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TOPOLOGICAL PROPERTIES OF DIFFERENCE SEQUENCE SPACE THROUGH ORLICZ-PARANORM FUNCTION

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Abstract

In this paper, we introduce the difference sequence space $\mathcal{S}(\mathcal{F}, M, \alpha, P)$ and $\mathcal{S}(\mathcal{F}, M, \alpha, P, G)$ of fuzzy real numbers using the Orlicz function. We also discussed some of the linear topological properties of the space and demonstrated $\mathcal{S}(\mathcal{F}, M, \alpha, P, G)$ is complete by defining a new paranorm on it.

Introduction

In 1965, L. A. Zadeh [24] developed the fuzzy set theory to address ambiguity and uncertainty in mathematics. Since then, both pure and applied fuzzy mathematics have been the subject of extensive research. The sequence space of fuzzy real numbers is the area where the majority of research has been conducted. Sequence space refers to a linear subspace of a vector space.

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According to Motloka [9], who has studied the boundedness of fuzzy numbers, every convergent sequence of fuzzy numbers is bounded. After that, Nanda [10] introduced a class of sequences of fuzzy numbers and studied their various properties.

Kizmaz [7], first introduced the idea of difference sequence space in 1981. Et and Colka [3] broadened the concept of Kizmaz. Then, Burch and Tripathy [1] applied this overarching concept to the fuzzy domain. The authors [4], [5], [6], [7], [15], [16] have studied difference sequence space.

A function $M : [0, \infty) \to [0, \infty)$ satisfying the following conditions

$$M(0) = 0,\ M(t) > 0 \text{ and } M(t) \to 0 \text{ as } t \to \infty$$

and is continuous, non-decreasing, and convex is called an Orlicz function.

Wladyslaw first discussed the Orlicz function in 1932. Then Lindenstrauss and Trzafriri [8] used the Orlicz function concept to define a new sequence space that is defined as

$$l_M = \{x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x|}{\rho}\right) < \infty \text{ for some } \rho > 0\}.$$

The space l_M together with the norm $\| x \|$ defined by $\| x \| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x|}{\rho}\right) < 1\right\}$ becomes a Banach space and it is called Orlicz sequence space.

In the concept of Orlicz function and fuzzy set theory, Sarma [17], Parashar and Choudhary [12], Paudel, Sahani and Pahari [14], Savas and Savas [18], Subraninan et al. [19], Tripathy and Borgohain [21-22], Tripathy and Sarma [23] and many more have contributed to the field of difference sequence spaces of fuzzy numbers through Orlicz function. In this paper, we introduce a class of sequence space and study some of its topological properties, and by defining a paranorm, we test the completeness property of the class. For this, we need some definitions, which are defined below.

Definition and Preliminaries

Let $\mathcal{S}$ be the universal set and $\mathcal{F} \subseteq \mathcal{S}$ then the collection of pairs

$$\mathcal{F} = \{(x, \psi_{\mathcal{F}}(x)) : \psi_{\mathcal{F}} : \mathcal{S} \to [0, 1], x \in \mathcal{S}\}$$

defines as fuzzy set $\mathcal{F}$ or $\mathcal{S}$.

The function $\psi_{\mathcal{F}}$ is called the membership function and $\psi_X(x)$ is called the degree of the element x belonging to the set $\mathcal{F}$. Here, if $\psi_{\mathcal{F}}(x) = 0$ then x is not included in $\mathcal{F}$, and if $\psi_x(x) = 1$ then x is fully included in $\mathcal{F}$. So that $\psi_{\mathcal{F}}(x)$ is defined as

$$\psi_{\mathcal{F}}(x) = \begin{cases} 0 & \text{if } x \notin \mathcal{F} \text{ and there is no ambiguity} \\ 1 & \text{if } x \in \mathcal{F} \text{ and there is no ambiguity} \\ (0, 1) & \text{if there is ambiguity whether } x \in \mathcal{F} \text{ or } x \notin \mathcal{F}. \end{cases}$$

Let $\mathcal{D}$ be the set of all bounded intervals $[x, y]$ on the real line $\mathbb{R}$. Then for any $D_1, D_2 \in \mathcal{D}$ with $D_1 = [x_1, y_1]$ and $D_2 = [x_2, y_2]$ then $D_1 \leq D_2$ if $x_2 \leq x_1$ and $y_1 \leq y_2$. Define a relation d on $\mathcal{D}$ by $d(D_1, D_2)$ $= \max\{| x_2 - x_1 |, | y_2 - y_1 |\}$.

Then clearly, d defines a metric in $\mathcal{D}$ and obviously $(\mathcal{D}, d)$ is a complete metric space.

A fuzzy real number $\mathcal{F}$ is a fuzzy set, or a mapping between each real number $(\mathbb{R})$ and its membership value $\mathcal{F}(t)$, where $\mathcal{F} : \mathbb{R} \to I = [0, 1]$ such that

The fuzzy number $\mathcal{F}$ is

i. normal if there exists $t \in \mathbb{R}$ such that $\mathcal{F}(t) = 1$

ii. convex if for $t, s \in \mathbb{R}$ and $0 \leq \theta \leq 1$, $\mathcal{F}(\theta t + (1 - \theta)s) \geq \min\{\mathcal{F}(t), \mathcal{F}(s)\}$

iii. $\mathcal{F}$ is upper semi-continuous if for $\varepsilon > 0$, $\mathcal{F}^{-1}([0, a + \varepsilon))$, for all $a \in I$, is open in the usual topology of $\mathbb{R}$.

The α-level [14] set on $\mathcal{F}$ is denoted by $\mathcal{F}^{\alpha}$ and defined by $F^{\alpha} = \{t \in \mathbb{R} : \mathcal{F}(t) \geq \alpha\}$.

The collection of all fuzzy numbers with membership values greater than zero is referred to as support fuzzy a number.

Assume that $\mathbb{R}(I)$ represents the collection of all fuzzy numbers with upper semi-continuity and compact support.

Now, let us consider a relation $\overline{\rho} : \mathbb{R}(I) \times \mathbb{R}(I) \to \mathbb{R}^*$ defined by

$$\overline{\rho}(\mathcal{F}, \mathcal{G}) = \underset{0 \leq \alpha \leq 1}{Sup}\, d(\mathcal{F}^{\alpha}, \mathcal{G}^{\alpha})$$

where $\mathbb{R}^* = \mathbb{R} \cup \{0\}$.

Then, $\overline{\rho}$ defines a metric on $\mathbb{R}(I)$ and $(\mathbb{R}(I), \overline{\rho})$ is a complete metric space.

Paranormed Space [11]**.** Let X be a vector space. A function $\xi : X \to \mathbb{R}$ satisfying the following

i. $\xi(0) = 0$

ii. $\xi(x) \geq 0$ for all $x \in X$.

iii. $\xi(-x) = \xi(x)$ for all $x \in X$.

iv. $\xi(x + y) \leq \xi(x) + \xi(y)$, for all $x, y \in X$.

v. if (a_n) is a sequence of scholars with $a_n \to a$ as $n \to \infty$ and $\{x_n\}$ is a sequence of such that $\xi(x_n - x) \to 0$ as $n \to \infty$ then $\xi(a_n x_n - ax) \to 0$ as $n \to \infty$ (continuity of scholar multiplication) is called paranormed and (X, ξ) is paranormed space.

We note that a paranorm ξ with $\xi(x) = 0$ implies $x = 0$ is called total.

Bounded fuzzy set. A fuzzy set A in $\mathcal{F}$ is said to be bounded above if there exists a fuzzy number M in $\mathcal{F}$ such that $a \leq M$ for all $a \in A$ and M is called upper bound for A.

The fuzzy number M is called the supremum of A if M is an upper bound of A and $M \leq W$ for any upper bound W of A and we write $M = \sup_{a \in A} a$.

Also, fuzzy set A in $\mathcal{F}$ is said to be bounded below if there exists a fuzzy number m in X such that $b \geq m$ for all $b \in A$ and m is called lower bound for A.

The fuzzy number m is called the infimum of A if m is an upper bound above of A and $m \geq W$ for any lower bound W of A and we write $m = \inf_{b \in A} b$.

A sequence of fuzzy numbers $\mathcal{F} = (\mathcal{F}_k)$ is said to be bounded if the set $\{\mathcal{F}_k : k \in \mathbb{N}\}$ is bounded.

A sequence $\mathcal{F} = (\mathcal{F}_k)$ of fuzzy numbers is said to converge to a fuzzy number $\mathcal{F}_o$ and we write $\lim_k \mathcal{F}_k = \mathcal{F}_o$ if, for every $\varepsilon > 0$, there exists a positive integer $n_0 = n(\varepsilon)$ such that $\bar{\rho}(\mathcal{F}_k, \mathcal{F}_o) < \varepsilon$ for all $k \geq n_o$.

Limit supremum and limit infimum[20]**.** Let $\mathcal{F} = \{\mathcal{F}_k\}$ be a bounded sequence of fuzzy numbers. The limit infimum and supremum of the sequence are defined as

$$\liminf \mathcal{F}_k = \lim_{n \to \infty} \inf_{k \geq n} \mathcal{F}_k \text{ and } \limsup \mathcal{F}_k = \lim_{n \to \infty} \sup_{k \geq n} \mathcal{F}_k.$$

We note that the limit infimum or limit supremum of the bounded sequence of fuzzy numbers may not exist.

Let $\omega(\mathcal{F})$ denote the set of all sequences of fuzzy numbers. Then any subsequence of $\omega(\mathcal{F})$ is sequence space and is called fuzzy sequence space.

Difference Sequence Space.

In 1981, Kizmaz [8] used the concept of difference sequence space for the first time as follow

$l_\infty(\Delta) = \{\mathcal{F} = \mathcal{F}_k : \Delta\mathcal{F} \in l_\infty\}$

$C(\Delta) = \{\mathcal{F} = \mathcal{F}_k : \Delta\mathcal{F} \in C\}$

$C_o(\Delta) = \{\mathcal{F} = \mathcal{F}_k : \Delta\mathcal{F} \in C_o\}$, where $\Delta\mathcal{F} = \mathcal{F}_k - \mathcal{F}_{k+1}$.

The concept of Kizmaz difference sequence spaces was generalized by Colka and Et [6] to define the sequence spaces $l_\infty(\Delta^m)$, $C(\Delta^m)$, $C_o(\Delta^m)$ in 1995.

To examine the diverse properties of the difference sequence spaces $l_{\infty}^{F}(\Delta^{m})$, $C^{F}(\Delta^{m})$ and $C_{o}^{F}(\Delta^{m})$ of fuzzy numbers, Baruah and Tripathy [1] established the notation of difference operator Δ_m in 2009.

To study topological properties of difference sequence space of fuzzy real numbers using Orlicz function we use the notation $\mathcal{S}(\mathcal{F}, M, \alpha, P)$ and $\mathcal{S}(\mathcal{F}, M, \alpha, P, G)$ for a classes of difference sequences.

Let $P = (p_k)$ and $Q = (q_k)$ be any two sequences of strictly positive real numbers and (λ_k) be a sequence of non-zero real numbers. Now we introduce the following classes of fuzzy sequences as follows:

$$\mathcal{S}(\mathcal{F}, M, \alpha, P) = \left\{X = (F_k) \in \omega^F : \sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k \Delta \mathcal{F}_k \|^{p_k}}{\eta}\right) < \infty \text{ for some } \eta > 0\right\}$$

$$\mathcal{S}(\mathcal{F}, M, \alpha, P, G) = \left\{X = (F_k) \in \omega^F : \sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k \Delta \mathcal{F}_k \|^{p_k}}{\eta}\right) < \infty \text{ for some } \eta > 0\right\}$$

where, $\sup_k p_k = G$.

Then, clearly $\mathcal{S}(\mathcal{F}, M, \alpha, P, G)$ is subset of $\mathcal{S}(\mathcal{F}, M, \alpha, P)$.

Main Work

Lemma 1[2]. *Let* $\{p_k\}$ *be a bounded sequence of strictly positive real numbers with* $0 < p_k \le \sup p_k = L$, $\mathcal{H} = \max\{1, 2^{-1}\}$ *then*

i. $| \mathcal{F} + \mathcal{G} |^{p_k} \le \mathcal{H}\{| \mathcal{F} |^{p_k} + | \mathcal{G} |^{p_k}\}$;

ii. $| a |^{p_k} \le \max(1, [\alpha]^L)$.

Theorem 1. *The class* $\mathcal{S}(\mathcal{F}, M, \alpha, P)$ *is a linear space.*

Proof. Suppose $\mathcal{F} = (\mathcal{F}_k)$ and $\mathcal{G} = (\mathcal{G}_k)$ be two elements of $\mathcal{S}(\mathcal{F}, M, \alpha, P)$. Then there exists $\rho_1 > 0$ and $\rho_2 > 0$ such that,

$$\sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k \Delta \mathcal{F}_k \|^{p_k}}{\eta_1}\right) < \infty \text{ and } \sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k \Delta \mathcal{G}_k \|^{p_k}}{\eta_2}\right) < \infty.$$

Let us choose $\eta_3 > 0$ for scholars a and b such that

$$\frac{\mathcal{H}}{\eta_3} \leq \frac{1}{2\eta_2 \max(1, |a|)} \quad \text{and} \quad \frac{\mathcal{H}}{\eta_3} \leq \frac{1}{2\eta_2 \max(1, |a|)}, \quad \text{where,} \quad \sup p_k = L$$

and $\mathcal{H} = \max\{1, 2^{L-1}\}$.

Then

$$\sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k (a\Delta \mathcal{F}_k + b\Delta \mathcal{G}_k) \|^{p_k}}{\eta_3}\right) = \sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k (a\Delta \mathcal{F}_k + b\Delta \mathcal{G}_k) \|^{p_k}}{\eta_3}\right)$$

$$= \sum_{k=1}^{\infty} M\left(\frac{\| (a\alpha_k \Delta \mathcal{F}_k) + (b\alpha_k \Delta \mathcal{F}_k) \|^{p_k}}{\eta_3}\right)$$

$$\leq \sum_{k=1}^{\infty} M\left(\frac{\mathcal{H}}{\eta_3} \| a\, \alpha_k \Delta \mathcal{F}_k \|^{p_k} + \frac{\mathcal{H}}{\eta_3} \| b\, \alpha_k \Delta \mathcal{G}_k \|^{p_k}\right)$$

$$\leq \sum_{k=1}^{\infty} M\left(\frac{1}{2\eta_1} \| a\, \alpha_k \Delta F_k \|^{p_k} + \frac{1}{2\eta_2} \| b\, \alpha_k \Delta G_k \|^{p_k}\right) < \infty$$

$\Rightarrow a\mathcal{F} + b\mathcal{G} \in \mathcal{S}(\mathcal{F}, M, \alpha, P)$ for $\mathcal{F}, \mathcal{G} \in \mathcal{S}(\mathcal{F}, M, \alpha, P)$ and hence the class is linear.

Theorem 2. *If* $0 < p_k \leq q_k < \infty$ *for all but finitely many values of* k, *then* $\mathcal{S}(\mathcal{F}, M, \alpha, P) \subseteq \mathcal{S}(\mathcal{F}, M, \alpha, Q)$.

Proof. Let $\mathcal{F} = (\mathcal{F}_k) \in \mathcal{S}(\mathcal{F}, M, \alpha, P)$, then there exists $\eta > 0$ such that

$$\sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k \Delta \mathcal{F}_k \|^{p_k}}{\eta}\right) < \infty. \tag{*}$$

This relation shows that there exists $K \geq 1$ such that $\| \lambda_k \Delta \mathcal{F}_k \| < 1$ for all $k \geq K$. Since, $p_k \leq q_k$, we have $\| \lambda_k \Delta \mathcal{F}_k \|^{q_k} \leq \| \lambda_k \Delta \mathcal{F}_k \|^{p_k}$ for every $k \geq K$.

Since M is a non-decreasing function, we have

$$M\left(\frac{\| \alpha_k \Delta \mathcal{F}_k \|^{q_k}}{\eta}\right) \le M\left(\frac{\| \alpha_k \Delta \mathcal{F}_k \|^{p_k}}{\eta}\right) \text{ for } \eta > 0$$

So, $\sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k \Delta \mathcal{F}_k \|^{q_k}}{\eta}\right) \le \sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k \Delta \mathcal{F}_k \|^{p_k}}{\eta}\right)$ for $\eta > 0$.

Hence, $\mathcal{F} = (\mathcal{F}_k) \in \mathcal{S}(\mathcal{F}, M, \alpha, Q)$ and hence $\mathcal{S}(\mathcal{F}, M, \alpha, P) \subseteq \mathcal{S}(\mathcal{F}, M, \alpha, Q)$ proved.

Theorem 3. *The sequence space* $\mathcal{S}(\mathcal{F}, M, \alpha, P)$ *is solid.*

Proof. Let $\mathcal{F} = (\mathcal{F}_k) \in \mathcal{S}(\mathcal{F}, M, \alpha, P)$, then there exists $\eta > 0$ such that

$$\sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k \Delta \mathcal{F}_k \|^{p_k}}{\eta}\right) < \infty.$$

Let $a = (a_k)$ be a sequence of scalars such that $| a_k | \le 1$ for all $k \ge 1$. Since M is non-decreasing, we have

$$\sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k (a_k \Delta \mathcal{F}_k) \|^{p_k}}{\eta}\right) = \sum_{k=1}^{\infty} M\left(\frac{| a_k |^{p_k} \| \alpha_k \Delta \mathcal{F}_k \|^{p_k}}{\eta}\right)$$

$$\le \sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k \Delta \mathcal{F}_k \|^{p_k}}{\eta}\right) < \infty.$$

Thus, $(\alpha_k \mathcal{F}_k) \in \mathcal{S}(\mathcal{F}, M, \alpha, P)$ for all sequences (a_k) of scholars with $| a_k | \le 1$, whenever $\mathcal{F} = (\mathcal{F}_k) \in \mathcal{S}(\mathcal{F}, M, \alpha, P)$. So $\mathcal{S}(\mathcal{F}, M, \alpha, P)$ is solid.

Theorem 4. *Let* $\mathcal{F} = (\mathcal{F}_k) \in \mathcal{S}(\mathcal{F}, M, \alpha, P, G)$, *then the space* $\mathcal{S}(\mathcal{F}, M, \alpha, P, G)$ *is a paranorm space with the paranorm defined by*

$$\xi(X) = \inf\left\{\eta > 0 : \sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k \Delta \mathcal{F}_k \|^{p_k / G}}{\eta}\right) \le 1\right\} (**)$$

where, $\sup_k p_k = G$.

Proof. i. $P \cdot N_1$, clearly, we can see that $\xi(0) = 0$.

ii. $P \cdot N_2 : \xi(-\mathcal{F}) = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k(-\Delta\mathcal{F}_k) \|^{p_k/G}}{\eta}\right) \leq 1\right\}$

$$= \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k\Delta\mathcal{F}_k \|^{p_k/G}}{\eta}\right) \leq 1\right\}$$

$$= \xi(X)$$

iii. For, $\mathcal{F}, \mathcal{G} \in \mathcal{S}(\mathcal{F}, M, \alpha, P, G)$, let us choose $\eta_1, \eta_2\mathcal{A}(\mathcal{F})$ in such that, $\sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k\Delta\mathcal{F}_k \|^{p_k/G}}{\eta_1}\right) \leq 1$ and $\sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k\Delta\mathcal{G}_k \|^{p_k/G}}{\eta}\right) \leq 1.$

Let $\eta_3 = \eta_1 + \eta_2$ then, $\eta_3 > 0$, and

$$\sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k(\Delta\mathcal{F}_k + \Delta\mathcal{G}_k) \|^{p_k/G}}{\eta_3}\right) = \sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k(\Delta\mathcal{F}_k + \Delta\mathcal{G}_k) \|^{p_k/G}}{\eta_1 + \eta_2}\right)$$

$$\leq \frac{\eta_1}{\eta_1 + \eta_2}\sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k\Delta\mathcal{F}_k \|^{p_k/G}}{\eta_1}\right) + \frac{\eta_2}{\eta_1 + \eta_2}\sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k\Delta\mathcal{G}_k \|^{p_k/G}}{\eta_2}\right)$$

$$\leq \frac{\eta_1}{\eta_1 + \eta_2} \cdot 1 + \frac{\eta_2}{\eta_1 + \eta_2} \cdot 1 = 1$$

$$\therefore \sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k(\Delta\mathcal{F}_k + \Delta\mathcal{G}_k) \|^{p_k/G}}{\eta_3}\right) \leq 1.$$

This relation shows that $\eta_1 + \eta_2 = \eta_3 \in \mathcal{A}(\mathcal{F})$. Thus we have,

$\xi(\mathcal{F} + \mathcal{G}) \leq \eta_1 + \eta_2$ for $\eta_1 \in \mathcal{A}(\mathcal{F})$ and $\eta_2 \in \mathcal{A}(\mathcal{G})$.

i.e., $\xi(\mathcal{F} + \mathcal{G}) \leq \eta_1 + \eta_2$

i.e., $\xi(\mathcal{F} + \mathcal{G}) \leq \xi(\mathcal{F}) + \xi(\mathcal{G})$.

Hence the triangle inequality holds.

P. N_4 : Suppose $\mathcal{F}^n = (\mathcal{F}_k^n)$ be a sequence in $\mathcal{S}(\mathcal{F}, M, \alpha, P, G)$ such that $\xi(\mathcal{F}^n) \to 0$ as $n \to \infty$ and (λ_n) be a sequence of scholar such that $\lambda_n \to \lambda$. Then

$$\xi(\lambda_n \mathcal{F}^n) = \inf\left\{\eta : \sum_{k=1}^{\infty} M\left(\left\| \frac{\lambda_n \alpha_k \Delta \mathcal{F}_k^n}{\eta} \right\|^{p_k/G}\right) \leq 1\right\}$$

$$= \inf\left\{\eta : \sum_{k=1}^{\infty} M\left(|\lambda_k|^{p_k}\left\| \frac{\alpha_k \Delta \mathcal{F}_k^n}{\eta} \right\|^{p_k/G}\right) \leq 1\right\}$$

$$\leq \inf\left\{\eta : \sum_{k=1}^{\infty} M\left(D^{p_k}\left\| \frac{\lambda_k \Delta \mathcal{F}_k^n}{\eta} \right\|^{p_k/G}\right) \leq 1\right\},$$

where $D = \sup_n |\lambda_n|$.

Then for $r = \max(1, D)$, we have,

$\xi(\lambda_n \mathcal{F}^n) = \inf\left\{\eta : \sum_{k=1}^{\infty} M\left(\frac{r\| \alpha_k \Delta \mathcal{F}_k^n \|^{p_k/G}}{\eta}\right) \leq 1\right\}$. Taking $\frac{\eta}{r} = s$ then $s > 0$ and $\xi(\lambda_n \mathcal{F}^n) \leq \inf\left\{rs : \sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k \Delta \mathcal{F}_k^n \|^{p_k/G}}{\eta}\right) \leq 1\right\}$.

$$= r\xi(\mathcal{F}^n) \to 0 \text{ as } n \to \infty.$$

$$\therefore \xi(\lambda_n \mathcal{F}^n) \to 0 \text{ as } n \to \infty.$$

Now, let (λ_n) be a sequence scalar such that $\lambda_n \to 0$ as $n \to \infty$ then for ε with $0 < \varepsilon < 1$, we can find a positive integer N such that $|\alpha_n| \leq \varepsilon$ for all $n \geq N$ and let $\mathcal{F} = (\mathcal{F}_k) \in \mathcal{S}(\mathcal{F}, M, \alpha, P, G)$.

$$\sum_{k=1}^{\infty} M\left(\frac{\| \lambda_n \alpha_k \Delta \mathcal{F}_k \|^{p_k/G}}{\eta}\right) = \sum_{k=1}^{\infty} M\left(\frac{|\lambda_n|^{p_k/G}\| \alpha_k \Delta \mathcal{F}_k \|^{p_k/G}}{\eta}\right)$$

$$\leq \sum_{k=1}^{\infty} M\left(\frac{\varepsilon^{l/G}\| \alpha_k \Delta \mathcal{F}_k \|^{p_k/G}}{\eta}\right), \text{ where } \lim_k p_k = l > 0.$$

This relation shows that if $\eta \in \mathcal{A}(\varepsilon^{l/G}\mathcal{A}(\mathcal{F}))$ then $\eta \in \mathcal{A}(\lambda_n \mathcal{F})$ and so $\mathcal{A}(\varepsilon^{l/G}\mathcal{F}) \subseteq \mathcal{A}(\lambda_n \mathcal{F})$.

$\Rightarrow \inf\{\eta > 0 : \eta \in \mathcal{A}(\lambda_n \mathcal{F})\} \leq \inf\{\eta > 0 : \eta \in \mathcal{A}(\varepsilon^{l/G}\mathcal{F})\}$

$= \varepsilon^{l/G} \inf\{\eta >: \eta \in \mathcal{A}(\mathcal{F})\}$

$\therefore \xi(\lambda_n \mathcal{F}) \leq \varepsilon^{l/G}\xi(\mathcal{F})$ for all $n \geq N$, which implies that $\xi(\lambda_n \mathcal{F}) \to 0$ as $n \to \infty$.

Thus ξ satisfies all the conditions of paranorm on $\mathcal{S}(\mathcal{F}, M, \alpha, P, G)$ and hence $\mathcal{S}((\mathcal{F}, M, \alpha, P, G), \xi)$ is a paranorm space.

Theorem 5. *The sequence space* $\mathcal{S}((\mathcal{F}, M, \alpha, P, G), \xi)$ *is complete paranormed space with the paranorm defined in* (**).

Proof. Let $\mathcal{F}^n = (\mathcal{F}_k^n)$ be a Cauchy sequence in $\mathcal{S}((\mathcal{F}, M, \alpha, P, G), \xi)$. Then there exists $\delta > 0$ such that for all $m, n > N \in Z_+$ we have

$$\xi(\mathcal{F}_k^n - \mathcal{F}_k^m) < \delta$$

By the definition of paranorm we see that,

$$\sum_{k=1}^{\infty} M\left(\frac{\| \lambda_k \Delta \mathcal{F}_k^n - \lambda_k \Delta \mathcal{F}_k^m \|^{p_k/G}}{\xi(\mathcal{F}_k^n) - \xi(\mathcal{F}_k^m)}\right) \leq 1 \text{ for all } m, n \geq N$$

Let us choose a fixed real number $r > 0$ such that $M(r) \geq 1$. Then $M\left(\frac{\| \lambda_k \Delta \mathcal{F}_k^n - \lambda_k \Delta \mathcal{F}_k^m \|^{p_k/G}}{\xi(\mathcal{F}_k^n) - \xi(\mathcal{F}_k^m)}\right) \leq M(r)$ for all $m, n \geq N$ and for all $k \geq 1$.

Since M non-decreasing, we have

$$\frac{\| \alpha_k \Delta \mathcal{F}_k^n - \alpha_k \Delta \mathcal{F}_k^m \|^{p_k/G}}{\xi(\mathcal{F}_k^n) - \xi(\mathcal{F}_k^m)} \leq r, \ \forall\, m, n \geq N$$

Or, $\| \Delta \mathcal{F}_k^n - \Delta \mathcal{F}_k^m \|^{p_k/G} \leq \dfrac{r}{| \alpha_k |^{p_k/G}} \cdot \xi(\mathcal{F}_k^n - \mathcal{F}_k^m)\ \forall\, m, n \geq N$

$$\therefore \| \Delta \mathcal{F}_k^n - \Delta \mathcal{F}_k^m \| < \frac{(r\delta)^{\frac{G}{p_k}}}{| \alpha_k |} = \varepsilon'(\text{say}), \ \forall\, m, n \geq N.$$

This shows that $\{\Delta\mathcal{F}_k^n\}$ is a Cauchy sequence in $\mathbb{R}(I)$. Since $\mathbb{R}(I)$ is complete, so the sequence $\{\Delta\mathcal{F}_k^n\}$ converges in $\mathbb{R}(I)$ say $\Delta\mathcal{F}_k^n \to \Delta\mathcal{F}_k$ as $k \to \infty$. To complete the proof we show that

$$\mathcal{F} = (\mathcal{F}_k) \in \mathcal{S}((\mathcal{F}, M, \alpha, P, G), \xi).$$

Let us choose $\eta > 0$ such that $\xi(X_k^n - X_k^m) < \eta < \varepsilon$ for all $m, n \geq N$. Then,

$$M\left(\frac{\| \alpha\Delta\mathcal{F}_k^n - \alpha_k\mathcal{F}_k^m \|^{p_k/G}}{\eta}\right) \leq \frac{\| \alpha_k(\Delta\mathcal{F}_k^n - \Delta\mathcal{F}_k^m) \|^{p_k/G}}{\xi(X_k^n - X_k^m)}$$

So that, $\sum_{k=1}^{\infty} M\left(\frac{\| \alpha\Delta\mathcal{F}_k^n - \alpha_k\mathcal{F}_k^m \|^{p_k/G}}{\eta}\right)$

$$\leq \sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k(\Delta\mathcal{F}_k^n - \Delta\mathcal{F}_k^m) \|^{p_k/G}}{\xi(X_k^n - X_k^m)}\right) \leq 1 \text{ for all } m, n \geq N.$$

Taking limit as $m \to \infty$ we get,

$$\sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k\Delta\mathcal{F}_k^n - \alpha_k\mathcal{F}_k^m \|^{p_k/G}}{\eta}\right) \leq 1 \text{ for all } n \geq N.$$

This relation is true for all such $\eta > 0$, so taking infimum of such η's, we get

$$\inf\left\{\eta > 0 : \sum_{l=1}^{\infty} M\left(\frac{\| \alpha_k(\Delta\mathcal{F}_k^n - \Delta\mathcal{F}) \|^{p_k/G}}{\eta}\right) \leq 1\right\}$$

Thus

$$\xi(\mathcal{F}^n - \mathcal{F}) = \inf\left\{\eta : \sum_{l=1}^{\infty} M\left(\frac{\| \alpha_k(\Delta\mathcal{F}_k^n - \Delta\mathcal{F}) \|^{p_k/G}}{\eta}\right) \leq 1\right\} \leq \eta < \varepsilon \quad \text{for}$$

all $n \geq N$.

$\Rightarrow \xi(\mathcal{F}^n - \mathcal{F}) < \varepsilon$ for all $n \geq N$.

This shows that $\mathcal{F}^n \to \mathcal{F}$.

Also, we have

$$\| \mathcal{F}^n - \mathcal{F} \| \to 0 \text{ as } n \to \infty$$

$$| \alpha_k | \| \Delta\mathcal{F}^n - \Delta\mathcal{F} \| \to 0 \text{ as } n \to \infty$$

$$\frac{\| \alpha_k(\Delta\mathcal{F}^n - \Delta\mathcal{F}) \|^{p_k/G}}{\eta} \to 0 \text{ as } n \to \infty$$

By the continuity of M, we have

$$M\left(\frac{\| \alpha_k(\Delta\mathcal{F}^n - \Delta\mathcal{F}) \|^{p_k/G}}{\eta}\right) \to M(0) \text{ as } n \to \infty$$

Then we have $\sum_{k=1}^{\infty} M\left(\frac{\| \alpha_k(\Delta\mathcal{F}^n - \Delta\mathcal{F}) \|^{p_k/G}}{\eta}\right) < \infty$

So that, $\mathcal{F}^n - \mathcal{F} \in \mathcal{S}(\mathcal{F}, M, \alpha, P, G)$.

Since, $\mathcal{S}(\mathcal{F}, M, \alpha, P, G)$ is linear space, $-(\mathcal{F}^n - \mathcal{F}) + \mathcal{F}^n \in \mathcal{S}(\mathcal{F}, M, \alpha, P, G)$, that is $\mathcal{F} \in \mathcal{S}(\mathcal{F}, M, \alpha, P, G)$. Hence $\mathcal{F}^n - \mathcal{F}$ in $\mathcal{S}(\mathcal{F}, M, \alpha, P, G)$. So the space $\mathcal{S}((\mathcal{F}, M, \alpha, P, G), \xi)$ is complete paranormed space.

Conclusion

In this paper, the Orlicz is used to investigate some topological properties classes $\mathcal{S}(\mathcal{F}, M, \alpha, P)$ and $\mathcal{S}(\mathcal{F}, M, \alpha, P, G)$ of fuzzy real numbers. We have investigated the solidity, inclusion relation, and linear property of these classes. Finally, we have defined a paranorm for the completeness properties of class $\mathcal{S}(\mathcal{F}, M, \alpha, P, G)$.

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