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BACKSTEPPING PROJECTIVE SYNCHRONIZATION SCHEME AND ADAPTIVE FUNCTION PROJECTIVE SYNCHRONIZATION SCHEME ON CHAOTIC DUMBBELL SATELLITE MODEL: A COMPARATIVE STUDY

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Abstract

In this paper, we have carried out a comparative study on the methodology and applicability of two control schemes: Backstepping Designing method and Adaptive Control scheme. In this study, Projective Synchronization (PS) using Backstepping Design Control method and Function Projective Synchronization (FPS) using Adaptive Control Scheme has been achieved between two identical chaotic Dumbbell Satellite systems which is under periodic external disturbances, and each is evolving from different initial conditions. PS is a special case of FPS, and we have shown a comparison of the results on PS by both control schemes. Numerical simulations are carried out in both cases, and they are in agreement with our analytical findings.

1. Introduction

Ever since the drive-response method for synchronization of two identical chaotic systems with different initial conditions (Pecora and Carroll, [8]) was

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proposed, Chaos Synchronization has become an active research subject in non-linear science and it has been intensively studied in the last two decades (Hu and Zhang, [7]; Cai et al., [4]; Elabbasy et al., [6]). Various types of chaos synchronization have been proposed such as Complete Synchronization (Kuntanapreeda, [9]), Phase Synchronization (Erjaee and Momani, [10]), Generalized Projective Synchronization (Farivar et al., [11]), Anti-Synchronization (El-Dessoky, [13]), Lag Synchronization (Shahverdiev and Sivaprakasam, [15]), Modified Projective Synchronization (Li, [16]) and Function Projective Synchronization (Du, [17]; Luo and Wei, [18]).

In particular, among all kinds of chaos synchronization (Mainieri and Rehacek, [19]), Projective Synchronization (PS) is one of the most noticeable ones that the drive and response vectors evolve in a proportional scale- the vectors become proportional. Function Projective Synchronization (FPS), is the more general definition of projective synchronization. As compared with projective synchronization, Function Projective Synchronization (FPS) means that the drive and response systems could be synchronized upto a scaling function. This feature can be used to get more secure communications.

Chaotic Dumbbell satellite (Celletti and Sidorenko [5]) is an interesting model which is an example of a system under external periodic forces and is a departure from the usual models which are taken into consideration. Also, while applying the Adaptive control scheme, it can be aptly modified into the required form for application without affecting its chaotic nature. Thus, we choose this as our model and have also performed numerical simulations on it to confirm our analytical findings (Arriaga-Camargo et al. [1]).

Backstepping design control has been used as an effective control method in many satellite problems. It has been used to present attitude control of satellites taking into consideration the presence of uncertainties caused by external disturbance (Babaei F. S. and Akbarzadeh K. A. [2]). Recently, the modeling and controlling the fuel slosh phenomenon in a satellite has been investigated in the field of satellite attitude control using various forms of backstepping control method. Further, adaptive control methodology has also been used in synchronization of satellite problems in the past. Synchronization of two identical non-integer order chaotic satellite systems has been carried out using adaptive control methodology (Kumar S. et al. [14]). The problem of decentralized attitude synchronization and tracking for a group of spacecraft subject to inter spacecraft communication resources constraints, model uncertainties and external disturbances has been investigated using adaptive control (Wu B. et al. [3]).

In this paper, we present analysis of both projective synchronization as well as function projective synchronization and compare the results of PS obtained from both schemes.

In section 2, Backstepping procedure for obtaining PS and Adaptive control scheme in systems with uncertain time-varying parameters for obtaining FPS are explained (Tarammim and Akter A, [21]).

In section 3, the chaotic Dumbbell Satellite model is briefly introduced.

In section 4, the two methods explained in section 2 are applied on the above model.

In section 5, Numerical simulations are performed.

In section 6, we make observations. We observe the simulations and show they confirm with our analytical findings.

Finally in section 7, we analyze the results and draw a comparison of these schemes. We provide a comparison of the results obtained for projective synchronization in both cases which in turn demonstrate the effectiveness of the proposed methods. The analytical results and the numerical simulations help us in understanding which of the schemes is more suitable and under what conditions. All this helps in drawing a conclusion.

2. Projective Synchronization and Function Projective Synchronization

2.1 Projective synchronization via Backstepping design

The Backstepping design is a recursive procedure that combines the choice of Lyapunov function with the design of a controller. In this manuscript, we apply it to obtain PS between two identical chaotic systems evolving from different initial conditions.

Remark 1. It represents a systematic procedure for selecting a proper controller in chaos synchronization.

Remark 2. It can be applied to any chaotic systems whether they are externally excited or not.

Remark 3. The backstepping approach stabilizes the error system by introducing a z-subsystem related to the error e-subsystem. In each i^{th} step, it designs controller u_i to control the strict-feedback "z_i-subsystem" which in turn controls the "e_i-subsystem". At the next step, the sub-system expands and this process continues till the entire "error dynamical system $(e_1, e_2, e_3, \ldots, e_n)$" is controlled and the controls $(u_1, u_2, u_3, \ldots, u_n)$ are designed.

Thus, we take the general form of the chaotic master system as

$$\dot{x}(t) = f(x, t) \tag{1}$$

Identical slave system is given by

$$\dot{y}(t) = f(y, t) + u(t) \tag{2}$$

where $x = (x_i; 1 = 1, 2, \ldots, n)^T$, $y = (y_i; 1 = 1, 2, \ldots, n)^T$, $u = (u_i; 1 = 1, 2, \ldots, n)^T \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is the matrix function containing all external disturbances which are known. Let there exists a non-zero constant S.

We define error functions as: $e(t) = x(t) - Sy(t)$ where $e(t) = (e_i; 1 = 1, 2, \ldots, n)^T \in \mathbb{R}^n$. Clearly, if $\| e \| \to 0$ as $t \to \infty$, we say PS is achieved between systems (1) and (2). Using (1) and (2), error dynamical system is given by:

$$\dot{e}(t) = f(x, t) - Sf(y, t) - Su(t) \tag{3}$$

We now stabilize (4) by introducing and stabilizing a z-subsystem, where z's and e's are connected as follows:

Let $z_1 = e_1$, then by first equation of (3), we have

$$\dot{z}_1 = F_1(e_2, u_1) \tag{4}$$

Next, we consider e_2 as virtual controller $S_1(z_1)$. Now, by approximately choosing u_1, there is a suitable Lyapunov function V_1 such that

$$V_1(z_1) > 0; z_1 \neq 0$$

$$V_1(z_1) = 0; z_1 = 0$$

Further,

$$\dot{V}_1(z_1) < 0; z_1 \neq 0$$

$$V_1(z_1) = 0; z_1 = 0$$

Thus, (4) is asymptotically stable. Next, as e_2 is a virtual controller, we define $z_2 = e_2 - S_1(z_1)$ so that new $z_1 - z_2$ subsystem is defined as

$$\dot{z}_1 = F_1(z_1, u_1)$$

$$\dot{z}_2 = F_1(z_1, z_2, e_3, u_2) \tag{5}$$

In order to stabilize this subsystem, we again choose u_2 in such a way its second Lyapunov function $V_2(z_1, z_2)$ exists, V_2 is positive definite and $\dot{V}_2$ is negative definite. So, it can be concluded that (5) is asymptotically stable. We continue this procedure, till we obtain the stable full dimension $(z_1, z_2, \ldots, z_n)$ via last Lyapunov function by the design of a control input function un. Finally, following the definition of the e_i's, it follows that $(e_1, e_2, \ldots, e_n)$ is also stable and converges to $(0, 0, \ldots, 0)$.

2.2. Function Projective synchronization via Adaptive Control Scheme

More often than not, the chaotic systems contain numerous parameters and in most practical situations, their values are uncertain and time-varying. To address these issues, we design adaptive controllers which have the additional property that they can be decomposed into a parameter estimation module together with control-law synthesis procedure. For this purpose, we re-write the chaotic master system as:

$$\dot{x}(t) = f_1(x, t) + f_2(x, t)\eta \tag{6}$$

and identical chaotic slave system is:

$$\dot{y}(t) = f_1(y, t) + f_2(y, t)\hat{\eta}(t) + u(t) \tag{7}$$

where $x = (x_i; 1 = 1, 2, \ldots, n)^T$, $y = (y_i; 1 = 1, 2, \ldots, n)^T$, $u = (u_i;\ 1 = 1, 2, \ldots, n)^T \in \mathbb{R}^n$; u is the controller vector, $f_1 : \mathbb{R}^n \rightarrow \mathbb{R}^n$,

$f_2 : \mathbb{R}^n \to \mathbb{R}^{n \times p}$, $\eta = (\eta_i; 1 = 1, 2, \ldots, p)^T$ is the known parameter vector and $\hat{\eta}(t) = (\hat{\eta}_i(t); 1 = 1, 2, \ldots, p)^T$ is the uncertain time-varying parameter vector.

Let $S(t) (\neq 0 \, \forall t \geq 0)$ be any continuous function. We define FPS error functions as $e(t) = x(t) - S(t)y(t)$ where $e(t) = (e_i; 1 = 1, 2, \ldots, p)^T$.

Definition. The systems defined by (1) and (2) are said to be *F.P.* synchronous upto the desired scaling function $S(t)$ if

$$\lim_{t \to \infty} \| e(t) \| = 0 \tag{8}$$

where $\| \cdot \|$ represents vector norm.

Remark. If $S(t) = \alpha I_n$, $\alpha \in \mathbb{R}$, the FPS problem reduces to projective synchronization (PS). Further, if $S(t) = I_n$, we obtain complete synchronization (CS) while $S(t) = -I_n$ implies anti-synchronization(AS). Thus, PS, CS and AS are special cases of FPS.

The error dynamical equation using (6) and (7) is given by:

$$\dot{e}(t) = f_1(x, t) - f_2(x, t)\eta + S(t)\{f_1(y, t) + f_2(y, t)\hat{\eta}(t) + u(t)\} - \dot{S}(t)y \tag{9}$$

For given continuous function $S(t)$ and any initial value (x(0), y(0)), systems (1) and (2) will be F.P. synchronous and further uncertain time-varying parameter $\hat{\eta}$ will be estimated if the adaptive control law and parameter update laws are designed as follows:

Adaptive control law:

$$- S(t)u(t) = -f_1(x) - f_2(x)\hat{\eta} + S(t)\{f_1(y, t) + f_2(y, t)\hat{\eta}(t)\} + \dot{S}(t)y - ke \tag{10}$$

Parameter update law:

$$\dot{\hat{\eta}} = \{f_2(x)\}^T e - k_n(\hat{\eta} - \eta) \tag{11}$$

where $k = diag(k_i; 1, 2, \ldots, n)$, $k_\eta = diag(\hat{k}_i; 1, 2, \ldots, p)$ and all k's are positive constant gains.

Proof. Substituting (10) into (9), we get

$$\dot{e}(t) = -f_2(x, t)(\hat{\eta} - \eta) - ke \tag{12}$$

We now choose Lyapunov function V as:

$$V(t) = \frac{1}{2} e^T e + \frac{1}{2} e_\eta^T e_\eta$$

where $e_\eta = \hat{\eta} - \eta$. Thus, $\dot{e}_\eta = \dot{\hat{\eta}}$. Finally, using (11) and (12) we obtain,

$$\begin{aligned}\dot{V}(t) &= e^T \dot{e} + e_\eta^T \dot{e}_\eta \\ &= e^T [- f_2(x, t)(\hat{\eta} - \eta) - ke] + e_\eta^T [\{f_2(x)\}^T e - k_n(\hat{\eta} - \eta)] \\ &= -ke^T e - k_\eta e_\eta^T e_\eta - [e_\eta^T \{f_2(x)\}^T e]^T + e_\eta^T \{f_2(x)\}^T e \\ &= -ke^T e - k_\eta e_\eta^T e_\eta \\ &< 0 \end{aligned} \tag{13}$$

So, by Lyapunov stability theorem, the error dynamical system is globally, asymptotically stable at equilibrium point $0 \in \mathbb{R}^n$. Thus, FPS is achieved between the systems (6) and (7).

3. Model Explanation

The equation of motion of a Dumbbell satellite in the central gravitational field of the earth under the influence of the solar radiation pressure together with the effects of Earth's shadow and phenomenological factor (Sharma et al. [20]) is given by:

$$(1 + e \cos v)\ddot{\phi} - 2e \sin v\dot{\phi} + 3 \sin \phi \cos \phi + \rho^3 k \cos \varepsilon \sin(\phi + \alpha)$$

$$= 2e \sin v + E \sin u$$

where the parameters are defined as follows:

$e =$ eccentricity of the orbit of the center of mass

$\rho =$ variable radius of circular orbit

$E =$ Phenomenological parameter characterizing the periodic term.

$\nu =$ The frequency of the external periodic force.

$\varepsilon =$ Inclination of the oscillating plane of the orbit of the center of mass of the system with the plane of ecliptic.

$\alpha =$ The angular separation of the solar position vector projected on the orbital plane.

$v =$ The true anomaly of the center of mass of the system in the ecliptic orbit.

$$k = \frac{\rho^3}{\pi\mu}\left[\frac{B_1}{m_1} - \frac{B_2}{m_2}\right]\delta_r \sin\theta$$

where $\mu =$ The product of gravitational constant and mass of the Earth

$B_i; (i = 1, 2, 3) =$ The absolute values of the forces due to direct solar radiation pressure exerted on masses of satellites m_1 and m_2 respectively.

$\delta_r =$ Earth's shadow function

$\theta =$ Angle between the axis of cylinder (shadow beam) and line joining the Earth's center and the end point of the orbit of the center of mass.

We note that in this model, instead of 't', true anomaly 'v' is the independent variable. Thus, in the next section, all dots will represent differentiation w.r.t. v.

4. Application of PS and FPS to Dumbbell Satellite Model

4.1 PS using Backstepping design

Let $\phi = x_1$, $\dot{\phi} = x_2$

Thus, system (14) can be now written as:

$$\dot{x}_1(v) = x_2 \tag{15}$$

$$\dot{x}_2(v) = \frac{1}{1 + e\cos v}[2e\sin v x_2 - 3\sin x_1 \cos x_1 - \rho^3 k \cos\varepsilon \sin(x_1 + \alpha)$$
$$+2e\sin v + E\sin(\nu v)]$$

(15) is the master system. We write down the slave system as:

$$\dot{y}_1(v) = y_2 + u_1 \tag{16}$$

$$\dot{y}_2(v) = \frac{1}{1 + e\cos v}[2e\sin v y_2 - 3\sin y_1 \cos y_1 - \rho^3 k \cos\varepsilon \sin(y_1 + \alpha)$$
$$+2e\sin v + E\sin(\nu v)]$$

where u_1, u_2 are the controllers to be determined. (16) is the slave system. PS errors e_1, e_2 are defined as:

$$e_1 = x_1 - Sy_1$$

$$e_2 = x_{21} - Sy_2 \tag{17}$$

$$\dot{e}_1(v) = e_1 - Su_1$$

$$\dot{e}_2(v) = \frac{1}{1 + e\cos v}[2e\sin v e_2 + \phi] - Su_2 \tag{18}$$

where

$$\phi = -3(\sin x_1 \cos x_1 - S\sin y_1 \cos y_1) - \rho^3 k\cos\varepsilon[\sin(x_1 + \alpha) - S\sin(y_1 + \alpha)]$$
$$+ (2e\sin v + E\sin(\nu v))(1 - S). \tag{19}$$

Now, we apply backstepping design as discussed in (2.1)

(i) Let

$$z_1 = e_1 \tag{20}$$

This implies

$$\dot{z}_1(v) = e_2 - Su_1 \tag{21}$$

Here $e_2 = S_1(z_1)$ is a virtual controller. S_1 is a control to stabilise z_1 z1-subsystem (21). In order to stabilise z_1-subsystem (21), we choose Lyapunov function V_1 as:

$$V_1 = \frac{1}{2}z_1^2$$

This gives

$$\frac{dV_1}{dv} = z_1 \frac{dz_1}{dv}$$

Further if

$$S_1 = 0 \tag{22}$$

and,

$$Su_1 = z_1 \tag{23}$$

Then,

$$\frac{dV_1}{dv} = -z_1^2$$

Thus, (7) is asymptotically stable.

(ii) As, S_1 is estimative, error between e_2 and $S_1(z_1)$ is given by

$$z_2 = e_2 - S_1(z_1) \tag{24}$$

Let us investigate sub-system:

$$\dot{z}_1(v) = z_2 - z_1$$

$$\dot{z}_2(v) = \frac{1}{1 + e\cos v}[2e\sin v e_2 + \phi] - Su_2 \tag{25}$$

In order to stabilise $z_1 - z_2$ subsystem, we choose a Lyapunov function V_2 as:

$$V_2 = V_1 + \frac{1}{2}z_2^2$$

This gives

$$\frac{dV_2}{dv} = z_1(z_2 - z_1) + z_2\left[\frac{1}{1 + e\cos v}[2e\sin v e_2 + \phi] - Su_2\right]$$

If

$$Su_2 = \frac{1}{1 + e\cos v}[2e\sin v e_2 + \phi] + z_2 + z_1 \tag{26}$$

Then,

$$\frac{dV_2}{dv} = -z_1^2 - z_2^2 < 0$$

Thus, (25) is asymptotically stable. In $z_1 - z_2$ coordinates, using Lyapunov's second theory, $(0, 0)$, which is the only equilibrium point is globally asymptotically stable. So, it follows that $z_1 = e_1, z_2 = e_2$, so, as all $(z_1, z_2) \to (0, 0)$ asymptotically.

Thus, PS is achieved.

4.2 FPS using Adaptive Control Scheme

We note that system (14) cannot be put in the form of (6). Thus, we propose certain modifications in the master system, which are valid near the origin and the system is still chaotic for a different set of parametric values. Near the origin, $\cos v \approx 1, \sin(\nu v) \approx \nu v$ where ν has a finite value.

We also re-define the parameters as:

$$a = \frac{2e}{1+e}, \; b = \frac{3}{1+e}, \; c = \frac{\rho^3 k \cos \varepsilon \cos \alpha}{1+e}, \; d = \frac{\rho^3 k \cos \varepsilon \sin \alpha}{1+e}, \; f = \frac{E\nu}{1+e}$$

So, the master system now changes to:

$$\dot{x}_1(v) = x_2$$

$$\dot{x}_2(v) = a \sin v(x_2 + 1) - b \sin x_1 \cos x_1 - c \sin x_1 - d \cos x_1 + fv \tag{27}$$

where a, b, c, d, f are the parameters.

The corresponding slave system is expressed as:

$$\dot{y}_1(v) = y_2 + u_1$$

$$\dot{y}_2(v) = a_1 \sin v(y_2 + 1) - b_1 \sin y_1 \cos y_1 - c_1 \sin y_1 - d_1 \cos y_1 + f_1 v + u_2 \tag{28}$$

where a_1, b_1, c_1, d_1, f_1 are the uncertain parameters (functions of v) to be estimated and u_1, u_2 are the non-linear controller functions such that the two chaotic systems are functional projective synchronized. Let $S(v)$ be the known scaling function. Then, we define the error functions as: $e_i = x_i - S(v)y_i, i = 1, 2.$ Clearly, FPS between (27) and (28) is achieved upto the desired scaling function $S(v)$ iff $\| e_i \| \to 0$ as $v \to \infty$.

The error dynamical system is given by:

$$\dot{e}_1(v) = e_2 - \dot{S}(v)y_1 - S(v)u_1$$

$$\dot{e}_2(v) = g(x_1, x_2) - \dot{S}(v)y_2 - S(v)g_1(y_1, y_2) - S(v)u_2 \tag{29}$$

where

$$g(z_1, z_2) = a \sin \nu(z_2 + 1) - b \sin z_1 \cos z_1 - c \sin z_1 - d \cos z_1 + fv$$

and

$$g_1(z_1, z_2) = a_1 \sin \nu(z_2 + 1) - b_1 \sin z_1 \cos z_1 - c_1 \sin z_1 - d_1 \cos z_1 + f_1 v$$

In order to stabilize the error variables at the origin, we propose the active control law as:

$$-S_1 u_1 = -e_2 + \dot{S}(v)y_1 - k_1 e_1$$

$$-S_2 u_2 = -g_1(x_1, x_2) + \dot{S}(v)y_2 + S(v)g_1(y_1, y_2) - k_2 e_2 \tag{30}$$

Parameter update law for uncertain parameters as:

$$\dot{a}_1(v) = \sin v(x_2 + 1)e_2 - k_3 e_a$$

$$\dot{b}_1(v) = -\sin x_1 \cos x_1 e_2 - k_4 e_b$$

$$\dot{c}_1(v) = -\sin x_1 e_2 - k_5 e_c$$

$$\dot{d}_1(v) = -\cos x_1 e_2 - k_6 e_d$$

$$\dot{f}_1(v) = v e_2 - k_7 e_f \tag{31}$$

where $e_a = a_1 - a$, $e_b = b_1 - b$, $e_c = c_1 - c$, $e_d = d_1 - d$, $e_f = f_1 - f$ and $k_i > 0; i = 1, 2, \ldots, 7$ are constant gains.

The Lyapunov function is constructed as:

$$V = \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2 + \frac{1}{2}e_a^2 + \frac{1}{2}e_b^2 + \frac{1}{2}e_c^2 + \frac{1}{2}e_d^e + \frac{1}{2}e_f^2$$

Using (29), (30) and (31), we find that

$$\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_a^2 - k_4 e_b^2 - k_5 e_c^2 - k_6 e_d^2 - k_7 e_f^2$$

which is negative-definite. Thus, the error dynamical system (29) is globally and asymptotically stable at the origin and hence we have, $e_1, e_2, e_a, e_b, e_c, e_d, e_f \to 0$ as $\upsilon \to \infty$. FPS between master system (27) and slave system (28) is achieved and the uncertain parameter is also identified in the receiver end simultaneously under the controllers given by (30) and parameter update law given by (31).

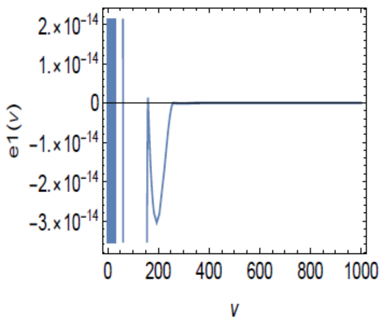
5. Numerical Simulations

We now present some numerical simulations carried out using Mathematica.

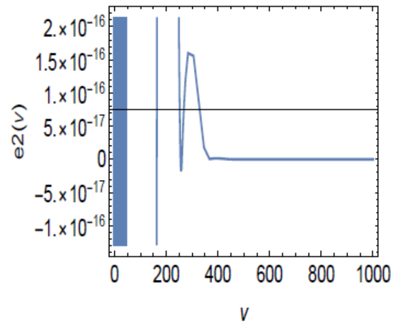
1. For showing projective synchronization using, we take the initial conditions for master system as $x_1(0) = 0.001, x_2(0) = 0.0001$ and the values of the parameters are chosen as $e = 0.01, \rho = 1.9005, k = 0.0025, \varepsilon = 0.1, \alpha = 1.35, \nu = 3.7, E = 0.00005$ while the initial conditions for the slave system are $y_1(0) = 5, y_2 = 0.05$. Figures 1 and 2 show the time series analysis of error variables e_1 and e_2. Figure 3 and 4 show the phase portraits of master system and slave system without controls. After the controls are added to the slave system, the phase portraits of the projective-synchronized slave systems for three different cases, $S = 0.005, 0.01$ and 0.02 are superimposed and shown in a single figure 5. Figure 6 and 7 show phase portraits of the slave system for $S = 1$ and -1 which correspond to the special cases of complete synchronization and anti-synchronization respectively.

2. For showing function projective synchronization, the initial conditions and parametric values for the master system are chosen as $x_1(0) = 1.23$, $x_2(0) = 0.1, a = 0.106061, b = 2.84091, c = 0.492778, d = -3.79271$, $f = 0.00020593$. The initial conditions and the initial values of the uncertain parametric values for the slave system are taken as: $y_1(0) = 0.6$, $y_2(0) = 0.003, a_1(0) = 0.0178394, b_1(0) = 2.97324, c_1(0) = 0.829319$, $d_1(0) = -1.81209, f_1(0) = 0.0000218038$. Phase portrait of the master system is given in figure 8 and phase portraits of synchronized slave system for different scaling factors is presented in figure 9. Let us choose the value of

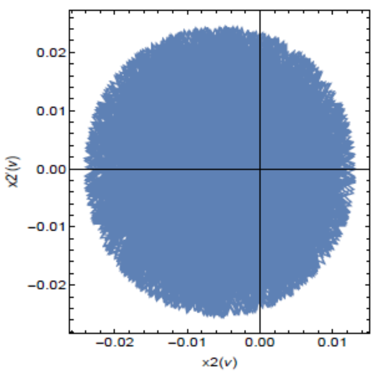
function $S(v) = 5$. Then, the time evolution of the estimated parameters a_1, b_1, c_1, d_1, f_1 are shown in figures 10 to 14. The time series of the error variables e_1 and e_2 are shown in figures 15 and 16.



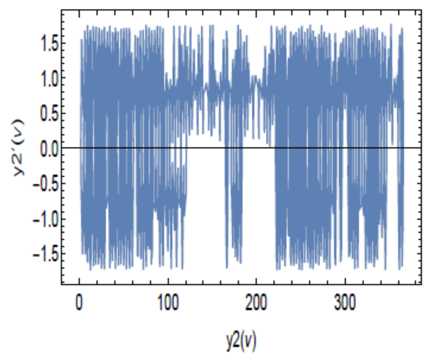
(a) **Figure 1.** Time Series Analysis of $e_1(v)$ under PS



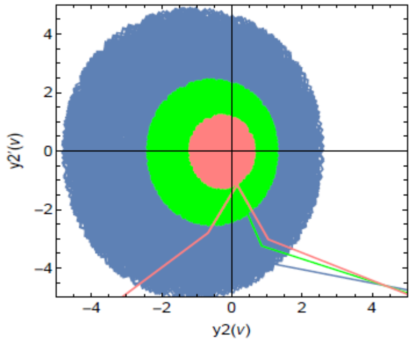
(b) **Figure 2.** Time Series Analysis of $e_2(v)$ under PS



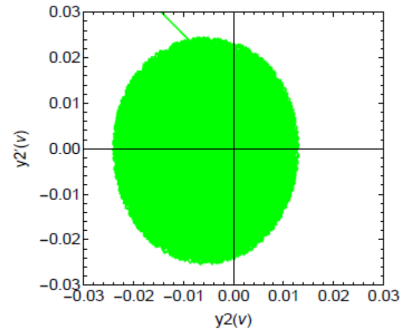
(c) **Figure 3.** Phase Portrait of master system (15)



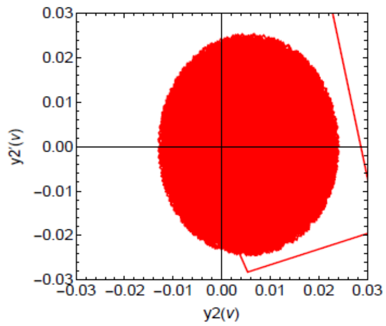
(d) **Figure 4.** Phase Portrait of slave system (16) (without controls)



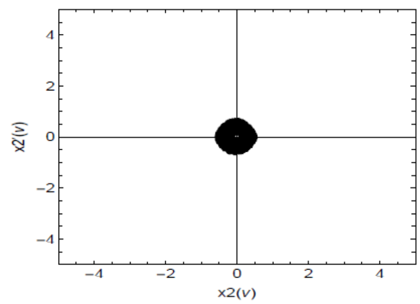
(e) **Figure 5.** Phase Portrait of slave system with Backstepping method (BM) controllers for $S = 0.005$ (blue), 0.01 (green) and 0.02(red)



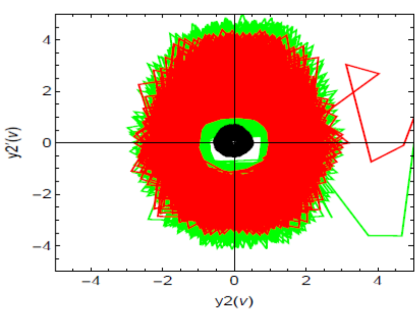
(f) **Figure 6.** Phase Portrait of slave system by BM for $S = 1$ implying complete synchronization



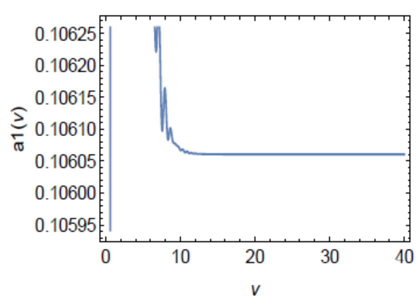
(g) **Figure 7.** Phase Portrait of slave system by BM for $S = -1$ implying antisynchronization



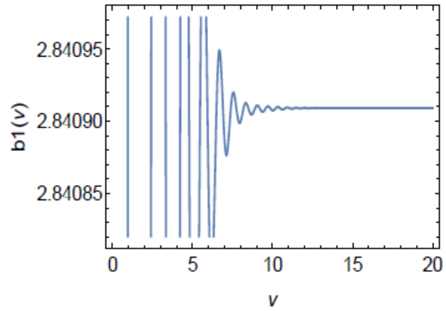
(h) **Figure 8.** Phase Portrait of master system (27)



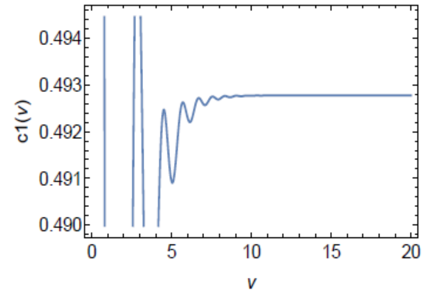
(i) **Figure 9.** Phase Portraits of slave system (28) by Adaptive control (AC) for $S(v) = 1$ (black), 0.1 (red), 0.01 (green)



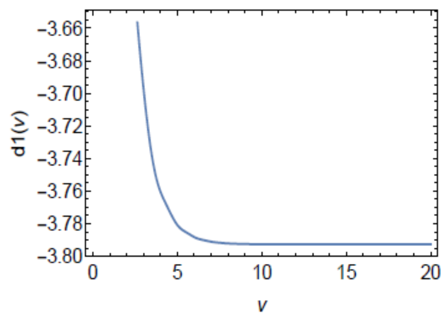
(j) **Figure 10.** Time Series Analysis of $a_1(v)(a_1 \to a = 0.106061)$



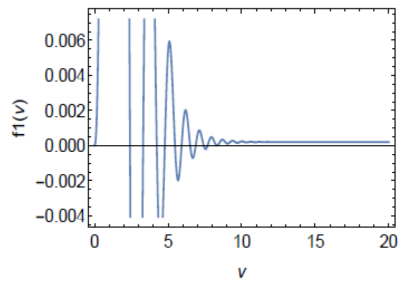
(k) **Figure 11.** Time Series Analysis of $b_1(v)(b_1 \to b = 2.84091)$



(l) **Figure 12.** Time Series Analysis of $c_1(v)(c_1 \to c = 0.492778)$

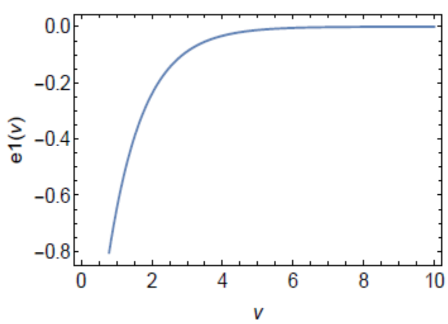


(m) **Figure 13.** Time Series Analysis of $d_1(v)(d_1 \to d = -3.79271)$

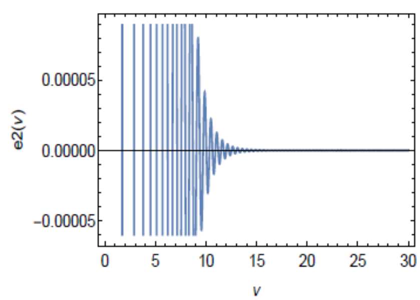


(n) **Figure 14.** Time Series Analysis of

$f_1(v)(f_1 \to f = 0.00020593)$



(o) **Figure 15.** Time Series Analysis of $e_1(v)$ by AC



(p) **Figure** 16. Time Series Analysis of $e_2(v)$ by AC

6. Observations

1. The simulations for Backstepping Method are presented by figures 1 to 7. We make the following observations:

(a) Comparing the phase portraits of the master system given by figure 3 and those of slave system given by figures 5 to 7, it can be clearly seen that not only the two systems (27) and (28) synchronize, they can be made to synchronize up to any factor by changing the value of the scaling factor S.

(b) When $S = 1$, we achieve complete synchronization while $S = -1$ leads to anti-synchronization between (27) and (28). Thus, PS is a more general synchronization phenomenon and complete synchronization and antisynchronization can be obtained as special cases.

(c) Time evolution of error variables given by figure 1 and 2 imply that we are successful in achieving PS between (15) and (16) asymptotically.

2. The simulations for Adaptive Control Method are presented by figures 8 to 16. We make the following observations:

(a) Figure 8 show the phase portrait for modified master system (27) and synchronized phase portraits of modified slave system (28) for different scaling factors is shown in figure 9. It effectively shows PS is achieved for different scaling factors.

(b) To observe the achievement of FPS, we note that figures 10 to 14 show the time evolution of the uncertain parameters a_1, b_1, c_1, d_1, f_1 of the slave system and we find that $a_1 \to a, b_1 \to b, c_1 \to c, d_1 \to d, f_1 \to f$ as $v \to \infty$ where a, b, c, d, f are the certain parametric values of the master system. Thus, parameter update law is verified.

(c) Figures 15 and 16 which show error time analysis, indicate that FPS is achieved between the two systems (27) and (28).

7. Conclusion

From the numerical simulations, it is clear that both the control methods have been successful in achieving PS between the respective master and slave system. While comparing their simulations, we note:

1. A major advantage of Backstepping control is, it can be applied to any dynamical system with first order differential equations, while for Adaptive control, modifications were needed to reduce it to the form of (6).

2. Figure 5 shows phase portraits of synchronized slave system for different scaling factors by Backstepping control while figure 9 shows the same by Adaptive control. We can clearly see that for the chaotic Dumbbell Satellite model, the simulations were better for Backstepping control.

3. A major advantage of the Adaptive control is, it has an additional feature of tackling uncertainties brought in by the time-varying parameters. Parameter update laws ensure that eventually the uncertain parameters of the slave system tend to the corresponding constant parametric values of the master system. Backstepping control does not have this feature.

4. Also, the error variables of the FPS scheme by Adaptive control converge faster to zero than their counterparts under PS scheme by Backstepping control.

So, we can conclude that both the control methods have their advantages as well as disadvantages. However, if the model under consideration has uncertainties due to parameters, Adaptive control method is a better method. Rate of synchronization is also better than backstepping control for this model. Whereas, Backstepping control is effective for most of the dynamical systems, even under external disturbances, but with certain parameters. It is a recursive method and the controllers can be designed step by step effectively and the method is gradually extended to larger and larger subsystems as needed. PS has been effectively achieved for the chaotic Dumbbell Satellite model by both the control methods and numerical simulations confirm with the analytical findings.

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