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SOME RESULTS ON CONTINUOUS FUNCTIONS IN THE CONTEXT OF IDEAL TOPOLOGICAL SPACES

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Abstract

In this paper, the concept of continuity in the context of ideal topological spaces are defined and discussed in a more natural and independent way; in follow the relationship between continuous functions on topological spaces and ones on ideal topological spaces is studied; consequently, some interesting results on real valued continuous functions in the context of ideal topological spaces are established.

1. Introduction

The notion of ideals in topological spaces is posted by Vaidyanathaswamy [15]. The concept of ideals in topological spaces are studied and developed by Noiri [4, 5], Levine [9] and many others [1, 9, 10, 12, 13, 14, 16]. In 1992, Jankovic and Hamlett [3] introduced the notion of I-open sets in topological spaces. El-monsef et al. [1] investigated I-open sets and I-continuous functions. Levine [9] introduced the concept of semi-open sets and semi-continuity. Hatir and Noiri [5] introduced and studied the concept of α-I-

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continuous, semi-I-continuous and so on.

Erdal and Noiri defined β^*-I-continuity, pre $*$-I-continuity and also studied their properties. Hatir and Jafari [4] introduced the notions of weakly semi-I-open sets and weakly semi-I-continuous functions. Dontchev [2] defined and discussed the notion of a contra-semicontinuous functions and further investigated the class of strongly S-closed spaces. Jafari [10] studied contra pre-I-continuous, contra semi-I-continuous, contra strong β-I-continuous and so on. Yüksel [16] introduced the concept of δ-I-continuous functions. Hatir and Noiri [6] introduced the concept of semi-$\mathcal{J}$-irresolute functions and prove some results.

In [11], several ideals on the same topological space $(X, \mathcal{T})$ were considered and the relationship among the topologies generated by these ideals were discussed and proved some results on ideal topological spaces in the context of connectedness. In this paper, we define and discuss continuous functions in ideal topological spaces and prove some results. In Section 2, we recall some definitions and results from the literature; in Section 3, we define continuous functions in the context of ideal topological spaces and prove some results.

2. Preliminaries

A closure operator on a set X is a function on the collection of all subsets of X taking A to $\overline{A}$ satisfying the following conditions:

- $\overline{\emptyset} = \emptyset$
- For each A, $A \subseteq \overline{A}$
- $\overline{\overline{A}} = \overline{A}$
- For any A and B, $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

These four conditions are called Kuratowski closure axioms [7]. This "$\bar{\ }$" is a closure operator on a set X. If

$$\mathcal{F} = \{A \subseteq X / \overline{A} = A\}$$

and if $\mathcal{T} = \{A \subseteq X / A^c \in \mathcal{F}\}$, then $\mathcal{T}$ is a topology on X and $\overline{A}$ is the $\mathcal{T}$-closure of A for each subset A of X. This topology is said to be the topology generated by the closure operator "¯".

Definition 2.1[15]**.** Let X be any set. An ideal in X is a nonempty collection $\mathcal{I}$ of subsets of X satisfying the following.

- If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.
- If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.

An ideal topological space, denoted by $(X, \mathcal{T}, \mathcal{I})$, is a topological space $(X, \mathcal{T})$ with an ideal $\mathcal{I}$ on X.

The collection $\mathcal{P}(X)$ of all subsets of X and the collection $\{\emptyset\}$ are some trivial examples of ideals. We call $\{\emptyset\}$ as the empty ideal. Throughout this paper, X will denote a topological space, $\mathcal{T}$ will denote a topology on X and $\mathcal{I}$ denote an ideal on X, unless otherwise specified. If $(X, \mathcal{T})$ is a topological space and $x \in X, \mathcal{T}(x)$ denote the set $\{U \in T / x \in U\}$, the collection of all open sets containing x. We denote the complement $X - A$ of A in X by A^c.

Definition 2.2[8]**.** For any subset A of X, define

$$A^*_{(\mathcal{I}, \mathcal{T})} = \{x \in X / U \cap A \notin \mathcal{I} \text{ for every } U \in \mathcal{T}(x)\}.$$

Let $\overline{A} = A \cup A^*_{(\mathcal{I}, \mathcal{T})}$. Then "¯" is a Kuratowski closure operator which gives a topology on X called the topology generated by $\mathcal{I}$ and denoted by $\mathcal{T}_{\mathcal{I}}$.

As there is no ambiguity, we denote $A^*_{(\mathcal{I}, \mathcal{T})}$ by A^*. We call A^* as the derived set of A, it is worthwhile to mention that the term derived set is used to refer to the set of all limit points (or accumulation points) of A in the literature. In fact, it is easy to see that when $\mathcal{I} = \{\emptyset\}$, $\mathcal{T}_{\mathcal{I}}$ coincides with $\mathcal{T}$; if $\mathcal{T}$ is the discrete topology, $\mathcal{I} = \{\emptyset\}$ and if A is the singleton set $\{x\}$, then $A^* = A$ whereas A has no limit point.

Let $(X, \mathcal{T})$ be a topological space, then the following results hold trivially.

- If $\mathfrak{I}_1 \subseteq \mathfrak{I}_2$ are two ideals on X, then $\mathfrak{T}_{\mathfrak{I}_1} \subseteq \mathfrak{T}_{\mathfrak{I}_2}$.
- If A is closed in $\mathfrak{T}_{\mathfrak{I}}$, then $A^* \subseteq A$.
- If $A \in \mathfrak{I}$, then $A^* = \emptyset$ and A is closed in $(X, \mathfrak{T}_{\mathfrak{I}})$.

3. Continuous Functions in Ideal Topological Spaces

We start this section with the definition of a continuous function in the context of ideal topological space. In the classical topology, a function $f : X \to Y$ is continuous if and only if for every subset B of Y, $cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$. Motivated by this we define the continuity in the context of ideal topological space as follows.

Definition 3.1. Let $(X_1, \mathfrak{T}_1)$ and $(X_2, \mathfrak{T}_2)$ be two topological spaces. Let $\mathfrak{I}_1$ and $\mathfrak{I}_2$ be two ideals on X_1 and X_2. A function $f : X_1 \to X_2$ is said to be $\mathfrak{I}$-continuous with respect to the ideals $(\mathfrak{I}_1, \mathfrak{I}_2)$ if $f^{-1}(I_2) \in \mathfrak{I}_1$ for every $I_2 \in \mathfrak{I}_2$ and

$$[f^{-1}(B)]^*_{(\mathfrak{I}_1, \mathfrak{T}_1)} \subseteq f^{-1}(B^*_{(\mathfrak{I}_2, \mathfrak{T}_2)})$$

for every $B \subseteq X_2$.

When there is no ambiguity, we write $\mathfrak{I}$-continuous instead of $\mathfrak{I}$-continuous with respect to the ideals $(\mathfrak{I}_1, \mathfrak{T}_1)$.

It can be proved that every $\mathfrak{I}$-continuous function with respect to the ideals $(\mathfrak{I}_1, \mathfrak{I}_2)$ from X_1 to X_2 is continuous from the topological space $(X_1, \mathfrak{T}_{1_{\mathfrak{I}_1}})$ to the topological space $(X_2, \mathfrak{T}_{2_{\mathfrak{I}_2}})$ (See Theorem 3.5). However there are functions that are continuous from the topological space $(X_1, \mathfrak{T}_{1_{\mathfrak{I}_1}})$ to the topological space $(X_2, \mathfrak{T}_{2_{\mathfrak{I}_2}})$ which are not $\mathfrak{I}$-continuous function with respect to $(\mathfrak{I}_1, \mathfrak{I}_2)$ (See Example 3.6).

Furthermore, at present, we see that there is no relationship between functions that are continuous from $(X_1, \mathfrak{T}_1)$ to $(X_2, \mathfrak{T}_2)$ and functions that are $\mathfrak{I}$-continuous with respect to the ideals $(\mathfrak{I}_1, \mathfrak{I}_2)$ from X_1 to X_2. In fact, there are functions that are continuous from $(X_1, \mathfrak{T}_1)$ to $(X_2, \mathfrak{T}_2)$ which are not $\mathfrak{I}$-continuous with respect to the ideals $(\mathfrak{I}_1, \mathfrak{I}_2)$ and there are functions that are $\mathfrak{I}$-continuous with respect to the ideals $(\mathfrak{I}_1, \mathfrak{I}_2)$ which are not continuous from $(X_1, \mathfrak{T}_1)$ to $(X_2, \mathfrak{T}_2)$. The following examples (Example 3.2 and Example 3.3) establish this. But we give a sufficient condition under which every continuous function is $\mathfrak{I}$-continuous (See Theorem 3.4).

Example 3.2. Let $X_1 = \{1, 2, 3, 4\}, \mathfrak{T}_1 = \{\emptyset, X_1, \{1\}, \{2\}, \{1, 2\}\}, \mathfrak{I}_1 = \{\emptyset, \{2\}\}$, $X_2 = \{1, 2, 3\}, \mathfrak{T}_2 = \{\emptyset, X_2, \{1\}, \{1, 2\}\}, \mathfrak{I}_2 = \{\emptyset, \{2\}\}$. Then clearly $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are the respective topologies on X_1 and X_2, where $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are corresponding ideals on X_1 and X_2. Let $f : X_1 \to X_2$ be defined as $f(1) = 2, f(2) = 1, f(3) = 3$ and $f(4) = 3$, then it can be easily verified that f is continuous. Let $B = \{2\} \subseteq X_2$, then $B \in \mathfrak{I}_2$ and hence $B^*_{(\mathfrak{I}_2, \mathfrak{T}_2)} = \emptyset$. Therefore

$$f^{-1}(B^*_{(\mathfrak{I}_2, \mathfrak{T}_2)}) = \emptyset,$$

also as $f^{-1}(B) = \{1\}$, we have $[f^{-1}(B)]^*_{(\mathfrak{I}_1, \mathfrak{T}_1)} = \{1, 3, 4\}$, which implies

$$[f^{-1}(B)]^*_{(\mathfrak{I}_1, \mathfrak{T}_1)} \not\subseteq f^{-1}(B^*_{(\mathfrak{I}_2, \mathfrak{T}_2)})$$

and hence f is not $\mathfrak{I}$-continuous. Thus a continuous function need not be an $\mathfrak{I}$-continuous function. Further, we note that f is not even continuous from $(X_1, \mathfrak{T}_{1_{\mathfrak{I}_1}})$ to $(X_2, \mathfrak{T}_{2_{\mathfrak{I}_2}})$.

Example 3.3. Let $X_1 = \{1, 2, 3, 4\}, \mathfrak{T}_1 = \{\emptyset, X_1, \{1\}, \{1, 2\}\}, X_2 = \{1, 2, 3\}$, $\mathfrak{T}_2 = \{\emptyset, X_2, \{1\}\}, \mathfrak{I}_1 = \{\emptyset, \{1\}\}, \mathfrak{I}_2 = \{\emptyset, \{2\}\}$. Then $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are the respective topologies on X_1 and X_2, where $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are corresponding ideals on X_1 and X_2. Let $f : X_1 \to X_2$ be defined as $f(1) = 2, f(2) = 1, f(3) = 3, f(4) = 3$, then f is $\mathfrak{I}$-continuous. As the inverse image of an open set $\{1\}$ in X_2 is not open in $\mathfrak{T}_1$, we have f is not continuous from $(X_1, \mathfrak{T}_1)$ to $(X_2, \mathfrak{T}_2)$. Thus an $\mathfrak{I}$-continuous function need not be continuous.

We also note that f is continuous from a topological space $(X_1, \mathfrak{T}_{1_{\mathfrak{I}_1}})$ to a topological space $(X_2, \mathfrak{T}_{2_{\mathfrak{I}_2}})$.

These two examples show that the concepts of continuity and $\mathfrak{I}$-continuity are independent to each other. This happens since the function f has no command over the ideals $\mathfrak{I}_1$ and $\mathfrak{I}_2$, the ideals appear independent of f. If we further impose the condition

$$C : f^{-1}(I_2) \in \mathfrak{I}_1 \text{ for every } I_2 \in \mathfrak{I}_2,$$

tying up the function and ideals, we get some interesting results.

Theorem 3.4. *If* $f : (X_1, \mathfrak{T}_1) \to (X_2, \mathfrak{T}_2)$ *is continuous and if the condition C holds, then f is* $\mathfrak{I}$*-continuous with respect to* $(\mathfrak{I}_1, \mathfrak{I}_2)$.

Proof. It is enough to show that,

$$[f^{-1}(B)]^*_{(\mathfrak{I}_1, \mathfrak{T}_1)} \subseteq f^{-1}(B^*_{(\mathfrak{I}_2, \mathfrak{T}_2)})$$

for any $B \subseteq X_2$. For, let

$$B \subseteq X_2 \text{ and } x \notin f^{-1}(B^*_{(\mathfrak{I}_2, \mathfrak{T}_2)}),$$

then $f(x) \notin B^*_{(\mathfrak{I}_2, \mathfrak{T}_2)}$, consequently, there exists an open set $V \in \mathfrak{T}_2$ containing $f(x)$ such that $V \cap B \in \mathfrak{I}_2$. In addition, by condition C, we have $f^{-1}(V \cap B) \in \mathfrak{I}_1$, which results that $f^{-1}(V) \cap f^{-1}(B) \in \mathfrak{I}_1$. Let $U = f^{-1}(V)$, then U is open in $\mathfrak{T}_1$, as f is continuous; also $x \in U$. Thus there exists an open set $U \in \mathfrak{T}_1(x)$ such that $U \cap f^{-1}(B) \in \mathfrak{I}_1$, which in turn results our need that $x \notin [f^{-1}(B)]^*_{(\mathfrak{I}_1, \mathfrak{T}_1)}$ as desired.

Theorem 3.5. *If* $f : (X_1, \mathfrak{T}_1) \to (X_2, \mathfrak{T}_2)$ *is* $\mathfrak{I}$*-continuous with respect to the ideals* $(\mathfrak{I}_1, \mathfrak{I}_2)$, *then* $f : (X_1, \mathfrak{T}_{1_{\mathfrak{I}_1}}) \to (X_2, \mathfrak{T}_{2_{\mathfrak{I}_2}})$ *is continuous*.

Proof. Let $f : (X_1, \mathfrak{T}_1, \mathfrak{I}_1) \to (X_2, \mathfrak{T}_2, \mathfrak{I}_2)$ be $\mathfrak{I}$-continuous. We wish to show that $f : (X_1, \mathfrak{T}_{1_{\mathfrak{I}_1}}) \to (X_2, \mathfrak{T}_{2_{\mathfrak{I}_2}})$ is continuous. Let B be closed in

$(X_2, \mathcal{T}_{2_{\mathcal{I}_2}})$, then since B is closed in $(X_2, \mathcal{T}_{2_{\mathcal{I}_2}})$, we have $B^*_{(\mathcal{I}_2, \mathcal{T}_2)} \subseteq B$ which implies

$$f^{-1}(B^*_{(\mathcal{I}_2, \mathcal{T}_2)}) \subseteq f^{-1}(B).$$

Also as f is $\mathcal{I}$-continuous, we get

$$[f^{-1}(B)]^*_{(\mathcal{I}_1, \mathcal{T}_1)} \subseteq f^{-1}(B^*_{(\mathcal{I}_2, \mathcal{T}_2)}),$$

which implies

$$[f^{-1}(B)]^*_{(\mathcal{I}_1, \mathcal{T}_1)} \subseteq f^{-1}(B)$$

and thus it results that $f^{-1}(B)$ is closed in $(X_1, \mathcal{T}_{1_{\mathcal{I}_1}})$ as needed.

Note that the converse of the above statement is not true; that is, a continuous function from $(X_1, \mathcal{T}_{1_{\mathcal{I}_1}})$ to $(X_2, \mathcal{T}_{2_{\mathcal{I}_2}})$ need not be $\mathcal{I}$-continuous.

Example 3.6. Let $X_1 = \{1, 2, 3, 4\}$, $\mathcal{T}_1 = \{\emptyset, X_1, \{1\}, \{2\}, \{1, 2\}\}$, $X_2 = \{1, 2, 3\}$, $\mathcal{T}_2 = \{\emptyset, X_2, \{1\}\}$, $\mathcal{I}_1 = \{\emptyset, \{2\}\}$, $\mathcal{I}_2 = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ are the respective topologies on X_1 and X_2, where $\mathcal{I}_1$ and $\mathcal{I}_2$ are corresponding ideals on X_1 and X_2. Further, we obtain

$$\mathcal{T}_{1_{\mathcal{I}_1}} = \{\emptyset, X_1, \{1\}, \{2\}, \{1, 2\}, \{1, 3, 4\}\}$$

and

$$\mathcal{T}_{2_{\mathcal{I}_2}} = \{\emptyset, X_2, \{1\}, \{3\}, \{2, 3\}, \{1, 3\}\}.$$

Let $f : X_1 \to X_2$ be a mapping defined as $f(1) = 3$, $f(2) = 1$, $f(3) = 2$, $f(4) = 2$, then $f : (X_1, \mathcal{T}_{1_{\mathcal{I}_1}}) \to (X_2, \mathcal{T}_{2_{\mathcal{I}_2}})$ is continuous. Let $B = \{2\} \subseteq X_2$, then $B \in \mathcal{I}_2$, which implies $B^*_{(\mathcal{I}_2, \mathcal{T}_2)} = \emptyset$, consequently, we get

$$f^{-1}(B^*_{(\mathcal{I}_2, \mathcal{T}_2)}) = \emptyset.$$

But it is visible that

$$[f^{-1}(B)]^*_{(\mathcal{I}_1, \mathcal{T}_1)} = \{3, 4\},$$

as $f^{-1}(B) = \{3, 4\}$, which implies

$$[f^{-1}(B)]^*_{(\mathcal{I}_1, \mathcal{T}_1)} \not\subseteq f^{-1}(B^*_{(\mathcal{I}_2, \mathcal{T}_2)})$$

and hence f is not $\mathcal{I}$-continuous.

Theorem 3.7. *Let f be $\mathcal{I}$-continuous from $(X_1, \mathcal{T}_1, \mathcal{I}_1)$ to $(X_2, \mathcal{T}_2, \mathcal{I}_2)$, then for any closed set F in $\mathcal{T}_2$ and $x \notin f^{-1}(F)$, there exists an open set $U \in \mathcal{T}_1(x)$ such that $U \cap f^{-1}(F) \in \mathcal{I}_1$.*

Proof. In order to prove our claim, it is enough to prove that $x \notin [f^{-1}(F)]^*_{(\mathcal{I}_1, \mathcal{T}_1)}$. From the given hypothesis that F is closed in $\mathcal{T}_2$ and $x \notin f^{-1}(F)$, we have F is closed in $\mathcal{T}_{2_{\mathcal{I}_2}}$ and $F^*_{(\mathcal{I}_2, \mathcal{T}_2)} \subseteq F$, which results that

$$f^{-1}(F^*_{(\mathcal{I}_2, \mathcal{T}_2)}) \subseteq f^{-1}(F).$$

Also as f is $\mathcal{I}$-continuous, we have

$$[f^{-1}(F)]^*_{(\mathcal{I}_1, \mathcal{T}_1)} \subseteq f^{-1}(F^*_{(\mathcal{I}_2, \mathcal{T}_2)}) \subseteq f^{-1}(F),$$

consequently, we get $x \notin [f^{-1}(F)]^*_{(\mathcal{I}_1, \mathcal{T}_1)}$ as required.

Note that the condition given in the above theorem is just a necessary condition but not a sufficient condition for a function to be an $\mathcal{I}$-continuous function as seen in the following example.

Example 3.8. Let $X_1 = \{1, 2, 3, 4\}$, $\mathcal{T}_1 = \{\emptyset, X_1, \{1\}, \{2\}, \{1, 2\}\}$, $\mathcal{I}_1 = \{\emptyset, \{4\}\}$, $X_2 = \{1, 2, 3\}$, $\mathcal{T}_2 = \{\emptyset, X_2, \{1\}, \{1, 2\}\}$, $\mathcal{I}_2 = \{\emptyset, \{1\}\}$. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ are the respective topologies on X_1 and X_2, where $\mathcal{I}_1$ and $\mathcal{I}_2$ are corresponding ideals on X_1 and X_2. Let $f : X_1 \to X_2$ be defined as $f(1) = 2$, $f(2) = 1$, $f(3) = 2$, $f(4) = 3$, then clearly f satisfies the hypothesis of the above theorem. If we let $B = \{1\} \subseteq X_2$, then $B \in \mathcal{I}_2$ and $B^*_{(\mathcal{I}_2, \mathcal{T}_2)} = \emptyset$. Therefore

$$f^{-1}(B^*_{(\mathcal{I}_2, \mathcal{T}_2)}) = \emptyset.$$

Also as $f^{-1}(B) = \{2\}$, we have $[f^{-1}(B)]^*_{(\mathcal{I}_1, \mathcal{T}_1)} = \{2, 3, 4\}$. Thus

$$[f^{-1}(B)]^*_{(\mathcal{I}_1, \mathcal{T}_1)} \not\subseteq f^{-1}(B^*_{(\mathcal{I}_2, \mathcal{T}_2)})$$

and hence f is not $\mathcal{I}$-continuous.

Definition 3.9. Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be two topological spaces, let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two ideals on X_1 and X_2. Let $x_0 \in X_1$, then a function $f : X_1 \to X_2$ is said to be $\mathcal{I}$-continuous with respect to the ideals $(\mathcal{I}_1, \mathcal{I}_2)$ at the point x_0 if

- for all $I_2 \in \mathcal{I}_2$ containing $f(x_0)$, we have $f^{-1}(I_2) \in \mathcal{I}_1$.
- if $f(x_0) \notin B^*_{(\mathcal{I}_2, \mathcal{T}_2)}$, then $x_0 \notin [f^{-1}(B)]^*_{(\mathcal{I}_1, \mathcal{T}_1)}$ for all $B \subseteq X_2$.

Theorem 3.10. *A function* $f : X_1 \to X_2$ *is* $\mathcal{I}$*-continuous with respect to the ideals* $(\mathcal{I}_1, \mathcal{I}_2)$ *if and only if it is* $\mathcal{I}$*-continuous with respect to the ideals* $(\mathcal{I}_1, \mathcal{I}_2)$ *at all points in* X_1.

Proof. Assume that f is $\mathcal{I}$-continuous with respect to the ideals $(\mathcal{I}_1, \mathcal{I}_2)$. Let $x_0, X_1, B \subseteq X_2, I_2 \in \mathcal{I}_2$ and $f(x_0) \in I_2$. We aim to prove that f is $\mathcal{I}$-continuous at x_0. Since f is $\mathcal{I}$-continuous with respect to the ideals $(\mathcal{I}_1, \mathcal{I}_2)$, we have $f^{-1}(I_2) \in \mathcal{I}_1$. Let $f(x_0) \notin B^*_{(\mathcal{I}_2, \mathcal{T}_2)}$, then $x_0 \notin f^{-1}(B^*_{(\mathcal{I}_2, \mathcal{T}_2)})$, also as f is $\mathcal{I}$-continuous,

$$[f^{-1}(B)]^*_{(\mathcal{I}_1, \mathcal{T}_1)} \subseteq f^{-1}(B^*_{(\mathcal{I}_2, \mathcal{T}_2)}),$$

which implies $x_0 \notin [f^{-1}(B)]^*_{(\mathcal{I}_1, \mathcal{T}_1)}$ and therefore f is $\mathcal{I}$-continuous at x_0.

Conversely, let $I_2 \in \mathcal{I}_2$, then clearly $f^{-1}(I_2) \in \mathcal{I}_1$. Let $B \subseteq X_2$. We have to show that

$$[f^{-1}(B)]^*_{(\mathcal{I}_1, \mathcal{T}_1)} \subseteq f^{-1}(B^*_{(\mathcal{I}_2, \mathcal{T}_2)}).$$

For, let $x_0 \notin f^{-1}(B^*_{(\mathcal{I}_2, \mathcal{T}_2)})$ and then $f(x_0) \notin B^*_{(\mathcal{I}_2, \mathcal{T}_2)}$. Since f is $\mathcal{I}$-continuous at x_0, we have $x_0 \notin [f^{-1}(B)]^*_{(\mathcal{I}_1, \mathcal{T}_1)}$ and therefore

$$[f^{-1}(B)]^*_{(\mathcal{I}_1, \mathcal{T}_1)} \subseteq f^{-1}(B^*_{(\mathcal{I}_2, \mathcal{T}_2)}).$$

Theorem 3.11. *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two ideals on topological spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$. Let $f : X_1 \to X_2$ be a function, then*

$$[f^{-1}(B)]^*_{(\mathcal{I}_1, \mathcal{T}_1)} \subseteq f^{-1}(B^*_{(\mathcal{I}_2, \mathcal{T}_2)})$$

for every $B \subseteq X_2$ if and only if

$$[f(A)]^*_{(\mathcal{I}_1, \mathcal{T}_1)} \supseteq f(A^*_{(\mathcal{I}_2, \mathcal{T}_2)})$$

for every $A \subseteq X_1$.

Proof. Let $A \subseteq X_1$, then $f(A) \subseteq X_2$ and

$$[f^{-1}(f(A))]^*_{(\mathcal{I}_1, \mathcal{T}_1)} \subseteq f^{-1}([f(A)]^*_{(\mathcal{I}_2, \mathcal{T}_2)}).$$

Since $A^*_{(\mathcal{I}_1, \mathcal{T}_1)} \subseteq [f^{-1}(f(A))^*_{(\mathcal{I}_2, \mathcal{T}_2)}$, we have

$$A^*_{(\mathcal{I}_1, \mathcal{T}_1)} \subseteq f^{-1}([f(A)]^*_{(\mathcal{I}_2, \mathcal{T}_2)})$$

and therefore

$$f(A^*_{(\mathcal{I}_1, \mathcal{T}_1)}) \subseteq f(f^{-1}([f(A)]^*_{(\mathcal{I}_2, \mathcal{T}_2)})).$$

But as $f(f^{-1}([f(A)]^*_{(\mathcal{I}_2, \mathcal{T}_2)})) \subseteq [f(A)]^*_{(\mathcal{I}_1, \mathcal{T}_1)}$, we have

$$f(A^*_{(\mathcal{I}_2, \mathcal{T}_2)}) \subseteq [f(A)]^*_{(\mathcal{I}_1, \mathcal{T}_1)}.$$

To prove the converse, let $B \subseteq X_2$, then $f^{-1}(B) \subseteq X_1$ and

$$f([f^{-1}(B)]^*_{(\mathcal{I}_2, \mathcal{T}_2)}) \subseteq [f(f^{-1}(B))]^*_{(\mathcal{I}_1, \mathcal{T}_1)}.$$

Further as $[f(f^{-1}(B))]^*_{(\mathcal{I}_1, \mathcal{T}_1)} \subseteq B^*_{(\mathcal{I}_2, \mathcal{T}_2)}$, we have

$$f([f^{-1}(B)]^*_{(\mathcal{I}_1, \mathcal{T}_1)}) \subseteq B^*_{(\mathcal{I}_2, \mathcal{T}_2)}$$

and therefore

$$[f^{-1}(B)]^*_{(\mathcal{I}_1, \mathcal{T}_1)} \subseteq f^{-1}(B^*_{(\mathcal{I}_2, \mathcal{T}_2)}).$$

Result 3.12. Let $\mathcal{I}_1$ and $\mathcal{J}_1$ be ideals on $(X_1, \mathcal{T}_1)$ such that $\mathcal{I}_1 \subseteq \mathcal{J}_1$ and $\mathcal{I}_2$ be an ideal on $(X_2, \mathcal{T}_2)$. Let $f : (X_1, \mathcal{T}_1) \to (X_2, \mathcal{T}_2)$ be a function. If f is $\mathcal{I}$-continuous with respect to the ideals $(\mathcal{I}_1, \mathcal{I}_2)$, then f is $\mathcal{I}$-continuous with respect to the ideals $(\mathcal{J}_1, \mathcal{I}_2)$.

Proof. Since f is $\mathcal{I}$-continuous with respect to the ideals $(\mathcal{I}_1, \mathcal{I}_2)$, we have $f^{-1}(I_2) \in \mathcal{I}_1$ for every $I_2 \in \mathcal{I}_2$ and

$$f(A^*_{(\mathcal{I}_1, \mathcal{T}_1)}) \subseteq [f(A)]^*_{(\mathcal{I}_2, \mathcal{T}_2)}$$

for every $A \subseteq X_1$. Also as $\mathcal{I}_1 \subseteq \mathcal{J}_1$, we have $f^{-1}(I_2)$ is in $\mathcal{J}_1$ for every $I_2 \in \mathcal{I}_2$. Let $A \subseteq X_1$, then $A^*_{(\mathcal{I}_1, \mathcal{T}_1)} \subseteq A^*_{(\mathcal{I}_2, \mathcal{T}_2)}$ and therefore

$$f(A^*_{(\mathcal{I}_1, \mathcal{T}_1)}) \subseteq f(A^*_{(\mathcal{I}_1, \mathcal{T}_1)}) \subseteq [f(A)]^*_{(\mathcal{I}_2, \mathcal{T}_2)},$$

consequently, f is $\mathcal{I}$-continuous with respect to the ideals $(\mathcal{J}_1, \mathcal{I}_2)$.

Theorem 3.13. *Let $\mathcal{I}_1, \mathcal{I}_2$ and $\mathcal{I}_3$ be ideals on X_1, X_2 and X_3 respectively. If $f : X_1 \to X_2$ is $\mathcal{I}$-continuous with respect to the ideals $(\mathcal{I}_1, \mathcal{I}_2)$ and $g : X_2 \to X_3$ is $\mathcal{I}$-continuous with respect to the ideals $(\mathcal{I}_2, \mathcal{I}_3)$, then $g \circ f : X_1 \to X_3$ is $\mathcal{I}$-continuous with respect to the ideals $(\mathcal{I}_1, \mathcal{I}_3)$.*

Proof. Let $g \circ f = h$ and let $I_3 \in \mathcal{I}_3$, then we have

$$g^{-1}(I_3) \in \mathcal{I}_2 \text{ and } f^{-1}(g^{-1}(g^{-1}(I_3)) \in \mathcal{I}_1.$$

But as $f^{-1}(g^{-1}(g^{-1}(I_3)) = (g \circ f)^{-1}(I_3)$, we have $h^{-1}(I_3) \in \mathcal{I}_1$. We wish to show that

$$h(A^*_{(\mathcal{I}_1, \mathcal{T}_1)}) \subseteq [h(A)]^*_{(\mathcal{I}_2, \mathcal{T}_2)}$$

for every $A \subseteq X_1$. For, let $A \subseteq X_1$, then $h(A^*_{(\mathcal{I}_1, \mathcal{T}_1)}) \subseteq [h(A)]^*_{(\mathcal{I}_2, \mathcal{T}_2)}$, also as $f(A) \subseteq X_2$, we have

$$g([f(A)]^*_{(\mathcal{I}_2, \mathcal{T}_2)}) \subseteq [g(f(A))]^*_{(\mathcal{I}_3, \mathcal{T}_3)}. \tag{1}$$

Further, since $f(A^*_{(\mathcal{I}_1, \mathcal{T}_1)}) \subseteq [f(A)]^*_{(\mathcal{I}_2, \mathcal{T}_2)}$, we have

$$g(f(A^*_{(\mathcal{I}_1, \mathcal{T}_1)})) \subseteq g([f(A)]^*_{(\mathcal{I}_2, \mathcal{T}_2)}). \tag{2}$$

From (1) and (2), we get

$$(g \circ f)(A^*_{(\mathcal{I}_1, \mathcal{T}_1)}) = g(f(A^*_{(\mathcal{I}_1, \mathcal{T}_1)})) \subseteq [g(f(A))]^*_{(\mathcal{I}_1, \mathcal{T}_1)} = [(g \circ f)(A)]^*_{(\mathcal{I}_1, \mathcal{T}_1)}$$

as desired.

Theorem 3.14. *If $f : X_1 \to X_2$ is $\mathcal{I}$-continuous with respect to the ideals $(\mathcal{I}_1, \mathcal{I}_2)$ at x_0 and $g : X_2 \to X_3$ is $\mathcal{I}$-continuous with respect to the ideals $(\mathcal{I}_2, \mathcal{I}_3)$ at $f(x_0)$, then $g \circ f : X_1 \to X_3$ is $\mathcal{I}$-continuous with respect to the ideals $(\mathcal{I}_2, \mathcal{I}_3)$ at x_0.*

Proof. We skip the proof, as it is analogous to the above.

Remark 3.15. Note that the theory developed in this section is new and independent from the concepts like $\mathcal{I}$-continuous, semi-$\mathcal{I}$-continuous, Pre-$\mathcal{I}$-continuous, $\mathcal{I}$-irresolute mappings, which are already available in the literature. As it is a fact that those concepts of $\mathcal{I}$-continuous, semi-$\mathcal{I}$-continuous and so on are defined for functions whose codomains are topological spaces without any ideal structure, which we do not follow in our case of study.

4. Real Valued Continuous Functions

We define real valued continuous functions in the context of ideal topological spaces and prove some results.

Definition 4.1. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{I}$ be an ideal on X.

Let $x_0 \in X$. A function $f : X \to \mathbb{R}$ is said to be $\mathfrak{I}_R$-continuous at x_0 with respect to the ideal $\mathfrak{I}$ if given $\epsilon > 0$, there exists $U \in \mathfrak{T}(x_0)$ such that

$$\{x / | f(x) - f(x_0) | \geq \epsilon\} \cap U \in \mathfrak{I}.$$

The function *f* is said to be $\mathfrak{I}_R$-continuous with respect to an ideal $\mathfrak{I}$ if it is $\mathfrak{I}_R$-continuous at all points in *X*.

If we consider $\mathbb{R}$ with usual topology and the empty ideal consisting of only the empty set as the ideal, then the above definition coincides with the definition of $\mathfrak{I}$-continuity with respect to the ideals $(\mathfrak{I}_1, \mathfrak{I}_2)$ where $\mathfrak{I}_1 = \mathfrak{I}$ and $\mathfrak{I}_2 = \{\emptyset\}$ as seen in Theorem 4.2. Clearly constant functions, in particular the zero function defined by $0(x) = 0$ and the unit function defined by $1(x) = 1$, are $\mathfrak{I}_R$-continuous functions with respect to any ideal $\mathfrak{I}$ because $\{x / | f(x) - f(x_0) | \geq \epsilon\} = \emptyset$.

Theorem 4.2. *Let* $(X_1, \mathfrak{T}_1, \mathfrak{I}_1)$ *be an ideal topological space, then a function* $f : X_1 \to \mathbb{R}$ *is* $\mathfrak{I}_R$*-continuous at* x_0 *if and only if f is* $\mathfrak{I}$*-continuous with respect to the ideals* $(\mathfrak{I}_1, \mathfrak{I}_2)$ *at* x_0, *where* $\mathfrak{I}_2 = \{\emptyset\}$ *and the topology on* $\mathbb{R}$ *is the usual topology*.

Proof. Let *f* be $\mathfrak{I}_R$-continuous at x_0, then clearly the first condition of Definition 4.1 is satisfied. Let $\mathfrak{T}_2$ be the usual topology of $\mathbb{R}$. Let $B \subseteq \mathbb{R}$ and $f(x_0) \notin B^*_{(\mathfrak{I}_2, \mathfrak{T}_2)}$, then we have $B^*_{(\mathfrak{I}_2, \mathfrak{T}_2)} = cl(B)$ with respect to $\mathfrak{T}_2$, as $\mathfrak{I}_2 = \{\emptyset\}$, which results that $f(x_0) \notin cl(B)$. Thus there exists $\epsilon > 0$ such that

$$(f(x_0) - \epsilon, f(x_0) + \epsilon) \cap B = \emptyset \tag{3}$$

Now we claim that $f^{-1}(B) \subseteq \{x / | f(x) - f(x_0) | \geq \epsilon\}$. Indeed, let $y \notin \{x / | f(x) - f(x_0) | \geq \epsilon\}$, then $| f(x) - f(x_0) | < \epsilon$. Therefore $f(y) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$ and hence $y \notin f^{-1}(B)$ as needed. Since *f* is $\mathfrak{I}_R$-continuous at x_0, there exists $U \in \mathfrak{T}_1(x_0)$ such that

$$\{x / | f(x) - f(x_0) | \geq \epsilon\} \cap U \in \mathfrak{I}_1,$$

consequently, $x_0 \notin [\{x / | f(x) - f(x_0) | \geq \epsilon\}]^*_{(\mathfrak{I}_1, \mathfrak{T}_1)}$. Also since

$$f^{-1}(B) \subseteq \{x / | f(x) - f(x_0) | \geq \epsilon\},$$

we have $[f^{-1}(B)]^*_{(\mathfrak{I}_1, \mathfrak{T}_1)} \subseteq [\{x / | f(x) - f(x_0) | \geq \epsilon\}]^*_{(\mathfrak{I}_1, \mathfrak{T}_1)}$ and therefore $x_0 \notin [f^{-1}(B)]^*_{(\mathfrak{I}_1, \mathfrak{T}_1)}$. Thus f is $\mathfrak{I}$-continuous with respect to $(\mathfrak{I}_1, \mathfrak{I}_2)$ at x_0.

To prove the converse, we have to prove that there exists $U \in \mathfrak{T}_1(x_0)$ such that $\{x / | f(x) - f(x_0) | \geq \epsilon\} \cap U \in \mathfrak{I}_1$ for every $\epsilon > 0$. For, let $\epsilon > 0$ and $B = [(f(x_0) - \epsilon, f(x_0) + \epsilon)]^c$, then clearly $cl(B) = B$. Since $\mathfrak{I}_2 = \{\emptyset\}$ and $f(x_0) \notin B$, it follows that $f(x_0) \notin B^*_{(\mathfrak{I}_1, \mathfrak{T}_1)}$, further as f is $\mathfrak{I}$-continuous at x_0, we have $x_0 \notin [f^{-1}(B)]^*_{(\mathfrak{I}_1, \mathfrak{T}_1)}$. Thus there must exists an open set $U \in \mathfrak{T}_1(x_0)$ such that

$$f^{-1}(B) = \{x / | f(x) - f(x_0) | \geq \epsilon\} \cap U \in \mathfrak{I}_1$$

as desired.

Corollary 4.3. *Let $(X_1, \mathfrak{T}_1, \mathfrak{I}_1)$ be an ideal topological space, then a mapping $f : X_1 \to \mathbb{R}$ is $\mathfrak{I}_R$-continuous on X_1 if and only if f is $\mathfrak{I}$-continuous with respect to the ideals $(\mathfrak{I}_1, \mathfrak{I}_2)$ where $\mathfrak{I}_2 = \{\emptyset\}$ and the topology on $\mathbb{R}$ is the usual topology.*

Let us have a discussion on the algebraic sum, product and scalar product of $\mathfrak{I}_R$-continuous functions. Let f and g be two real valued functions on $(X, \mathfrak{T}, \mathfrak{I})$, throughout this section.

Theorem 4.4. *Let f and g are $\mathfrak{I}_R$-continuous with respect to the ideal $\mathfrak{I}$ at x_0, then*

1. *$f + g$ is $\mathfrak{I}_R$-continuous with respect to the ideal $\mathfrak{I}$ at x_0.*

2. *cf is $\mathfrak{I}_R$-continuous with respect to the ideal $\mathfrak{I}$ at x_0, where c is a constant.*

3. *fg is $\mathfrak{I}_R$-continuous with respect to the ideal $\mathfrak{I}$ at x_0.*

Proof. Let $x_0 \in X$ and let f and g be $\mathfrak{I}_R$-continuous with respect to the ideal $\mathfrak{I}$ at x_0. Let $\epsilon > 0$. To prove (1), we have to show that there exists an open set W in $\mathfrak{T}(x_0)$ such that $\{x / |(f+g)(x) - (f+g)(x_0)| \geq \epsilon\} \cap W \in \mathfrak{I}$. Since f and g are $\mathfrak{I}_R$-continuous at x_0, there exists $U_1, U_2 \in \mathfrak{T}(x_0)$ such that

$$\left\{x / |f(x) - f(x_0)| \geq \frac{\epsilon}{2}\right\} \cap U_1 \in \mathfrak{I} \text{ and } \left\{x / |g(x) - g(x_0)| \geq \frac{\epsilon}{2}\right\} \cap U_2 \in \mathfrak{I}$$

and therefore

$$\{x / |(f+g)(x) - (f+g)(x_0)| \geq \epsilon\}$$

$$\subseteq \left\{x / |f(x) - f(x_0)| \geq \frac{\epsilon}{2}\right\} \cup \left\{x / |g(x) - g(x_0)| \geq \frac{\epsilon}{2}\right\}.$$

Let $W = U_1 \cap U_2$, then W is an open set in $\mathfrak{I}$ containing x_0, further we have

$$\{x / |(f+g)(x) - (f+g)(x_0)| \geq \epsilon\} \cap \left(\left\{x / |f(x) - f(x_0)| \geq \frac{\epsilon}{2}\right\}\right.$$

$$\left.\cup \left\{x / |g(x) - g(x_0)| \geq \frac{\epsilon}{2}\right\}\right) \cap W$$

$$\cup \left\{x / |g(x) - g(x_0)| \geq \frac{\epsilon}{2}\right\}$$

$$= \left(\left\{x / |f(x) - f(x_0)| \geq \frac{\epsilon}{2}\right\} \cap W\right) \cup \left(\left\{x / |g(x) - g(x_0)| \geq \frac{\epsilon}{2}\right\} \cap W\right) \in \mathfrak{I}$$

as required.

To prove (2), we have to prove that for every $c \neq 0$, there exists an open set W in $\mathfrak{T}(x_0)$ such that $\{x / |(cf)(x) - (cf)(x_0)| \geq \epsilon\} \cap W \in \mathfrak{I}$. Now since f is $\mathfrak{I}_R$-continuous at x_0, there exists $U \in \mathfrak{T}(x_0)$ such that

$$\left\{x / |f(x) - f(x_0)| \geq \frac{\epsilon}{|c|}\right\} \cap U \in \mathfrak{I}.$$

If we let $U = W$, then clearly W is an open set in $\mathfrak{I}$ containing x_0 with the required property.

To prove (3), we have to show that there exists an open set $W \in \mathfrak{T}(x_0)$ such that $\{x/|(fg)(x)-(fg)(x_0)| \geq \epsilon\} \cap W \in \mathfrak{I}$. Now since f is $\mathfrak{I}_R$-continuous at x_0, there exist $U_1, U_2 \in \mathfrak{T}(x_0)$ such that

$$\left\{x/|f(x)-f(x_0)| \geq \frac{\sqrt{\epsilon}}{2}\right\} \cap U_1 \in \mathfrak{I}$$

and

$$\left\{x/|f(x)-f(x_0)| \geq \frac{\epsilon}{4(|f(x_0)|+1)}\right\} \cap U_2 \in \mathfrak{I}.$$

Also since g is $\mathfrak{I}_R$-continuous at x_0, there exist $U_3, U_4 \in \mathfrak{T}(x_0)$ such that

$$\left\{x/|g(x)-g(x_0)| \geq \frac{\sqrt{\epsilon}}{2}\right\} \cap U_3 \in \mathfrak{I}$$

and

$$\left\{x/|g(x)-g(x_0)| \geq \frac{\epsilon}{4(|g(x_0)|+1)}\right\} \cap U_4 \in \mathfrak{I}.$$

Thus if we let $W = U_1 \cap U_2 \cap U_3 \cap U_4$, then clearly W is an open set in $\mathfrak{I}$ containing x_0 and

$$\{x/|(fg)(x)-(fg)(x_0)| \geq \epsilon\} \cap W \in \mathfrak{I}$$

as desired.

Corollary 4.5. *Let f and g be $\mathfrak{I}_R$-continuous with respect to the ideal $\mathfrak{I}$ and $c \in \mathbb{R}$, then $f+g$, fg, cf are $\mathfrak{I}_R$-continuous with respect to the ideal $\mathfrak{I}$.*

Conclusion

The concept of continuity in the domain of ideal topological spaces are defined and discussed in a natural way; further the relationship between continuous functions on classical topological spaces and on the ideal topological spaces is studied; parallel justifications are provided to establish the consistency of the theory developed. Sequentially, some interesting results are posted in the turf of real valued continuous functions in the context of ideal topological spaces.

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