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SOME FIXED POINT THEOREMS OF INTEGRAL TYPE IMPLICIT RELATION IN RANDOM CONE METRIC SPACE VIA RANDOM OPERATORS

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Abstract

The main goal of the paper, we discuss some different types of contractions in random cone metric spaces and prove some fixed point theorems of integral type implicit relations by use of random operators.

Introduction

Randomness raises various new problems about solution measurability, as well as probabilistic and statistical features of random solutions. Stochastic generalisations of classical common fixed point theorems are known as common random fixed point theorems. The financial markets have been transformed by random approaches. In 1976, Bharucha-Reid [4] came to the attention of many mathematicians and gave enormous theoretical approaches. Itoh [9] extended the results of Spacek and Hans in multi-valued contractive mappings. Papageorgiou [10,11] and Beg [2,3] found common random fixed points and random coincidence points of a pair of compatible random operators, as well as fixed point theorems for contractive random operators in Polish spaces.

Huang and Zhang [7] defined the cone metric spaces by generalizing the

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concept of metric spaces by replacing the set of real numbers with an ordered Banach space in 2007. They also developed the concept of completion in cone metric spaces and defined sequence of convergence. The assumption of normality of a cone and they proved certain fixed point theorems of contractive mappings on complete cone metric space. Other researchers [8, 11-13] investigated the existence of fixed points and common fixed points of mappings satisfying contractive type conditions on a normal cone metric space. Rezapour and Hamlbarani [12] removed the assumption of normality in cone normal spaces in 2008, which was a significant step forward in the development of fixed point theory in cone metric spaces.

Preliminaries

Definition 2.1 [18]**.** Let (E, τ) be a topological vector space and P a subset of E, P is called cone if

1. P is non empty and closed, $P \neq \{0\}$,

2. For $x, y \in P$ and $a, b \in R \Rightarrow ax + by \in P$ where $a, b \geq 0$,

3. If $x \in P$ and $-x \in P \Rightarrow x = 0$.

For a given cone $P \subseteq E$, a partial ordering $\leq$ with respect to P is defined by $x \leq y$ if and only if $y - x \in P$, $x < y$ if $x \leq y$ and $x \neq y$, while $x << y$ will stand for $y - x \in \operatorname{int} P$ denotes the interior of P.

Let E be a real banach space, $P \in E$ cone and $\leq$ partial ordering defined by P. Then P is called normal if there is a number $K > 0$ such that for all $x, y \in P$. $0 \leq x \leq y$ imply $\| x \| \leq k \| y \|$ or equivalently, if $(\forall n) x_n \leq y_n \leq z_n$ and $\lim_{n \to \infty} x_n = \lim_{n \to \infty} z_n = x$ imply $\lim_{n \to \infty} y_n = x$. The least positive number K satisfying (1) is called the normal constant of P. It is clear that $K \geq 1$.

Definition 2.2 [10] (Measurable function)**.** Let (Ω, Σ) be a measurable space with Σ a sigma algebra of subsets of Ω and M a non-empty subset of metric space $X = (X, d)$. Let 2^M be the family of all non-empty subsets of M where $C(M)$ the family of all non empty closed subsets of M. A mapping $G : \Omega \to 2^M$ is called measurable if, for each open subset U of M, such that

$G^{-1} \in \Sigma$, where $G^{-1}(U) = \{\omega \in \Omega, G(\omega) \cap U \neq \emptyset\}$.

Definition 2.3 [10] (Measurable selector). A mapping $\xi : \Omega \to M$ is called a measurable selector of a measurable mapping $G : \Omega \to 2M$ if $\xi(\omega) \in G(\omega)$ for each $\omega \in \Omega$.

Definition 2.4 [10] (Random operator). Mapping $T : \Omega \times M \to X$ is said to be a random operator if, for each fixed $x \in M, T(\cdot, x) : \Omega \to X$ is measurable.

Definition 2.5 [10] (Continuous random operator). A random operator $T : \Omega \times M \to X$ is said to be continuous random operator if, for each fixed $x \in M, T(\cdot, x) : \Omega \to X$ is continuous.

Definition 2.6 [10] (Random fixed point). A measurable mapping $\xi : \Omega \to M$ is a random fixed point of random operator $T : \Omega \times M \to X$ if $\xi(\omega) = T(\omega, \xi(\omega))$ for each $\omega \in \Omega$.

Definition 2.7 [9]. Let M be a non empty set and the mapping $d : \Omega \times M \to X$ and $P \subset X$ be a cone, $\omega \in \Omega$ be a selector, satisfies the following conditions:

(1) $d(x(\omega), y(\omega)) > 0 \; \forall x(\omega), y(\omega) \in \Omega \times M \Leftrightarrow x(\omega) = y(\omega)$

(2) $d(x(\omega), y(\omega)) = d(y(\omega), x(\omega)) \; \forall x, y \in X, \omega \in \Omega$ and $x(\omega), y(\omega) \in \Omega \times X$

(3) $d(x(\omega), y(\omega)) = d(x(\omega), z(\omega)) + d(z(\omega), y(\omega)) \; \forall x, y \in X,$ and $\omega \in \Omega$ be a selector.

(4) For any $x, y \in X, \omega \in \Omega, d(x(\omega), y(\omega))$ is a non decreasing and left continuous in α. Then d is called cone random metric in M and (M, d) is a cone random metric space.

Example [2]. Let $E = R^2, P = \{(x, y) \in R^2 : x \geq 0, y \geq 0\}, X = \mathbb{R}$ and $d : X \times X \to E$ defined by $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space with normal cone P where $K = 1$.

Example [2]. Let $E = \ell^2$, $P = \{\{x_n\}_{n\geq 1} \in E : x_n \geq 0, \forall n\}$, (X, ρ) a metric space and $d : X \times X \to E$ defined by $d(x, y) = \left\{\frac{\rho(x, y)}{2^n}\right\}_{n\geq 1}$. Then (X, d) is a cone metric space. Clearly the above examples show that class of cone metric spaces contains the class of metric spaces.

Example [15]. Let $M = R$ and $P = \{x \in M : x \geq 0\}$, also $\Omega = [0, 1]$ and Σ be the sigma algebra of Lebegsue's measurable subset of $[0, 1]$. Let $X = [0, \infty)$ and define a mapping $d : (\Omega \times X) \times (\Omega \times X) \to P$ by $d(x(\omega)y(\omega)) = |x(\omega) - y(\omega)|$. Then (X, d) is a random cone metric space.

Definition 2.8 [9]. Let (X, d) be a cone metric space. We say that $\{x_n\}$ is:

(i) A Cauchy sequence if for every $\epsilon \in E$ with $0 << \epsilon$, then there is an N such that for all $n, m > N$, $d(x_n, x_m) << \epsilon$.

(ii) A convergent sequence if for every $\epsilon \in E$ with $0 << \epsilon$, then there is an N such that for all $n > N$, $d(x_n, x) << \epsilon$ for some fixed x in X.

(iii) A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

In the following (X, d) will stands for a cone metric space with respect to a cone P with $P^0 \neq \emptyset$ in a real banach space E and $\leq$ is partial ordering in E with respect to P.

Definition 2.9 [16]. Suppose that P is a normal cone in $E. a, b \in E$ and $a < b$. We define

$$\begin{cases}[a, b] = \{x \in E; x = tb + (1-t)a \text{ for some } t \in [0, 1]\} \\ [a, b) = \{x \in E; x = tb + (1-t)a \text{ for some } t \in [0, 1)\}\end{cases}$$

Definition 2.10 [16]. The set $\{a = x_0, x_1, x_2, \ldots, x_n = b\}$ is called a partition for $[a, b]$ if and only if the sets $\{x_{t-1}, x_t\}_{i=1}^{n}$ are pairwise disjoint and $[a, b] = \{\bigcup_{t=1}^{n} \{x_{t-1}, x_t\} \cup \{b\}\}$.

Definition 2.11 [16]. For each partition Q of $[a, b]$ and each increasing function $\zeta : [a, b] \to P$, we define cone lower summation and cone upper summation as

$$L_n^{con}(\zeta, Q) = \sum_{t=0}^{n-1} \zeta(x_t) \| x_t - x_{t+1} \| \text{ and}$$

$$U_n^{con}(\zeta, Q) = \sum_{t=0}^{n-1} \zeta(x_{t+1}) \| x_t - x_{t+1} \| \tag{2.2}$$

respectively.

Definition 2.12 [16]. Suppose that p is normal cone in $E. = \zeta : [a \cdot b] \to p$ is called a integrable function on $[a, b]$ with respect to cone P or to simplicity. Cone integrable function, if and only if for all partition Q of $[a, b]$, $\lim_{n \to \infty} L_n^{con}(\zeta, Q) = S^{con} = \lim_{n \to \infty} U_n^{con}(\zeta, Q)$. Where S^{con} must be unique. We show the common value S^{con} by $\int_a^b \zeta(x) d_p(x)$ to simplicity $\int_a^b \zeta d_p$.

Definition 2.13 [15]. The function $\zeta : P \to E$ is called sub additive cone integrable function if and only if for all $a, b \in P, \int_a^{a+b} \zeta d_p \leq \int_0^a \zeta d_p + \int_0^b \zeta d_p$

Example [15]. Let $E = X = R, d(x, y) = | x - y |, P = (0, \infty)$ and $\zeta(t) = \frac{1}{(t+1)}$ for all $t > 0$ then for all $a, b \in P, \int_0^{a+b} \frac{dt}{(t+1)} = In(a + b + 1)$, $\int_0^a \frac{dt}{(t+1)} = In(a+1)$ and $\int_0^b \frac{dt}{(t+1)} = In(b+1)$. Since $ab \geq 0$, then $a + b + 1 \leq a + b + 1 + ab = (a+1)(b+1)$. Therefore $In(a + b + 1) \leq In(a+1)$ $\leq In(b+1)$. This shows that ζ is an example of sub additive cone integrable function.

Definition 2.14 (implicit relation). Let ϕ be the class of real valued continuous functions $\phi : (R^+)^4 \to R^+$ non-decreasing in the first argument and satisfying the following conditions: For $x, y \geq 0, x \leq$

$\phi\left(y, \frac{x+y}{2}, 0, x+y\right)$ or $x \leq \phi\left(y, \frac{x}{2}, y, x\right)$ or $x \leq \phi(x, y, x, x)$ there exists a real number $0 < h < 1$ such that $x \leq hy$.

Main Results

Theorem 3.1. *Let (X, d) be a complete cone random metric space and let M be a non-empty separable closed subset of cone metric space X and let T be a continuous random on M such that for all $\omega \in \Omega, T(\omega, \cdot) : \Omega \times M \to M$ satisfying contraction*

$$\int_0^{d(T(x(\omega)),\, Ty(\omega))} \zeta(t)dt \leq$$

$$\varphi\begin{pmatrix} \int_0^{d(x(\omega),\, y(\omega))} \zeta(t)dt, \int_0^{d(y(\omega),\, Tx(\omega))} \zeta(t)dt \\ + \int_0^{d(y(\omega),\, ty(\omega))} \zeta(t)dt, \int_0^{d(x(\omega),\, Tx(\omega))} \zeta(t)dt + \\ \int_0^{d(y(\omega),\, Tx(\omega))} \zeta(t)dt, \int_0^{d(x(\omega),\, T(y(\omega),\, T(y(\omega),\, T(x(\omega)))))} \zeta(t)dt \end{pmatrix} \tag{3.1}$$

For all $y \in X, \omega \in \Omega$. Then T has a fixed point in X.

Proof. For each $x_0(\omega) \in \Omega \times X$ and $n \geq 1$, let $x_1 = Tx_0$ and $x_{n+1}(\omega) = T(x_n(\omega)) = T^{n+1}x_0(\omega)$. Then $\int_0^{d(x_n(\omega),\, x_{n+1}(\omega))} \zeta(t)dt$

$$= \int_0^{d(x_{n-1}(\omega),\, Tx_n(\omega))} \zeta(t)dt$$

$$\leq \varphi\begin{pmatrix} \int_0^{d(x_{n-1}(\omega),\, x_n(\omega))} \zeta(t)dt, \int_0^{d(x_{n-1}(\omega),\, Tx_{n-1}(\omega))} \zeta(t)dt + \int_0^{d(x_n(\omega),\, Tx_n(\omega))} \zeta(t)dt, \\ \int_0^{d(x_{n-1}(\omega),\, Tx_{n-1}(\omega))} \zeta(t)dt + \int_0^{d(x_n(\omega),\, Tx_{n-1}(\omega))} \zeta(t)dt, \\ \int_0^{d(x_{n-1}(\omega),\, Tx_n(\omega))} \zeta(t)dt + \int_0^{d(x_n(\omega),\, Tx_{n-1}(\omega))} \zeta(t)dt \end{pmatrix}$$

$$\leq \varphi\begin{pmatrix} \int_0^{d(x_{n-1}(\omega),\, x_n(\omega))} \zeta(t)dt,\ \int_0^{d(x_n(\omega),\, x_n(\omega))} \zeta(t)dt + \int_0^{d(x_n(\omega),\, x_{n+1}(\omega))} \zeta(t)dt, \\ \int_0^{d(x_n(\omega),\, x_n(\omega))} \zeta(t)dt + \int_0^{d(x_{n-1}(\omega),\, x_n(\omega))} \zeta(t)dt, \\ \int_0^{d(x_{n-1}(\omega),\, x_{n+1}(\omega))} \zeta(t)dt + \int_0^{d(x_n(\omega),\, x_n(\omega))} \zeta(t)dt \end{pmatrix}$$

$$\leq \varphi\begin{pmatrix} \int_0^{d(x_{n-1}(\omega),\, x_n(\omega))} \zeta(t)dt,\ 0 + \int_0^{d(x_n(\omega),\, x_{n+1}(\omega))} \zeta(t)dt, \\ \int_0^{d(x_{n-1}(\omega),\, x_n(\omega))} \zeta(t)dt + 0, \\ \int_0^{d(x_{n-1}(\omega),\, x_n(\omega))} \zeta(t)dt + \int_0^{d(x_n(\omega),\, x_{n+1}(\omega))} \zeta(t)dt + 0 \end{pmatrix}$$

Therefore by definition (2.8) we have

$$\int_0^{d(x_n(\omega),\, x_{n+1}(\omega))} \zeta(t)dt \leq h\left(\int_0^{d(x_{n-1}(\omega),\, x_n(\omega))} \zeta(t)dt\right)$$

Similarly

$$\int_0^{d(x_{n-1}(\omega),\, x_n(\omega))} \zeta(t)dt \leq h\left(\int_0^{d(x_{n-2}(\omega),\, x_{n-1}(\omega))} \zeta(t)dt\right)$$

Hence

$$\int_0^{d(x_n(\omega),\, x_{n+1}(\omega))} \zeta(t)dt \leq h\left(\int_0^{d(x_{n-1}(\omega),\, x_n(\omega))} \zeta(t)dt\right)$$

$$\leq h^2\left(\int_0^{d(x_{n-2}(\omega),\, x_{n-1}(\omega))} \zeta(t)dt\right)$$

On continuing this process

$$\int_0^{d(x_n(\omega),\, x_{n+1}(\omega))} \zeta(t)dt \leq h^n\left(\int_0^{d(x_0(\omega),\, x_1(\omega))} \zeta(t)dt\right)$$

So for $n > m$

$$\int_0^{d(x_m(\omega),\, x_n(\omega))} \zeta(t)dt \le (h^m + h^{m+1} + h^{m+2} + \ldots + h^{n-1})\left(\int_0^{d(x_0(\omega),\, x_1(\omega))} \zeta(t)dt\right)$$

$$\le \frac{h^m}{1-h}\left(\int_0^{d(x_0(\omega),\, x_1(\omega))} \zeta(t)dt\right)$$

Let $0 << \epsilon$ be given, choose a natural number N such that $\frac{h^m}{1-h}\left(\int_0^{d(x_0(\omega),\, x_1(\omega))} \zeta(t)dt\right) << \epsilon$ for every $m \ge N$, thus $\int_0^{d(x_m(\omega),\, x_n(\omega))} \zeta(t)dt$ $\le \frac{h^m}{1-h}\left(\int_0^{d(x_0(\omega),\, x_1(\omega))} \zeta(t)dt\right) << \epsilon$ for every $n \ge m \ge N$. Therefore the sequence $\{x(\omega)\}$ is a Cauchy sequence in $\Omega \times X$ such that $x_n(\omega) \to z(\omega)$. Choose a natural number N_1 such that

$$\int_0^{d(z(\omega),\, Tz(\omega))} \zeta(t)dt \le \int_0^{d(z(\omega),\, x_{n+1}(\omega))} \zeta(t)dt + \int_0^{d(x_{n+1}(\omega),\, Tz(\omega))} \zeta(t)dt$$

$$= \int_0^{d(z(\omega),\, x_{n+1}(\omega))} \zeta(t)dt + \int_0^{d(Tx_n(\omega),\, Tz(\omega))} \zeta(t)dt$$

$$\le \int_0^{d(z(\omega),\, x_{n+1}(\omega))} \zeta(t)dt + \int_0^{\varphi(d(x_n(\omega),\, z(\omega)))} \zeta(t)dt, \int_0^{d(z(\omega),\, Tx_n(\omega))} \zeta(t)dt$$

$$+ \int_0^{d(z(\omega),\, Tz(\omega))} \zeta(t)dt + \int_0^{d(x_n(\omega),\, Tx_n(\omega))} \zeta(t)dt$$

$$+ \int_0^{d(z(\omega),\, Tx_n(\omega))} \zeta(t)dt + \int_0^{d(x_0(\omega),\, Tz(\omega))} \zeta(t)dt, \int_0^{d(z(\omega),\, Tx_n(\omega))} \zeta(t)dt$$

$$\le \int_0^{d(z(\omega),\, x_{n+1}(\omega))} \zeta(t)dt + \phi(\int_0^{d(x_n(\omega),\, z(\omega))} \zeta(t)dt, \int_0^{d(z(\omega),\, x_{n+1}(\omega))} \zeta(t)dt$$

$$+ \int_0^{d(z(\omega),\, Tz(\omega))} \zeta(t)dt + \int_0^{d(x_n(\omega),\, x_{n+1}(\omega))} \zeta(t)dt$$

$$\le \int_0^{d(z(\omega),\, x_{n+1}(\omega))} \zeta(t)dt, \int_0^{d(x_n(\omega),\, Tz(\omega))} \zeta(t)dt + \int_0^{d(z(\omega),\, x_{n+1}(\omega))} \zeta(t)dt)$$

Taking $n \to \infty$ we have

$$\int_0^{d(z(\omega),\, Tz(\omega))} \zeta(t)dt \leq 0 + \varphi(0, \int_0^{d(z(\omega),\, Tz(\omega))} \zeta(t)dt + 0,\ 0 + 0,$$

$$\int_0^{d(z(\omega),\, Tz(\omega))} \zeta(t)dt + 0)$$

$$\int_0^{d(z(\omega),\, Tz(\omega))} \zeta(t)dt \leq 0 + \varphi(0,\ 0, \int_0^{d(z(\omega),\, Tz(\omega))} \zeta(t)dt,\ 0 + 0,$$

$$\int_0^{d(z(\omega),\, Tz(\omega))} \zeta(t)dt + 0))$$

$\int_0^{d(z(\omega),\, Tz(\omega))} \zeta(t)dt \leq 0.$ Thus $-\int_0^{d(z(\omega),\, Tz(\omega))} \zeta(t)dt \in P,$ but $\int_0^{d(z(\omega),\, Tz(\omega))} \zeta(t)dt \in P.$

Therefore $\int_0^{d(z(\omega),\, Tz(\omega))} \zeta(t)dt = 0$ and so $Tz(\omega) = z(\omega).$

Example. Let $M = R$ and $P = \{x \in M : x \geq 0\}$, also $\Omega = [0, 1]$ and Σ be the sigma algebra of lebesgue's measurable subset of $[0, 1]$. Let $X = [0, \infty)$ and define mapping as $d : (\Omega \times X) \times (\Omega \times X) \to M$ by $d(x(\omega), y(\omega)) = [x(\omega) - y(\omega)]$. Then (X, d) is a cone random metric space. Define random operator T from $\Omega \times X$ to X as $T(x(\omega)) = \frac{x(\omega)}{2}$. Also sequence of mapping $x_n : \Omega \to X$ is defined by $x_n(\omega) = \left\{\left(1 - \left(\frac{\omega}{2}\right)^2\right)^{1+\frac{1}{2}}\right\}$ for every $\omega \in \Omega$ and $n \in N$. Defined measurable mapping $x : \Omega \to X$ as $(\Omega) = \left\{1 - \left(\frac{\omega}{2}\right)^2\right\}$ is fixed point of the space.

Theorem 3.2. *Let (X, d) be a complete cone random metric space and let M be a non empty separable closed subset of cone metric space X and let S and T be continuous random operators defined on M such that for*

$\omega \in \Omega$, $T(\omega, \cdot) : \Omega \times M \to M$ *satisfying contraction*

$$\int_0^{d(S^r(x(\omega),\, T^r y(\omega)))} \zeta(t)dt \le \varphi(\int_0^{d(x(\omega),\, y(\omega))} \zeta(t)dt,$$

$$\left[\int_0^{d(x(\omega),\, T^r y(\omega))} \zeta(t)dt + \varphi(\int_0^{d(y(\omega),\, T^r y(\omega))} \zeta(t)dt\right],$$

$$\int_0^{d(x(\omega),\, T^r y(\omega))} \zeta(t)dt + \int_0^{d(S^r x(\omega),\, T^r y(\omega))} \zeta(t)dt \tag{3.2(a)}$$

For all $x, y \in X, \omega \in \Omega$ *and* $r > 0$. *Then S and T has a common fixed point in X.*

Proof. For each $x_0(\omega) \in \Omega \times X$, let us choose $x_1(\omega) = S^r x_0(\omega)$ and $x_2(\omega) = T^r x_1(\omega)$. In general $n \ge 1, x_{n+1}(\omega) = S^r(x_n(\omega))$ and $x_{n+2}(\omega) = T^r(x_{n+1}(\omega))$. Then

$$\int_0^{d(x_{n+1}(\omega),\, x_{n+2}(\omega))} \zeta(t)dt = \int_0^{d(S^r(x_n(\omega),\, S^r(x_n(\omega))} \zeta(t)dt$$

$$\le \varphi\begin{pmatrix} \int_0^{d((x_n(\omega),\, (x_{n+1}(\omega))))} \zeta(t)dt, \left[\int_0^{d(x_n(\omega),\, S^r(x_n(\omega)))} \zeta(t)dt \right. \\ \left. + \int_0^{d(x_{n+1}(\omega),\, T^r(x_{n+2}(\omega)))} \zeta(t)dt \right], \\ \left[\int_0^{d(x_n(\omega),\, (x_n(\omega)))} \zeta(t)dt + \int_0^{d(S^r(x_n(\omega),\, x_{n+2}(\omega)))} \zeta(t)dt\right] \end{pmatrix}$$

$$\le \varphi\begin{pmatrix} \int_0^{d((x_n(\omega),\, (x_{n+1}(\omega))))} \zeta(t)dt, \left[\int_0^{d(x_n(\omega),\, (x_{n+1}(\omega)))} \zeta(t)dt \right. \\ \left. + \int_0^{d(x_{n+1}(\omega),\, (x_{n+2}(\omega)))} \zeta(t)dt \right], \\ \left[\int_0^{d(x_n(\omega),\, (x_{n+1}(\omega)))} \zeta(t)dt\right] \end{pmatrix}$$

Therefore by definition (2.1) we have

$$\int_0^{d(x_{n+1}(\omega),\, (x_{n+2}(\omega)))} \zeta(t)dt \leq h\left(\int_0^{d(x_n(\omega),\, (x_{n+1}(\omega)))} \zeta(t)dt\right)$$

Similarly

$$\int_0^{d(x_{n-1}(\omega),\, (x_n(\omega)))} \zeta(t)dt \leq h\left(\int_0^{d(x_{n-2}(\omega),\, (x_{n-1}(\omega)))} \zeta(t)dt\right)$$

Hence

$$\int_0^{d(x_n(\omega),\, (x_{n+1}(\omega)))} \zeta(t)dt \leq h\left(\int_0^{d(x_{n-1}(\omega),\, (x_n(\omega)))} \zeta(t)dt\right)$$

$$\leq h^2\left(\int_0^{d(x_{n-2}(\omega),\, (x_{n-1}(\omega)))} \zeta(t)dt\right)$$

On continuing this process

$$\int_0^{d(x_n(\omega),\, (x_{n+1}(\omega)))} \zeta(t)dt \leq h^n\left(\int_0^{d(x_n(\omega),\, (x_{n+1}(\omega)))} \zeta(t)dt\right)$$

So for $n > m$

$$\int_0^{d(x_m(\omega),\, x_n(\omega))} \zeta(t)dt \leq (h^m + h^{m+1} + h^{m+2} + h^{m+3} + \ldots + h^{n-1})$$

$$\left(\int_0^{d(x_0(\omega),\, x_1(\omega))} \zeta(t)dt\right) \leq \frac{h^m}{1-h}\left(\int_0^{d(x_0(\omega),\, x_1(\omega))} \zeta(t)dt\right)$$

Let $0 << c$ be given. Choose a natural number N such that $\frac{h^m}{1-h}\left(\int_0^{d(x_0(\omega),\, x_1(\omega))} \zeta(t)dt\right) << c$ for every $m \geq N$. Thus $\int_0^{d(x_m(\omega),\, x_n(\omega))} \zeta(t)dt \leq \frac{h^m}{1-h}\left(\int_0^{d(x_0(\omega),\, x_1(\omega))} \zeta(t)dt\right) << c$ for every $m \geq N$.

Therefore the sequence $\{(\omega)\}$ is a Cauchy sequence in $\Omega \times X$. Since (X, d) is complete, there exists $z(\omega) \in \Omega \times X$ such that $x_n(\omega) \to z(\omega)$. Choose a natural number N_1 such that

$$\int_0^{d(z(\omega),\, Tz(\omega))} \zeta(t)dt \leq \int_0^{d(z(\omega),\, x_{n+1}(\omega))} \zeta(t)dt + \int_0^{d(x_{n+1}(\omega),\, Tz(\omega))} \zeta(t)dt$$

$$= \int_0^{d(z(\omega),\, x_{n+1}(\omega))} \zeta(t)dt + \int_0^{d(S^r(x_{n+1}(\omega),\, T^r z(\omega))} \zeta(t)dt$$

$$\leq \int_0^{d(z(\omega), x_{n+1}(\omega))} \zeta(t)dt + \varphi\left(\begin{array}{l} \displaystyle\int_0^{d(x_n(\omega), z(\omega))} \zeta(t)dt, \left[\int_0^{d(x_n(\omega), S^r(x_n(\omega)), z(\omega))} \zeta(t)dt\right], \\ \displaystyle\left[\int_0^{d(x_n(\omega), T^r z(\omega))} \zeta(t)dt + \int_0^{d(S^r(x_n(\omega)), z(\omega))} \zeta(t)dt\right] \end{array} \right)$$

$$\leq \int_0^{d(z(\omega),\, x_{n+1}(\omega))} \zeta(t)dt + \varphi\left(\begin{array}{l} \displaystyle\int_0^{d(x_n(\omega),\, z(\omega))} \zeta(t)dt, \\ \displaystyle\left[\int_0^{d(z(\omega),\, x_{n+1}(\omega))} \zeta(t)dt + \int_0^{d(z(\omega),\, T^r z(\omega))} \zeta(t)dt\right], \\ \displaystyle\left[\int_0^{d(x_n(\omega),\, T^r z(\omega))} \zeta(t)dt + \int_0^{d(x_{n+1}(\omega),\, z(\omega))} \zeta(t)dt\right] \end{array} \right)$$

Taking $n \to \infty$ we have

$$\int_0^{d(z(\omega),\, T^r z(\omega))} \zeta(t)dt \leq 0 + \varphi\left(0, \left[0 + \int_0^{d(z(\omega),\, T^r z(\omega))} \zeta(t)dt\right], \left[\int_0^{d(z(\omega),\, T^r z(\omega))} \zeta(t)dt + 0\right]\right)$$

$$\int_0^{d(z(\omega),\, T^r z(\omega))} \zeta(t)dt \leq 0 + \varphi\left(0, \int_0^{d(z(\omega),\, T^r z(\omega))} \zeta(t)dt, \int_0^{d(z(\omega),\, T^r z(\omega))} \zeta(t)dt\right)$$

$$\int_0^{d(z(\omega),\, T^r z(\omega))} \zeta(t)dt \leq 0$$

Thus $-\int_0^{d(z(\omega),\, T^r z(\omega))} \zeta(t)dt \in P$. But $\int_0^{d(z(\omega),\, T^r z(\omega))} \zeta(t)dt \in P$.

Therefore $\int_0^{d(z(\omega),\, T^r z(\omega))} \zeta(t)dt = 0$ and so $T^r z(\omega) = z(\omega)$.

Similarly

$$\int_0^{d(S^r z(\omega),\, z(\omega))} \zeta(t)dt \le \int_0^{d(S^r z(\omega),\, z(\omega))} \zeta(t)dt + \int_0^{d(x_{n+2}(\omega),\, z(\omega))} \zeta(t)dt$$

$$= \int_0^{d(S^r z(\omega)),\, T^r x_{n+1}(\omega))} \zeta(t)dt + \int_0^{d(x_{n+2}(\omega),\, z(\omega))} \zeta(t)dt$$

$$\le \varphi\left(\begin{array}{l} \int_0^{d(z(\omega),\, x_{n+1}(\omega))} \zeta(t)dt, \left[\int_0^{d(z(\omega),\, S(z(\omega)))} \zeta(t)dt + \int_0^{d(x_{n+1}(\omega),\, T^r x_{n+1}(\omega))} \zeta(t)dt\right] \\ \left[\int_0^{d(z(\omega),\, T^r x_{n+1}(\omega))} \zeta(t)dt + \int_0^{d((S^r z(\omega)),\, x_{n+1}(\omega))} \zeta(t)dt\right] \end{array}\right)$$

$$+ \int_0^{d(x_{n+2}(\omega),\, z(\omega))} \zeta(t)dt$$

$$\le \varphi\left(\begin{array}{l} \int_0^{d(z(\omega),\, x_{n+1}(\omega))} \zeta(t)dt, \left[\int_0^{d(z(\omega),\, S(z(\omega)))} \zeta(t)dt + \int_0^{d(x_{n+1}(\omega),\, x_{n+1}(\omega))} \zeta(t)dt\right], \\ \left[\int_0^{d(z(\omega),\, x_{n+1}(\omega))} \zeta(t)dt + \int_0^{d(S(z(\omega),\, x_{n+1}(\omega))))} \zeta(t)dt\right] \end{array}\right)$$

$$+ \int_0^{d(x_{n+2}(\omega),\, z(\omega))} \zeta(t)dt$$

Taking as $n \to \infty$ we have

$$\int_0^{d((S^r z(\omega)),\, z(\omega))} \zeta(t)dt \le \varphi\left(\begin{array}{l} \int_0^{d(z(\omega),\, z(\omega))} \zeta(t)dt, \left[\int_0^{d(z(\omega),\, z(\omega))} \zeta(t)dt\right], \\ \left[\int_0^{d((x(\omega)),\, S^r z(\omega))} \zeta(t)dt + \int_0^{d(z(\omega),\, z(\omega))} \zeta(t)dt\right], \\ \left[\int_0^{d(z(\omega), z(\omega))} \zeta(t)dt + \int_0^{d(S^r z(\omega),\, z(\omega))} \zeta(t)dt\right] \end{array}\right)$$

$$+ \int_0^{d(z(\omega),\, z(\omega))} \zeta(t)dt$$

$$\int_0^{d(S^r z(\omega),\, z(\omega))} \zeta(t)dt \leq \phi(0, [d(z(\omega), S^r z(\omega))) + 0], [0 + d(S^r z(\omega), z(\omega))] + 0$$

$$d(S^r z(\omega), z(\omega)) \leq \phi(0, d(z(\omega), S^r z(\omega))), d(S^r z(\omega)), z(\omega)))$$

$$d(S^r z(\omega), z(\omega)) \leq 0$$

Thus $-(d(S^r z(\omega), z(\omega))) \in P$. But $(d(S^r z(\omega), z(\omega))) \in P$.

Therefore $(d(S^r z(\omega), z(\omega))) = 0$ and so $S^r(z(\omega)) = z(\omega)$.

Hence $S^r z(\omega) = z(\omega) = T^r(z(\omega))$.

Theorem 3.3. *Let (X, d) be a complete cone random metric space and let M be a non-empty separable closed subset of cone metric space X and let T and f be two continuous random operators defined on M. Assume that T is a injective mapping and mapping T and f be such that for $\omega \in \Omega, (\omega, \cdot) : \Omega \times X \to M$ satisfying the contraction*

$$\int_0^{d(Tx(\omega),\, Tf(y(\omega)))} \zeta(t)dt \leq$$

$$\varphi\begin{pmatrix} \int_0^{d(Tx(\omega),\, Ty(\omega))} \zeta(t)dt, \int_0^{d(Tx(\omega),\, Tf(y(\omega)))} \zeta(t)dt \\ + \int_0^{d(Tx(\omega),\, Tf(y(\omega)))} \zeta(t)dt, \int_0^{d(Ty(\omega),\, Tf(y(\omega)))} \zeta(t)dt \\ + \int_0^{d(Tx(\omega),\, Tf(y(\omega)))} \zeta(t)dt \end{pmatrix} \tag{3.3(a)}$$

For all $x \in X, \omega \in \Omega$, then f has a unique fixed point in X. Moreover if (T, f) is a banach pair, then T and f have unique fixed point in X.

Proof. Let $x_0(\omega) \in \Omega \times X$ be arbitrary. Define a sequence $\{x_n\}$ $\int_0^{d(Tx_n(\omega),\, Tx_{n+1}(\omega))} \zeta(t)dt = \int_0^{d(Tf(x_{n-1}(\omega)),\, Tf(x_n(\omega)))} \zeta(t)dt$

$$\leq \varphi\begin{pmatrix} \int_0^{d(Tx_{n-1}(\omega),\, Tx_n(\omega))}\zeta(t)dt,\ \int_0^{d(Tx_{n-1}(\omega),\, Tf(x_{n-1}(\omega)))}\zeta(t)dt \\ +\int_0^{d(Tx_n(\omega),\, Tf(x_n(\omega)))}\zeta(t)dt,\ \int_0^{d(Tx_{n-1}(\omega),\, Tf(x_n(\omega)))}\zeta(t)dt \\ \int_0^{d(Tx_{n-1}(\omega),\, Tf(x_n(\omega)))}\zeta(t)dt,\ \int_0^{d(Tx_{n-1}(\omega),\, Tf(x_n(\omega)))}\zeta(t)dt \\ +\int_0^{d(Tx_{n-1}(\omega),\, Tf(x_{n-1}(\omega))))}\zeta(t)dt \end{pmatrix}$$

$$\leq \varphi\begin{pmatrix} \int_0^{d(Tx_{n-1}(\omega),\, Tx_n(\omega))}\zeta(t)dt,\ \int_0^{d(Tx_{n-1}(\omega),\, Tx_n(\omega))}\zeta(t)dt \\ +\int_0^{d(Tx_n(\omega),\, Tx_{n+1}(\omega))}\zeta(t)dt,\ \int_0^{d(Tx_{n-1}(\omega),\, Tx_{n+1}(\omega))}\zeta(t)dt \\ +\int_0^{d(Tx_{n-1}(\omega),\, Tx_n(\omega))}\zeta(t)dt,\ \int_0^{d(Tx_n(\omega),\, Tx_{n+1}(\omega))}\zeta(t)dt \\ ,\int_0^{d(Tx_n(\omega),\, Tx_{n+1}(\omega))}\zeta(t)dt+\int_0^{d(Tx_{n-1}(\omega),\, Tx_n(\omega))}\zeta(t)dt \end{pmatrix}$$

Hence we get,

$$\int_0^{d(Tx_n(\omega),\, Tx_{n+1}(\omega))}\zeta(t)dt \leq h\int_0^{d(Tx_{n-1}(\omega),\, Tx_n(\omega))}\zeta(t)dt$$

Similarly we can show that,

$$\int_0^{d(Tx_n(\omega),\, Tx_{n+1}(\omega))}\zeta(t)dt \leq h^n\int_0^{d(Tx_{n-2}(\omega),\, Tx_{n-1}(\omega))}\zeta(t)dt$$

In general we can write,

$$\int_0^{d(Tx_n(\omega),\, Tx_{n+1}(\omega))}\zeta(t)dt \leq h^n\int_0^{d(Tx_0(\omega),\, Tx_1(\omega))}\zeta(t)dt$$

So for $n > m$

$$\int_0^{d(Tx_m(\omega),\, Tx_n(\omega))}\zeta(t)dt \leq (h^m + h^{m+1} + h^{m+2} + h^{m+3} + \ldots + h^{n-1})$$

$$\int_0^{d(Tx_0(\omega),\, Tx_1(\omega))}\zeta(t)dt \leq \frac{h^m}{1-h}\int_0^{d(Tx_0(\omega),\, Tx_1(\omega))}\zeta(t)dt << a \text{ for every } m \geq n$$

Thus $$\int_0^{d(Tx_n(\omega),\, Tx_m(\omega))} \zeta(t)dt \le \frac{h^m}{1-h} \int_0^{d(Tx_0(\omega),\, Tx_1(\omega))} \zeta(t)dt << a,$$

$n > m \ge N$. Therefore the sequence $\{x_n(\omega)\}$ is a Cauchy sequence in $\Omega \times X$. Since (X, d) is complete, there exists $u(\omega) \in \Omega \times X$ such that $Tx_n(\omega) \to Tu(\omega)$. Since T is subsequently convergent, $\{x_n(\omega)\}$ is such that $\lim_{n\to\infty} x_n(\omega) \to z(\omega)$.

As T is continuous $\lim_{n\to\infty} Tx_n(\omega) \to Tz(\omega)$.

By uniqueness of limit $z(\omega) \to Tu(\omega)$.

Since f is continuous $\lim_{n\to\infty} fx_n(\omega) \to fz(\omega)$.

Again as T is continuous $\lim_{n\to\infty} Tfx_n(\omega) \to Tfz(\omega)$.

Therefore $\lim_{n\to\infty} Tx_{n+1}(\omega) \to Tfz(\omega)$. Choose a natural number N_1 such that for every $N\int_0^{d(Tz(\omega),\, Tx_{n+1}(\omega))} \zeta(t)dt << \frac{a}{2}$ and $\int_0^{d(Tx_n(\omega),\, Tx_{n+1}(\omega))} \zeta(t)dt$ $<< \frac{a}{2}$. Hence for $n \ge N_1$ we have

$$\int_0^{d(Tf(z(\omega)),\, Tz(\omega))} \zeta(t)dt \le \int_0^{d(Tf(z(\omega)),\, Tx_{n+1}(\omega))} \zeta(t)dt + \int_0^{d(Tx_{n+1}(\omega),\, Tz(\omega))} \zeta(t)dt$$

$$= \int_0^{d(Tf(z(\omega)),\, Tf(x_n(\omega)))} \zeta(t)dt + \int_0^{d(Tx_{n+1}(\omega),\, Tz(\omega))} \zeta(t)dt$$

$$\le \varphi\begin{pmatrix} \int_0^{d(Tz(\omega),\, Tx_n(\omega))} \zeta(t)dt, \int_0^{d(Tz(\omega),\, Tf(z(\omega)))} \zeta(t)dt + \int_0^{d(Tx_n(\omega),\, Tf(x_n(\omega)))} \zeta(t)dt \\ + \int_0^{d(Tz(\omega),\, Tf(x_n(\omega)))} \zeta(t)dt + \int_0^{d(Tz(\omega),\, Tf(z(\omega)))} \zeta(t)dt, \\ \int_0^{d(Tx_n(\omega),\, Tf(z(\omega)))} \zeta(t)dt + \int_0^{d(Tz(\omega),\, Tf(z(\omega)))} \zeta(t)dt + \int_0^{d(Tx_{n-1}(\omega),\, Tz(\omega))} \zeta(t)dt \end{pmatrix}$$

$$= \varphi\left(\begin{array}{l} \int_0^{d(Tz(\omega),\, Tx_n(\omega))} \zeta(t)dt,\ \int_0^{d(Tz(\omega),\, Tf(z(\omega)))} \zeta(t)dt + \int_0^{d(Tx_n(\omega),\, Tf(x_{n+1}(\omega)))} \zeta(t)dt, \\ + \int_0^{d(Tz(\omega),\, Tf(x_{n+1}(\omega)))} \zeta(t)dt + \int_0^{d(Tz(\omega),\, Tf(z(\omega)))} \zeta(t)dt, \\ \int_0^{d(Tx_n(\omega),\, Tf(z(\omega)))} \zeta(t)dt + \int_0^{d(Tz(\omega),\, Tf(z(\omega)))} \zeta(t)dt + \int_0^{d(Tx_{n+1}(\omega),\, Tz(\omega))} \zeta(t)dt \end{array}\right)$$

$$\int_0^{d(Tz(\omega),\, Tf(z(\omega)))} \zeta(t)dt \le \int_0^{d(Tz(\omega),\, Tx_{n+1}(\omega))} \zeta(t)dt,\ \int_0^{d(Tx_n(\omega),\, Tf(x_{n+1}(\omega)))} \zeta(t)dt$$

$$<< \frac{a}{2} + \frac{a}{2} = a \text{ for every } n \ge N_1$$

Thus $\int_0^{d(Tz(\omega),\, Tf(z(\omega)))} \zeta(t)dt << \frac{a}{m}$ for all $m \ge 1$.

So $\frac{a}{m} - \int_0^{d(Tz(\omega),\, Tf(z(\omega)))} \zeta(t)dt \in P$ for all $m \ge 1$.

Since $\frac{a}{m} \to 0$ as $m \to \infty$, and P is closed.

$$-\int_0^{d(Tz(\omega),\, Tf(z(\omega)))} \zeta(t)dt \in P.$$

But $\int_0^{d(Tz(\omega),\, Tf(z(\omega)))} \zeta(t)dt \in P$.

Therefore $\int_0^{d(Tz(\omega),\, Tf(z(\omega)))} \zeta(t)dt = 0$. And so $Tz(\omega) = Tf(z(\omega))$.

As T is injective $z(\omega) = f(z(\omega))$. Thus $z(\omega)$ is the fixed point of f.

Uniqueness. If $u(\omega)$ is another fixed point of f, then $u(\omega) = f(u(\omega))$.

$$\int_0^{d(Tz(\omega),\, Tz(\omega))} \zeta(t)dt = \int_0^{d(Tf(u(\omega)),\, Tf(z(\omega)))} \zeta(t)dt$$

$$\leq \varphi\left(\begin{array}{l}\int_0^{d(Tu(\omega),\, Tz(\omega))}\zeta(t)dt,\ \int_0^{d(Tu(\omega),\, Tf(u(\omega)))}\zeta(t)dt \\ +\int_0^{d(Tz(\omega),\, Tf(z(\omega)))}\zeta(t)dt,\ \int_0^{d(Tz(\omega),\, Tf(z(\omega)))}\zeta(t)dt \\ +\int_0^{d(Tu(\omega),\, Tf(u(\omega)))}\zeta(t)dt,\ \int_0^{d(Tz(\omega),\, Tf(u(\omega)))}\zeta(t)dt+\int_0^{d(Tu(\omega),\, Tf(u(\omega)))}\zeta(t)dt\end{array}\right)$$

$$= \varphi\left(\begin{array}{l}\int_0^{d(Tu(\omega),\, Tz(\omega))}\zeta(t)dt,\ \int_0^{d(Tu(\omega),\, Tu(\omega))}\zeta(t)dt \\ +\int_0^{d(Tz(\omega),\, Tz(\omega))}\zeta(t)dt,\ \int_0^{d(Tu(\omega),\, Tz(\omega))}\zeta(t)dt \\ +\int_0^{d(Tu(\omega),\, Tu(\omega))}\zeta(t)dt,\ \int_0^{d(Tz(\omega),\, Tu(\omega))}\zeta(t)dt+\int_0^{d(Tu(\omega),\, Tu(\omega))}\zeta(t)dt\end{array}\right)$$

$\int_0^{d(Tu(\omega),\, Tz(\omega))}\zeta(t)dt$ as $h < 1,$ a contraction.

Hence $\int_0^{d(Tu(\omega),\, Tz(\omega))}\zeta(t)dt = 0$ which implies $Tu(\omega) = Tz(\omega)$. As T is injective, $u(\omega) = z(\omega)$ is the unique fixed point of f. As (T, f) is a banach pair, T and f commutes at fixed point of f which implies that $Tfz(\omega) = fTz(\omega)$ i.e. $Tz(\omega) = fTz(\omega)$ which implies that $Tz(\omega)$ is another fixed point of f. By uniqueness of fixed point of f, $z(\omega) = Tz(\omega)$. Hence $z(\omega) = fz(\omega) = Tz(\omega)$ is the unique fixed point of f and T in X.

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