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EXTENDED RELAXED SKOLEM MEAN LABELING OF WHEEL RELATED GRAPHS

A. RUBY PRISCILLA[1] and S. FIRTHOUS FATIMA[2]

[1]Research Scholar, Reg. No.18221192092003
Department of Mathematics
Sadakathullah Appa College (Autonomous)
Rahmath Nagar, Tirunelveli-627011, India

[1]Assistant Professor
Department of Mathematics
Sarah Tucker College (Autonomous)
Tirunelveli-627007, India
Affiliated to Manonmaniam Sundaranar University
Abishekapatti, Tirunelveli-627012, Tamil Nadu, India

[2]Assistant Professor
Department of Mathematics
Sadakathullah Appa College, (Autonomous)
Rahmath Nagar, Tirunelveli-627011, India
E-mail: kitherali@yahoo.co.in

Abstract

In this paper, we evince the existence of an extended relaxed skolem mean labeling of helm graph H_n, closed helm graph CH_n, and flower graph Fl_n for $n \geq 3$. We also probe the necessary conditions for the extended relaxed skolem mean labeling of helm, closed helm and flower graphs.

1. Introduction

Graphs regarded here are finite, undirected and simple. The symbols $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G. A graph

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*Corresponding author; E-mail: ruby@sarahtuckercollege.edu.in

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labeling is an assignment of integer to the vertices or edges or both subject to certain conditions. Most graph labeling methods are derived from the descendants of by Rosa's [9] findings of the year 1967. A graph labeling is an assignment of integer to the vertices or edges or both subject to certain conditions. Labelled graph has many branch out applications such as coding theory, missile guidance, X-ray, crystallography analysis, communication network addressing systems, astronomy, radar, circuit design, database management etc. Labelled graph has many branch out applications such as coding theory, missile guidance, X-ray, crystallography analysis, communication network addressing systems, astronomy, radar, circuit design, database management etc. D. S. T. Ramesh et al. [8] introduced relaxed skolem mean labeling for four star. A. Manshathi et al. [4] have proved non-existence of relaxed skolem mean labeling for star graphs. K. Murugan [5] introduced relaxed skolem mean labeling of bistars. Ruby Priscilla. A et al. [10,11] introduced an extended relaxed skolem mean labeling of crown, tadpole and gear graphs. In this paper, we probe an extended relaxed skolem mean labeling of helm graph H_n, closed helm graph CH_n, and flower graph Fl_n for $n \geq 3$.

Definition 2.1. A graph G is a non - empty set of objects called vertices together with a set of unordered pairs of distinct vertices of G called edges. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$ respectively. $|E(G)| = q$ is called the size of G and $|V(G)| = p$ is called the order of G. A graph of order p and size q is known as a (p, q)-graph. If $e = uv$ is an edge of G, we say that u and v are adjacent and that u and v are incident with e.

Definition 2.2. [7]**.** A vertex labeling of a graph G is an assignment of labels to the vertices of G that induces for each edge xy a label depending on the vertex labels $f(x)$ and $f(y)$. Similarly, an edge labeling of a graph G is an assignment of labels to the edges of G that induces for each vertex v a label depending on the edge labels incident on it. Total labeling involves a function from the vertices and edges to some set of labels.

Definition 2.3 [7]**.** A graph G with p vertices and q edges is called a mean graph if it is possible to label the vertices $x \in V$ with distinct elements

$f(x)$ from $f(x)$ in such a way that when each edge $e = uv$ is labeled with $\frac{f(u)+f(v)}{2}$ if $f(u)+f(v)$ is even and $\frac{f(u)+f(v)+1}{2}$ if $f(u)+f(v)$ is odd, then the resulting edge labels are distinct. The labeling f concedes a mean labeling of G.

Definition 2.4 [7]**.** A graph $G = (V, E)$ with p vertices and q edges is said to be a skolem mean graph if there exists a function f from the vertex set of G to $\{1, 3, \ldots, p\}$ such that the induced map f^* from the edge set of G to $\{2, 3, \ldots, p\}$ defined by $f^*(e = uv) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd} \end{cases}$ is odd then the resulting edges get distinct labels from the set $\{2, 3, \ldots, p\}$.

Definition 2.5 [8]**.** A graph $G = (V, E)$ with p vertices and q edges is said to concede a relaxed skolem mean graph if there exists a function f from the vertex set of G to $\{1, 2, 3, \ldots p+1\}$ such that the induced map f^* from the edge set of G to $\{2, 3, 4, \ldots, p+1\}$ defined by $f^*(e = uv) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd} \end{cases}$ is odd then the resulting edges get distinct labels from the set $\{2, 3, \ldots, p+1\}$.

Definition 2.6. The helm W_n is the graph obtained from a wheel W_n by attaching a pendant edge to each rim vertex. It contains three types of vertices: an apex of degree n, n vertices of degree 4 and n pendant vertices.

Definition 2.7. The closed helm CH_n is the graph obtained from a helm H_n by joining each pendant vertex to form a cycle. It contains three types of vertices: an apex of degree n, n vertices of degree 4 and n vertices degree 3.

Definition 2.8. The flower Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to the apex of the helm. It contains three types of vertices: an apex of degree $2n$, n vertices of degree 4 and n vertices of degree 3.

Theorem 2.9 [10]. *The Crown* $C_n \odot K_1$, $n \geq 3$ *admits an extended relaxed skolem mean labeling if there exists a function f from the vertex set of* G *to* $\{1, 2, 3, \ldots, p + 2n,$ *where* $2 \leq 2n \leq 2p + 4$ *such that the induced map* f^* *from the edge set of* G *to* $\{2, 3, 4, \ldots, p + 1\}$ *are distinct iff* $p + 2n = 2|V(C_n \odot K_1)|$, *where* $n \equiv 0, 1 (\text{mod } 2)$.

Theorem 2.10 [10]. *The tadpole* $T_{n,k}$ *graph admits an extended relaxed skolem mean labeling when* $n, k \equiv 0, 1 (\text{mod } 2)$, $n \geq 3$ *where* $n + k \geq 4$ *iff* $k \geq 1$.

Theorem 2.11 [11]. *Gear graph* C_n *admits an extended relaxed skolem mean labeling iff*

(i) $f(w_n) \equiv \begin{cases} 0 (\text{mod } 4)\, \textit{where n is odd} \\ 0 (\text{mod } 4)\, \textit{where n is even} \end{cases}$ *for* $n \geq 3$ *and for* $f(w_n) = 6n - 2$.

(ii) $f(u) = 4i + 6$ *where* $n = i + 2$, $n \geq 3$ *and* $i \geq 1$ *where* w_n, $n \in G_n$.

3. Main Results

Definition 3.1. A graph $G = (V, E)$ with p vertices and q edges is cleped to be an extended relaxed skolem mean graph if there exists an injection γ from the vertex set of G to $\{1, 2, 3, \ldots |V(G)| + V_L$ where $2 \leq V_L \leq 2p + 4\}$ such that the induced map λ^* from the edge set of G to $\{1, 2, 3, \ldots |V(G)| + V_L$ where $1 \leq V_L \leq \frac{5p + 3}{2}\}$ defined by

$$\gamma^*(e = uv) = \begin{cases} \frac{\gamma(u) + \gamma(v)}{2} & \text{if } \gamma(u) + \gamma(v) \text{ is even} \\ \frac{\gamma(u) + \gamma(v) + 1}{2} & \text{if } \gamma(u) + \gamma(v) \text{ is odd} \end{cases}$$

then the resulting edges get distinct labels from the set $\{1, 2, 3, \ldots |V(G)| + V_L$ where $1 \leq V_L \leq \frac{5p + 3}{2}\}$.

The motivation behind the study of an extended relaxed skolem mean labeling is predominantly based on heterogeneous network. Assume that the set of nodes in the heterogeneous pharmaceuticals network like research nodes, manufacturing nodes, merchandising nodes, pharmaceuticals nodes and development nodes are labelled as $\{r_1, r_2, \ldots, r_n\}, \{m_1, m_2, \ldots, m_q\}$ $\{s_1, s_2, \ldots, s_b\}, \{t_1, t_2, \ldots, t_q\}$ respectively $\{d_1, d_2, \ldots, d_n\}$. The edges like expenditure edges exist between research and development, employee edges exist between merchandising nodes and pharmaceuticals nodes, manufacturer edges among development nodes, manufacturer edges among pharmaceuticals nodes. We have designed our extended relaxed skolem mean graph in such a way that it reflects the heterogeneous network model, by dividing the set of vertex labels which follows similar labeling fashion. All the edge labels are distinct in such a way that a link exist between the nodes having similar and distinct labeling pattern as well.

Heterogeneous network necessitates the use of the extended relaxed skolem mean graphs for the following reasons (i) Duplication of links in the network can be figured out by existence of the reoccurrence of edge labels. (ii) Each type of node (For example: research node) in the heterogeneous network can receive similar labeling design in an extended relaxed skolem graph.

Theorem 3.2. *Helm graph H_n concedes an extended relaxed skolem mean labeling for every $n \geq 3$ iff* (i) $\gamma(w_1) \equiv 0 (\bmod 4)$ *for* $n \geq 3$ $\gamma(w_1) \geq 4n$ (ii) $\gamma(u) = 2k \equiv 0 (\bmod 2)$ *where* $k = 2n + 1$ *for* $k \equiv 1 (\bmod 2)$ *where* $k \geq 7$ *and* $n \geq 3$ *and* $w_1, u \in H_n$.

Proof. Let "u" be the centre vertex. Let $v_1, v_2, \ldots, v_n$ be the vertices of degree 4 and $w_1, w_2, \ldots, w_n$ be the pendant vertices of helm graph H_n. Then the order of the graph H_n is $|V(H_n)| = 2n + 1$ and the size of the helm graph is $|E(H_n)| = 3n$.

Define $f : V(H_n) \to \{1, 2, 3, \ldots (2n + 1) + (2n + 1)\}$ where $2 \leq 2n + 1 \leq (2n + 1) + 4$

Sufficiency

Let the centre vertex be labeled as $\gamma(u) = 4n + 2$ and $\gamma(w_1) = 4n$

Case 1. $n \equiv 1 (\bmod 2)$

$\gamma(v_1) = 1$

$\gamma(v_i) = 2i - 1$; for $2 \leq i \leq \frac{n+1}{2}$

$\gamma(w_i) = 2i - 3$; for $i \leq \frac{n+1}{2}$, for $n > 3$

$\gamma(w_i) = 2i - 2$; for $2 \leq i \leq \frac{n+1}{2} - 1$, for $n > 3$

$\gamma(w_i) = 2i$; for $3 < i \leq \frac{n+1}{2}$, for $n > 3$

$\gamma(v_i) = 2i$; for $\frac{n+3}{2} \leq i < n$

$\gamma(w_i) = 2i - 1$;

If $i = n$

$\gamma(v_i) = 2i - 1; \gamma(w_i) = 2i$

Case 2. For $n \equiv 0 (\bmod 2)$

$\gamma(v_1) = 1$

For $2 \leq i \leq \frac{n}{2}$

$\gamma(v_i) = 2i - 1$

$\gamma(w_i) = 2i - 2$

For $\frac{n+2}{2} \leq i \leq n - 1$

$\gamma(v_i) = 2i$

$\gamma(w_i) = 2i - 1$

For $i = n$

$\gamma(v_i) = 2i - 1$

$\gamma(w_i) = 2i$

The corresponding edge labelling γ^* is defined as

$$\gamma^*(e = wv) = \begin{cases} \dfrac{\gamma(w) + \gamma(v)}{2} & \text{if } \gamma(w) + \gamma(v) \text{ is even} \\ \dfrac{\gamma(w) + \gamma(v) + 1}{2} & \text{if } \gamma(w) + \gamma(v) \text{ is odd} \end{cases}$$

The repercussion edge labels of G are $\{2, 3, (2n+1) + 4n\}$ distinct labels with $1 \le 4n \le \frac{5p+3}{2}$. Hence the aforementioned labeling design yields an extended relaxed skolem mean labeling of helm graph.

Necessity

(i) For $n = 3$

Let $\gamma(v_1) = 1; \gamma(v_i) = 2i - 1;$ for $i = 2, 3; \gamma(w_2) = 2i + 3; \gamma(w_3) = 6$

Suppose $2n + 1 < \gamma(w_1) < 4n$

Let 8, 9, 10, 11 be the labels exist between $2n + 1$ and $4n.$ Then the edge labels $\gamma^*(e) = \frac{\gamma(v_1) + \gamma(w_1)}{2} = \frac{\gamma(v_2) + \gamma(w_2)}{2} = 5$ and $\gamma^*(e) = \frac{\gamma(v_1) + \gamma(w_1)}{2}$ $= \frac{\gamma(v_3) + \gamma(w_3)}{2} = 6$

For $n > 3$. For $n \equiv 0, 1 (\text{mod}\, 2)$, Suppose $2n < \gamma(w_1) < 4n.$

We get that the edge labels of c and the edge labels of $(v_2, v_3), (v_3, w_3),$ $(v_3, v_4), (v_4, v_4), (v_4, v_5), (v_5, w_5), \ldots, (v_{\frac{n+1}{2}}, w_{\frac{n+1}{2}}), (v_{\frac{n+1}{2}}, v_{\frac{n+3}{2}}), (v_{\frac{n+3}{2}}, v_{\frac{n+3}{2}}),$ $(v_{\frac{n+3}{2}}, v_{\frac{n+5}{2}}), (v_{\frac{n+5}{2}}, v_{\frac{n+5}{2}}), (v_{\frac{n+5}{2}}, w_{\frac{n+7}{2}}), (v_{\frac{n+7}{2}}, w_{\frac{n+7}{2}}), \ldots, (v_{n-1}, v_n), (v_n, w_n)$ are same. Which is a contradiction (ii) Suppose $\gamma(u) = 2k - 1 \equiv 1 (\text{mod}\, 2)$ where $k \equiv 1 (\text{mod}\, 2)$ with $k \ge 7$, where $n = k - 3$ for $n \ge 3$. That is there exist label under γ between $4n$ and for $n \ge 3$. Then the induced edge label $* (e) = \frac{\gamma(v_1) + \gamma(w_1)}{2} = \frac{\gamma(v_1) + \gamma(u)}{2} = 2n + 1$. Which is a contradiction to the definition of an extended relaxed skolem mean graph.

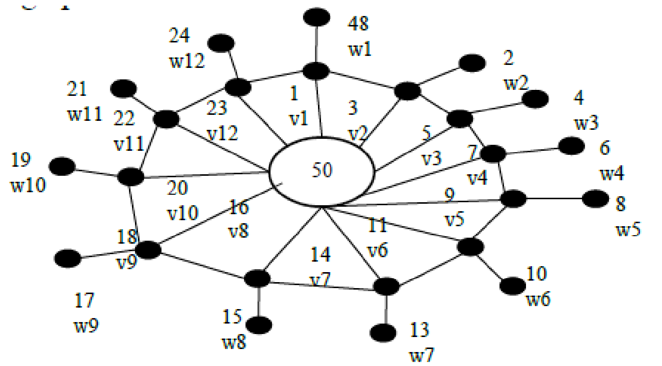

Figure 3.1. An extended relaxed skolem mean labeling of helm graph H_{12}.

Theorem 3.3. *Prove that closed helm graph* CH_n *concedes an extended relaxed skolem mean labeling for* $n \geq 3$ *iff* (i) *for* $n = 3$, $\gamma(s_1) = 2n + 2$ *and for* $n \equiv 0 (\bmod 2)$, $\gamma(s_1) = 4i + 6$ *where* $i = \frac{n-2}{2}$, $n \geq 4$. (ii) *for* $n \equiv 0, 1 (\bmod 2)$, $\gamma(s_2) = 6i + 8$ *where* $i = n - 2; n \geq 3$. (iii) $n \geq 5$, $n \equiv 1 (\bmod 2) \gamma(s_1) = 4i + 10$ *where* $i = \frac{n-3}{2}$ (iv) $\gamma(u) = 4n - 2$, *for* $n \geq 3$.

Proof. Let 'u' be the centre vertex and $t_1, t_2, \ldots, t_n$ be the vertices of degree 4 and $s_1, s_2, \ldots, s_n$ be the vertices of degree 3 of CH_n. Then the order of the closed helm graph CH_n is $|V(CH_n)| = 2n + 1$ and $|E(CH_n)| = 4n$.

Define $f : V(CH_n) \to \{1, 2, 3, \ldots (2n+1) + (6n-9)\}$ where $2 \leq 6n - 9 \leq 2(2n+1) + 4$

For Sufficiency assume the label for the centre vertex as $\gamma(u) = 4n - 2$.

Case 1. $n \equiv 0 (\bmod 2)$

Let $\gamma(t_1) = 1$. Assign the label 3 to the vertex 't_2', 5 to the vertex 't_3', 7 to the vertex t_4 and so on upto $\frac{t_{n+1}}{2}$ as $2i - 1$ and assign the vertices $t_{\frac{n+3}{2}}, t_{\frac{n+3}{2}+1}, \ldots$ and t_{n-1} as $n+3, n+4, n+5, \ldots 2(n-1)$ respectively and the vertex 't_n' as $2i - 1$.

Let the vertices $s_1, s_2, \ldots s_n$ of degree 3 be labeled as follows

For $n = 3 \geq \gamma(s_1) = 2n + 2$

For $n = 5 \geq \gamma(s_1) = 2n + 4$

$f(s_i) = 6n + (i - 2) - 4$; for $2 \leq i \leq 3$;

for $n > 3$

$\gamma(s_i) = 2n + 4$ for $i = 1$

for $i = 1$

$\gamma(s_i) = 6n + 2(i - 2) - 4$

Case 2. $n \equiv 0 (\bmod 2)$

$\gamma(u) = 4n - 2$

$\gamma(t_1) = 1$

For $2 \leq i \leq \frac{n}{2}$

$\gamma(t_i) = 2i - 1$

For $\frac{n+2}{2} \leq i \leq n - 1$

$\gamma(t_i) = 2i$

For $i = n$

$\gamma(t_i) = 2i - 1$

Let the vertices $s_1, s_2, \ldots, s_n$ of degree 3 be labeled as follows:

$\gamma(s_1) = 2n + 2$

$\gamma(s_i) = 6n + 2(i - 2) - 4$ for $2 \leq i \leq n$

The corresponding edge labeling γ^* is defined as

$$\gamma^*(e = st) = \begin{cases} \frac{\gamma(s) + \gamma(t)}{2} & \text{if } \gamma(s) + \gamma(t) \text{ is even} \\ \frac{\gamma(s) + \gamma(t) + 1}{2} & \text{if } \gamma(s) + \gamma(t) \text{ is odd} \end{cases}$$

The repercussion edge labels of G are $\{2, 3, (2n+1) + (6n-10)\}$ distinct labels with $1 \leq 6n - 10 \leq \frac{5p+3}{2}$.

Hence the aforementioned labeling design yields an extended relaxed skolem mean labeling of the closed helm graph CH_n.

Following cases bring forth the proof of Necessity

Case 1. For $n = 3$

Suppose $\gamma(s_1) \neq 2n + 2$

i.e., Assume $2n - 1 < \gamma(s_1) < 2n + 2$.

Let $2n, 2n+1$ be the labels exist between $2n - 1 < \gamma(s_1) < 2n + 2$.

Then $\gamma^*(e) = \frac{\gamma(t_1) + \gamma(s_1)}{2} = \frac{\gamma(t_2) + \gamma(t_3)}{2} = n + 1$. Which contradicts the definition of an extended relaxed skolem mean labeling.

Case 2. For $n \equiv 0 (\bmod 2)$,

Suppose, $\gamma(s_1) \neq 4i + 6$ where $i = \frac{n-2}{2}$, $n \geq 4$

i.e., Assume that $2n - 1 < \gamma(s_1) < 2n + 2$.

Then $\gamma^*(e) = \frac{\gamma(t_1) + \gamma(s_1)}{2} = \frac{\gamma(t_2) + \gamma(t_3)}{2} = \frac{\gamma(t_3) + \gamma(t_4)}{2} = \frac{\gamma(t_4) + \gamma(t_s)}{2}$ $= \ldots = \frac{\gamma(t_{n-1}) + \gamma(t_n)}{2} = n + 1$. Which contradicts the definition of an extended relaxed skolem mean labeling.

Case 3. For $n \equiv 1 (\bmod 2)$ where $n \geq 5$

Suppose $\gamma(s_1) < 4i + 10$ where $i = \frac{n-3}{2}$

i.e., Suppose $2n-1<\gamma(s_1)<2n+4$.

Then the edge label $\gamma^*(e)=\frac{\gamma(s_1)+\gamma(s_n)}{2}=\frac{\gamma(t_n)+\gamma(s_n)}{2}=\frac{10n-9}{2}$;

$\gamma^*(e)=\frac{\gamma(s_1)+\gamma(s_2)}{2}=\frac{\gamma(t_4)+\gamma(t_1)}{2}=\frac{\gamma\left(\frac{t_{n+1}}{2}+1\right)+\gamma(\frac{s_{n+1}}{2}+1)}{2}=4n-1$ and $=\frac{\gamma(t_1)+\gamma(s_1)}{2}=\frac{\gamma\left(\frac{t_{n+1}}{2}+1\right)+\gamma(\frac{s_{n+1}}{2}+1)}{2}=n+2$. Which contradicts.

Case 4. For $n\equiv 1(\bmod 2)$

Suppose $\gamma(u)<n-2$. i.e., Suppose $2n+4<\gamma(u)<4-2$. Then the edge label $\gamma^*(e)=\frac{\gamma(t_1)+\gamma(u)}{2}=\frac{\gamma(t_1)+\gamma(s_1)}{2}=n+3; \gamma^*(e)=\frac{\gamma(t_1)+\gamma(u)}{2}$ $=\ldots=\frac{\gamma(t_{n-1})+\gamma(t_n)}{2}=n+4$ and also $\gamma^*(e)=\frac{\gamma(t_2)+\gamma(u)}{2}=\frac{\gamma(t_1)+\gamma(u)}{2}$ $=\frac{\gamma(t_{n-1})+\gamma(t_n)}{2}=n+6$. Which is a contradiction.

Case 5. $n\equiv 0, 1(\bmod 2)$

Suppose $\gamma(s_2)\neq 6i+8$ where $i=n-2; n\geq 3$

i.e., Suppose $4n-2<\gamma(s_2)<6n-4$. Then the edge label of (t_2, s_2) and the edge labels of (t_i, u) are identical where $2\leq i\leq n$. Which is a contradiction.

Case 6. For $n\equiv 0(\bmod 2)$

Suppose $2n+2<\gamma(u)<n-2$.

Then the edge labels of (t_1, u) is same as that of (t_1, s_1) for all $n\geq 4$ and (t_1, u) is same as that of (t_4, t_5) for $n=4$ and for $n=6$, (t_1, u) is same as that of (t_4, t_5) and (t_5, t_6) and for $n=8$, (t_1, u) is same as that (t_5, t_6) and (t_2, u) is same as that (t_6, t_7) and (t_7, t_8) and so on.

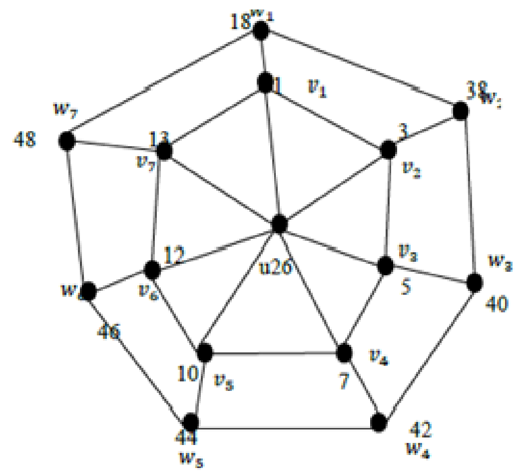

Figure 3.2. An extended relaxed skolem mean labeling of closed helm graph CH_7.

Theorem 3.4. *Flower graph* Fl_n *concedes an extended relaxed skolem mean labeling for* $n \geq 3$ *iff* (i) *For* $n \equiv 0 (\bmod 2)$, $\gamma(r_1) = 11 + 4i$, $\gamma(u) = 8i + 8$, $n \geq 4$, *where* $i = \frac{n-2}{2}$; $\gamma(r_7) = 10i + 19$, $\gamma(r_8) = 12i + 23$ *where* $i = \frac{n-4}{2}$, $n \geq 6$ (ii) *For* $n \equiv 1 (\bmod 2)$, $\gamma(r_2) = 4i + 9$, $\gamma(u) = 8i + 12$, $\gamma(r_7) = 12i + 23$ *where* $i = \frac{n-3}{2}$ *and* $n \geq 5$, $r_1, u, r_7, r_8 \in V(Fl_n)$.

Proof. Let u be the centre vertex, $d_1, d_2, \ldots, d_n$ be the vertices of degree 4 and $r_1, r_2, \ldots, r_n$ be the vertices of degree 2 of flower graph F_l. Then the order of the flower graph $|V(F_l)| = 2n + 1$ and $|E(F_l)| = 4n$. To define $f : V(F_l) \to \{1, 2, 3, \ldots, (2n+1) + 5(n-1) = 7n - 4\}$ with $2 \leq 5(n-1) \leq 2(2n+1) + 4$ we consider following four cases.

Case 1. $n = 3$.

Label the centre vertex as 12 and the vertices d_1, d_2, d_3 of degree 4 are labeled as 1, 3, 4 respectively and the vertices r_1, r_2, r_3 of degree 2 are labeled as 22, 18, and 14 respectively. Then all the induced edge labels are distinct.

Case 2 $n = 4$.

Label the centre vertex as 16 and the vertices d_1, d_2, d_3, d_4 of degree 4 are labeled as 1, 3, 6, 7 respectively and the vertices r_1, r_2, r_3, r_4 of degree 2 are labeled as 30, 26, 22, 18, 16 respectively. Then all the induced edge labels are distinct.

For the sufficiency assume the label for the centre vertex as $\gamma(u) = 4n$.

Case 3. $n \equiv 1(\text{mod}\, 2),\ n \geq 5$

$\gamma(d_1) = 1$

$\gamma(v_i) = 2i - 1;$ for $2 \leq i \leq \frac{n+1}{2}$

$\gamma(v_i) = 2i;$ for $\frac{n+3}{2} 2 \leq i \leq n$

$\gamma(v_i) = 2i - 1;$ if $i = n$

Let the vertices $r_1, r_2, \ldots r_n$ of degree 2 be labelled as follows

For $1 \leq i \leq \frac{n+1}{2}$

$\gamma(r_i) = (2n+1) + 2(i-1)$

For $\frac{n+1}{2} + 1 \leq i \leq n$

$\gamma(r_i) = 5n + 2i - 4.$

Case 4. $n \equiv 0(\text{mod}\, 2),\ n \geq 6$

$\gamma(d_1) = 1$

$\gamma(d_i) = 2i - 1;$ for $2 \leq i \leq \frac{n}{2}$

$\gamma(d_i) = 2i;$ For $\frac{n+2}{2} \leq i \leq n - 1$

$\gamma(d_i) = 2i - 1;$ for $i = n$

Let the vertices $r_1, r_2, \ldots r_n$ of degree 2 be labeled as follows:

$$\gamma(r_i) = (2n+3) + 2(i-1); \text{ For } 1 \le i \le \frac{n}{2}$$

$$\gamma(r_i) = 5n - 1; \text{ for } i = \frac{n}{2} + 1$$

$$\gamma(r_i) = 6n - 1 + 2\left[i - \left(\frac{n}{2} + 2\right)\right]; \text{ for } \frac{n}{2} + 2 \le i \le n$$

The corresponding edge labelling γ^* is defined as

$$\gamma^*(e = ed) = \begin{cases} \dfrac{\gamma(r) + \gamma(d)}{2} & \text{if } \gamma(r) + \gamma(d) \text{ is even} \\ \dfrac{\gamma(r) + \gamma(d) + 1}{2} & \text{if } \gamma(r) + \gamma(d) \text{ is odd.} \end{cases}$$

The repercussion edge labels of G are $\{2, 3, \ldots, [(2n+1)]5n - 6\}$ distinct labels where $1 \le 5n - 6 \le \frac{5p+3}{2}$. Hence the aforementioned labeling design yields an extended relaxed skolem mean labeling of flower graph.

Necessity

Case 1. For $n \equiv 0 (\bmod 2), n \ge 6$

Subcase 1A. To Prove: $\gamma(r_1) = 11 + 4i$, where $i = \frac{n-4}{2}$.

Suppose $\gamma(r_1) \ne 11 + 4i$ i.e., Assume that $2n - 1 < \gamma(r_1) < 2n + 3$. Then there exist vertex labels $2n, 2n+1, 2n+2$ such that the edge labels of $(\gamma(d_1), \gamma(r_1))$ and $(\gamma(d_{\frac{n}{2}}), \gamma(d_{\frac{n}{2}+1}))$ and $(\gamma(u), (\gamma(r_1)))$ and $(\gamma(d_{\frac{n}{2}+1}), \gamma(r_{\frac{n}{2}+1})$ are identical. Which is a contradiction.

Subcase 1B. To Prove: $\gamma(u) = 8i + 12$, where $i = \frac{n-2}{2}$, $n \ge 4$. Suppose $3n + 1 < \gamma(u) < 4n$ for $n \ge 6$. Then the edge labels of (d_i, u) and $(d_3, r_3), (d_3, r_3), (d_5, r_5), (d_6, r_6), \ldots, (d_{\frac{n}{2}}, r_{\frac{n}{2}}), (d_{\frac{n}{2}+5}, r_{\frac{n}{2}+5}), (d_{\frac{n}{2}+5}, d_n)$ are identical. Which is a contradiction.

Subcase 1C. To Prove: $\gamma(r_7) = 10i + 19$, where $i = \frac{n-4}{2}$. Suppose labels exist between $4n < \gamma(r_7) < 5n - 1$ for $n \geq 6$. Then the edge labels of $(d_{\frac{n}{2}+1}, r_{\frac{n}{2}+1})$ and $(d_5, u), (d_6, u), \ldots, (d_{\frac{n}{2}+2}, u)$ are identical which is a contradiction.

Subcase 1D. To Prove: $\gamma(r_8) = 12i + 23$ where $i = \frac{n-4}{2}$ and $n \geq 6$. Suppose label exist between $5n - 1 < \gamma(r_8) < 6n - 1$. Then the edge labels of $(d_{\frac{n}{2}+2}, r_{\frac{n}{2}+2})$ and $(r_1, u), (r_2, u), \ldots, (r_{\frac{n}{2}}, u)$ are same which is a contradiction.

Case 2. For $n \equiv 1 (\mathrm{mod}\, 2)$, $n \geq 5$.

Subcase 2A. To Prove: $\gamma(r_2) = 4i + 9$, where $i = \frac{n-3}{2}$ and $n \geq 5, r_2, u, r_7, r_8 \in V(Fl_n)$. Suppose $\gamma(r_2) \neq 4i + 9$, where $i = \frac{n-3}{2}$. i.e., Assume that $2n + 1 < \gamma(r_2) < 2n + 3$. Resulted edge labels of (u, r_1) and (u, r_2) are identical.

Subcase 2B. To Prove: $\gamma(u) = 8i + 12$ where $i = \frac{n-3}{2}$., $n \geq 5$. Suppose $3n < \gamma(u) < 4n$. Then the edge labels of $(d_{\frac{n+1}{2}+1}, r_{\frac{n+1}{2}+1})$ and $(d_2, u), (d_3, u)$ $(d_4, u), (d_5, u), (d_6, u), \ldots, (d_{\frac{n+1}{2}+1}, u)$ are identical. Which is a contradiction.

Subcase 2C. To Prove: $\gamma(r_7) = 12 + 23$ where $i = \frac{n-3}{2}$. Suppose $4n < \gamma(r_7) < 6n - 1$ Then the edge labels of $(d_{\frac{n+1}{2}+1}, r_{\frac{n+1}{2}+1})$ and $(d_6, u), (d_7, u), \ldots, (d_n, u)$ and also $(d_{\frac{n+1}{2}+1}, r_{\frac{n+1}{2}+1})$ and (r_1, u), $(r_2, u), \ldots, (r_{\frac{n+1}{2}}, u)$ are identical which is a contradiction.

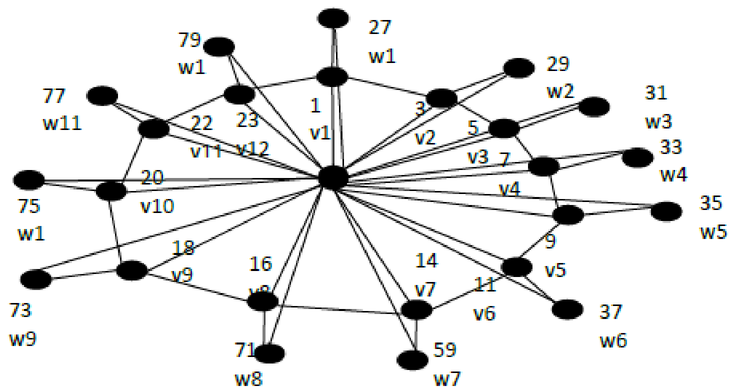

Figure 3.3. An extended relaxed skolem mean labeling of flower graph Fl_{12}.

Conclusion

We have evinced helm, closed helm and flower graphs conceding an extended relaxed skolem mean labeling and also investigated the necessary conditions.

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