendship graph, denoted by F_p can be constructed by identifying p copies of the cycle C_3 at a common vertex. Subdivide every edge in the graph G, join subdivided vertices that are adjacent to a common vertex, the obtained graph is called middle graph $M(G)$. The triangular snake graph TS_p is obtained from a path $(a_1, a_2, \ldots, a_p)$ by joining a_i and a_{i+1} to a new vertex b_i for $i = 1, 2, \ldots, n-1$. That is every edge of a path is replaced by a triangle C_3. The mirror graph is $M_r = P_2 \times G$. Central graph is obtained by subdividing every edge and obtain the original graph G. The shadow graph $D_2(G)$ of a connected graph G is obtained by taking two copies of G say G' and G'' join each vertex u' in G' to the neighbours of corresponding vertex u'' in G''.

3. Complementary Triple Connected at Most Twin Domination Number of a Graph

Definition 3.1. A set $S \subseteq V$ is called a complementary triple connected at most twin dominating set $(CTATD(G))$ if every vertex $v \in V - S$; $1 \leq |N(v) \bigcap S| \leq 2$ and $\langle V - S \rangle$ is triple connected. The minimum cardinality of a CTATD-set is called the complementary triple connected at most twin domination number (CTATD-number) and is denoted by $CTATD(G)$.

Observation 3.1. Complementary triple connected at most twin domination number does not exist for path, cycle, star graph, bistar graph, friendship graph.

Observation 3.2. For any connected graph G, $\gamma(G) \leq \gamma_{[1,2]cc}(G) \leq CTATD(G)$ and the bounds are sharp.

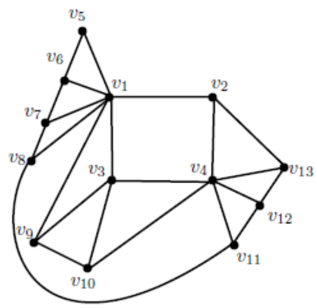

Figure 3.3.

In figure 3.3, $S_1 = \{v_1, v_4\}$ is a dominating set of smallest size, so that $\gamma(G) = 2$.

$S_2 = \{v_1, v_3, v_4, v_9, v_{10}\}$ is a $[1, 2]cc$ dominating set of minimum cardinality, so that $\gamma_{[1,2]cc}(G) = 4$.

$S_3 = \{v_1, v_3, v_4, v_9, v_{10}\}$ is a complementary triple connected at most twin dominating set of minimum cardinality, so that $CTATD(G) = 5$.

Observation 3.3. There exists a graph G for which, $\gamma(G) = \gamma_{[1,2]cc} = CTATD(G)$.

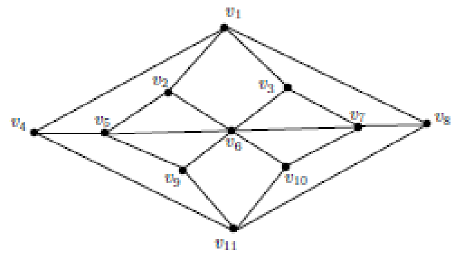

Figure 3.4.

In figure 3.4, $S = \{v_4, v_6, v_8\}$ is a dominating set, $[1, 2]$-complementary connected dominating set and complementary triple connected at most twin dominating set of smallest size. Hence $\gamma(G) = \gamma_{[1,2]cc} = CTATD(G)$.

Theorem 3.1. *For a connected graph G with* $p \geq 3, \left\lceil \frac{p}{\Delta + 1} \right\rceil \leq CTATD(G)$ *and the bounds is sharp*.

Proof. Since, $\left\lceil \frac{p}{\Delta + 1} \right\rceil \leq \gamma(G)$ and $\gamma(G) \leq CTATD(G)$ and the result follows. □

Example 3.1.

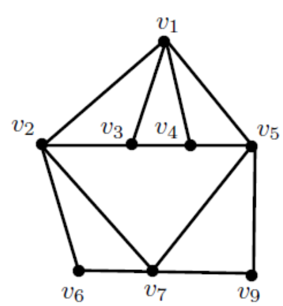

Figure 3.5.

In figure 3.5, $p = 8$ and $\Delta = 4$. Hence $S = \{v_1, v_7\}$ is a CTATD set of minimum cardinality. Hence $CTATD(G) = \left\lceil \frac{p}{\Delta + 1} \right\rceil = \left\lceil \frac{8}{4 + 1} \right\rceil = 2$.

4. Exact Value of $CTATD(G)$-Number for Some Standard Graphs

(1) $CTATD(W_{1,n}) = 1$.

(2) $CTATD(K_p) = 1$.

(3) $CTATD(H_n) = n + 1$.

(4) $CTATD(K_{r,s}) = 2$, $r \geq s \geq 2$.

(5) $CTATD(TS_p) = p - 1$.

(6) For a path P_p, $p \geq 3$, $CTATD(D_2(P_p)) = p$.

(7) For a cycle C_p, $p \geq 3$, $CTATD(D_2(C_p)) = p$.

(8) $CTATD(Fl_p) = p - 1$.

(9) $CTATD(K_p \cup K_p + e) = 2$, $p \geq 3$.

5. Complementary Triple Connected at Most Twin Domination Number for Peculiar Types of Graphs

Observation 5.1.

(1) $CTATD(M_d(P_p)) = p$, $p \geq 4$.

(2) $CTATD(M_d(C_p)) = p$, $p \geq 4$.

(3) $CTATD(C(P_p)) = p - 1$, $p \geq 3$.

(4) $CTATD(C(C_p)) = p$, $p \geq 3$.

Theorem 5.1. *For a path* P_p, $p \geq 3$, $CTATD(T(P_p)) = \left\lceil \frac{2p-1}{5} \right\rceil$.

Proof. Let $P_p = (v_1, v_2, \ldots, v_p)$, $p \geq 3$. This gives $V(T(P_p)) = \{v_1, v_2, \ldots, v_p, u_1, u_2, \ldots, u_{p-1}\}$, $E(T(P_p)) = \{v_iu_i, v_iv_{i+1}, u_iv_{i+1}, u_ju_{j+1}; 1 \leq i \leq p-1; 1 \leq i \leq p-2\}$. Let $S_1 = \{v_i, u_j : i \equiv 2 (\text{mod } 5); j \equiv 4 (\text{mod } 5)\}$. Assume

$S = \begin{cases} S_1 & \text{if } p \equiv 0 \text{ or } 2 \text{ or } 3 (\bmod 5) \\ S_1 \cup \{v_p\} & \text{if } p \equiv 1 \text{ or } 4 (\bmod 5). \end{cases}$ Then S is a CTATD-set of $T(P_p)$ and hence $CTATD(T(P_p)) \leq |S| = \left\lceil \frac{2p-1}{5} \right\rceil$. Let S' be a CTATD-set of $T(P_p)$. Since any set D of cardinality at most $k = \left\lceil \frac{2p-1}{5} \right\rceil - 1$ is not a dominating set. We have $|S'| \geq k+1 = \left\lceil \frac{2p-1}{5} \right\rceil$. Hence the result follows. □

Theorem 5.2. *For a cycle* C_p, $p \geq 3$, $CTATD(T(C_p)) = \left\lceil \frac{2p}{5} \right\rceil$.

Proof. Let $C_p = (v_1, v_2, \ldots, v_p, v_1)$, $p \geq 3$. This gives $V(T(C_p)) = \{v_1, v_2, \ldots, v_p, u_1, u_2, \ldots, u_{p-1}\}$, $E(T(C_p)) = \{v_i u_i, v_i v_{i+1}, u_i v_{i+1}, u_j u_{j+1}, u_i u_{i+1}, v_p v_1, u_p v_1; 1 \leq i \leq p-1\}$.

Let $S_1 = \{v_i, u_j : i \equiv 2 (\text{mod } 5); j \equiv 4 (\text{mod } 5)\}$.

Assume $S = \begin{cases} S_1 & \text{if } p \equiv 0 \text{ or } 2 \text{ or } 3 (\bmod 5) \\ S_1 \cup \{v_p\} & \text{if } p \equiv 1 \text{ or } 3 (\bmod 5). \end{cases}$

Then S is a CTATD-set of $T(C_p)$ and hence $CTATD(T(C_p)) \leq |S| = \left\lceil \frac{2p}{5} \right\rceil$. Let S' be a CTATD-set of $T(C_p)$. Since any set D of cardinality at most $k = \left\lceil \frac{2p}{5} \right\rceil - 1$ is not a dominating set, we have $|S'| \geq k+1 = \left\lceil \frac{2p}{5} \right\rceil$. Hence the result follows. □

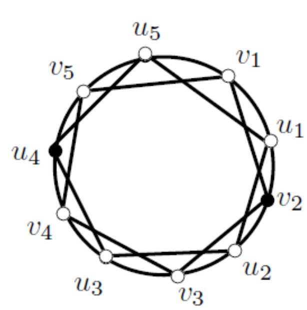

Figure 5.6.

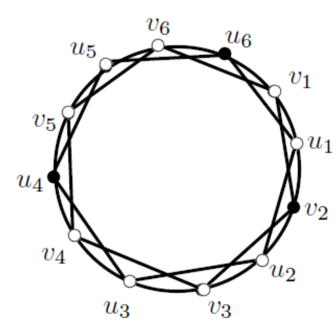

Figure 5.7.

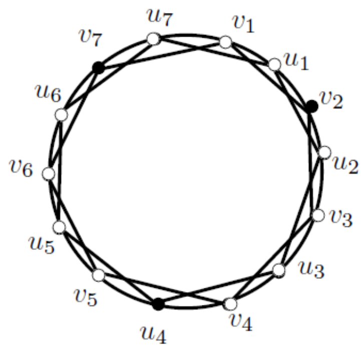

Figure 5.8.

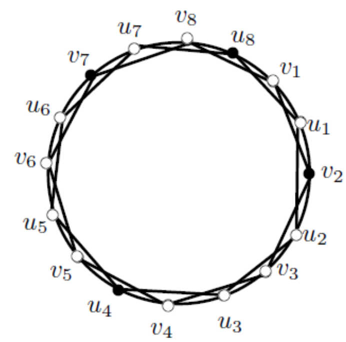

Figure 5.9.

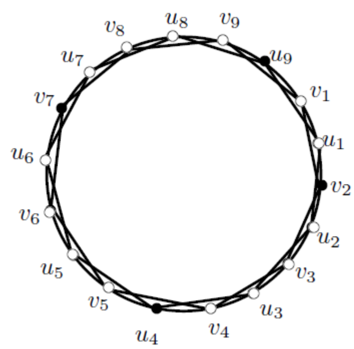

Figure 5.10.

Demonstration

Here, darked vertices are the CTATD-set, which is our S

In figure 5.6, $|S| = 2$.

As $p = 5$, $CTATD(T(C_p)) = \left\lceil \frac{2p}{5} \right\rceil$ implies $CTATD(T(C_5)) = \left\lceil \frac{2 \times 5}{5} \right\rceil = 2$.

In figure 5.7, $|S| = 3$.

As $p = 6$, $CTATD(T(C_p)) = \left\lceil \frac{2p}{5} \right\rceil$ implies $CTATD(T(C_6)) = \left\lceil \frac{2 \times 6}{5} \right\rceil = 3$.

In figure 5.8, $|S| = 3$.

As $p = 7$, $CTATD(T(C_p)) = \left\lceil \frac{2p}{5} \right\rceil$ implies $CTATD(T(C_7)) = \left\lceil \frac{2 \times 7}{5} \right\rceil = 3$.

In figure 5.9, $|S| = 4$.

As $p = 8$, $CTATD(T(C_p)) = \left\lceil \frac{2p}{5} \right\rceil$ implies $CTATD(T(C_8)) = \left\lceil \frac{2 \times 8}{5} \right\rceil = 4$.

In figure 5.10, $|S| = 4$.

As $p = 9$, $CTATD(T(C_p)) = \left\lceil \frac{2p}{5} \right\rceil$ implies $CTATD(T(C_9)) = \left\lceil \frac{2 \times 9}{5} \right\rceil = 4$.

Theorem 5.3. *For a path* P_p, $p \geq 3$, $CTATD(M_r(P_p)) =$

$$\begin{cases} 2\left\lfloor \frac{p}{4} \right\rfloor + 1 & if\ p \equiv 0\ or\ 1(\operatorname{mod} 4) \\ 2\left\lceil \frac{p}{4} \right\rceil & if\ p \equiv 2\ or\ 3(\operatorname{mod} 4). \end{cases}$$

Proof. Let $P_p = (v_1, v_2, \ldots, v_p)$ and let the copies of $p'_p = (u_1, u_2, \ldots, u_p)$.

This gives $E(M_r(P_p)) = \{u_i v_i, u_j u_{j+1}, v_j v_{j+1} : 1 \leq i \leq p; 1 \leq j \leq p - 1\}$.

Let $S_1 = \{v_i, u_j : i \equiv 1(\text{mod } 4); j \equiv 3(\text{mod } 4)\}$.

Assume $S = \begin{cases} S_1 \cup \{v_p\} & \text{if } p \equiv 0(\operatorname{mod} 4) \\ S_1 & \text{if } p \equiv 1 \text{ or } 3(\operatorname{mod} 4) \\ S_1 \cup \{v_p\} & \text{if } p \equiv 0(\operatorname{mod} 4) \end{cases}$

Then S is a CTATD-set of $M_r(P_p)$ and hence

$$CTATD(M_r(P_p)) \leq |S| = \begin{cases} 2\left\lfloor \frac{p}{4} \right\rfloor + 1 & \text{if } p \equiv 0 \text{ or } 1(\operatorname{mod} 4) \\ 2\left\lceil \frac{p}{4} \right\rceil & \text{if } p \equiv 2 \text{ or } 3(\operatorname{mod} 4) \end{cases}$$

Let S' be a CTATD-set of $M_r(P_p)$. Since $D \subseteq V$ such that

$$|D| \leq k = \begin{cases} 2\left\lfloor \frac{p}{4} \right\rfloor & \text{if } p \equiv 0 \text{ or } 1(\operatorname{mod} 4) \\ 2\left\lceil \frac{p}{4} \right\rceil - 1 & \text{if } p \equiv 2 \text{ or } 3(\operatorname{mod} 4) \end{cases}$$

is not a dominating set, we have

$$|S'| \geq k+1 = \begin{cases} 2\left\lfloor \frac{p}{4} \right\rfloor + 1 & \text{if } p \equiv 0 \text{ or } 1 (\text{mod } 4) \\ 2\left\lceil \frac{p}{4} \right\rceil & \text{if } p \equiv 2 \text{ or } 3 (\text{mod } 4). \end{cases}$$

Hence the result follows. □

Theorem 5.4. *For a cycle* $C_p,\ p \geq 3,\ CTATD(M_r(C_p)) =$

$$\begin{cases} 2\left\lfloor \frac{p}{4} \right\rfloor & \textit{if } p \equiv 0 \textit{ or } 2 \textit{ or } 3 (\text{mod } 4) \\ 2\left\lceil \frac{p}{4} \right\rceil + 1 & \textit{if } p \equiv 1 (\text{mod } 4). \end{cases}$$

Proof. Let $C_p = (v_1, v_2, \ldots, v_p, v_1)$ and let the copies of $C'_p = (u_1, u_2, \ldots, u_p, u_1)$. This gives $E(M_r(C_p)) = \{u_i v_i,\ u_j u_{j+1},\ v_j v_{j+1},\ v_p v_1,\ u_p u_1 : 1 \leq i \leq p; 1 \leq j \leq p-1\}$. Let $S_1 = \{v_i,\ u_j : i \equiv 1 (\text{mod } 4); j \equiv 3 (\text{mod } 4)\}$.

Assume $S = \begin{cases} S_1 & \text{if } p \equiv 0 \text{ or } 1 \text{ or } 3 (\text{mod } 4) \\ S_1 \cup \{v_p\} & \text{if } p \equiv 2 (\text{mod } 4). \end{cases}$

Then S is a CTATD-set of $M_r(C_p)$ and hence

$$CTATD(M_r(C_p)) \leq |S| = \begin{cases} 2\left\lfloor \frac{p}{4} \right\rfloor & \text{if } p \equiv 0 \text{ or } 2 \text{ or } 3 (\text{mod } 4) \\ 2\left\lceil \frac{p}{4} \right\rceil + 1 & \text{if } p \equiv 1 (\text{mod } 4) \end{cases}$$

Let S' be a CTATD-set of $M_r(C_p)$. Since $D \subseteq V$

$$|D| \leq k = \begin{cases} 2\left\lfloor \frac{p}{4} \right\rfloor - 1 & \text{if } p \equiv 0 \text{ or } 2 \text{ or } 3 (\text{mod } 4) \\ 2\left\lceil \frac{p}{4} \right\rceil & \text{if } p \equiv 1 (\text{mod } 4) \end{cases}$$

is not a dominating set, we have

$$|S'| \geq k+1 = \begin{cases} 2\left\lceil \frac{p}{4} \right\rceil & \text{if } p \equiv 0 \text{ or } 2 \text{ or } 3 (\text{mod } 4) \\ 2\left\lfloor \frac{p}{4} \right\rfloor & \text{if } p \equiv 1 (\text{mod } 4). \end{cases}$$

Hence the result follows. □

Observation 5.2.

(1) $CTATD(M_r(K_p)) = 2.$

(2) $CTATD(M_r(W_{1,n})) = 2.$

Theorem 5.5. *For a closed Helm graph* CH_n, *for* $n \geq 3$ *then*, $CTATD(CH_n) = \left\lceil \frac{n}{3} \right\rceil + 1.$

Proof. Let v_0 be apex vertex of the closed Helm graph CH_n, $(v_1, v_2, \ldots, v_n)$ be the inner cycle of CH_n and $(v_1', v_2', \ldots, v_n')$ be the outer cycle of CH_n. Let $S = \{v_i' : i \equiv 1 (\mathrm{mod}\, 3)\} \cup \{v_0\}$. Then S is a CTATD-set of (CH_n) and hence $CTATD(CH_n) \leq |S'| = \left\lceil \frac{n}{3} \right\rceil + 1$. Let S' be a CTATD-set of (CH_n). Since any set D of cardinality at most $k = \left\lceil \frac{n}{3} \right\rceil$ is not dominating set, we have $|S'| \geq k + 1 = \left\lceil \frac{n}{3} \right\rceil + 1.$

Hence the result follows. □

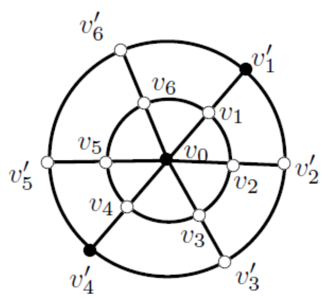

Figure 5.19.

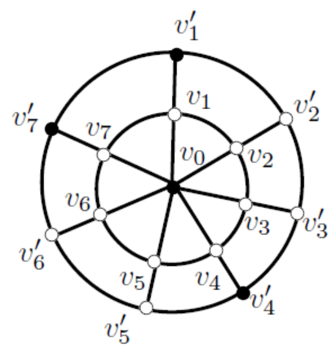

Figure 5.20.

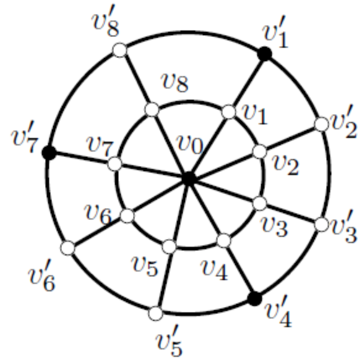

Figure 5.21.

Demonstration

Here, darked vertices are the CTATD-set, which is our S

In figure 5.19, $|S| = 3$.

As $p = 6$, $CTATD(CH_n) = \left\lceil \frac{n}{4} \right\rceil + 1$ implies $CTATD(CH_6) = \left\lceil \frac{6}{4} \right\rceil + 1 = 3$.

In figure 5.20, $|S| = 4$.

As $p = 7$, $CTATD(CH_n) = \left\lceil \frac{n}{4} \right\rceil + 1$ implies $CTATD(CH_7) = \left\lceil \frac{7}{4} \right\rceil + 1 = 4$.

In figure 5.21, $|S| = 4$.

As $p = 8$, $CTATD(CH_n) = \left\lceil \frac{n}{4} \right\rceil + 1$ implies $CTATD(CH_8) = \left\lceil \frac{8}{4} \right\rceil + 1 = 4$.

6. Conclusion

In this article we developed a new parameter called CTATD-number and found its exact values for some special types of graphs such as complete graph, wheel graph, Helm graph etc. The authors obtained results for various types of product graphs like Cartesian product, corona product, lexicographic product, strong product, which will be investigated in the subsequent articles.

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