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STABILITY ANALYSIS OF STOCHASTIC HIV/AIDS EPIDEMIC MODEL WITH VERTICAL TRANSMISSION

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Abstract

We adapt a deterministic Human Immunodeficiency Virus (HIV) epidemic model to a Stochastic Differential Equations (SDE) model in this study by including random disturbances. Using Lyapunov theory, we show that the SDE system has a unique positive global solution and that the threshold value $\widetilde{R}^S$ may be utilized to govern the stochastic dynamics of the SDE model. When we combine $\widetilde{R}^S < 1$ with additional conditions, we see that the solution of the stochastic system swings around the solution of the deterministic system. Furthermore, we explore the long-term behaviour of stochastic system near the endemic equilibrium of deterministic system when $\widetilde{R}^S > 1$. Analytical results are numerically validated.

1. Introduction

Epidemiology is a discipline of medicine that examines infectious diseases in communities and is concerned with all elements of an epidemic, including transmission, control, and vaccine strategy. Many models addressing the spread of infectious diseases are based on the famous SIR model of Kermack and Mckendrick [9].

One of the world's most critical health and development issues is AIDS (Acquired Immunodeficiency Syndrome) which is caused by HIV. HIV spreads through bodily fluids and gradually weakens the immune system by

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eliminating cells that fight pathogens, diseases, and bacteria like tuberculosis, especially $CD4^+T$ cells. When this happens, the immune system weakens, making it more difficult for the immune system to combat infections and other diseases. The person is thus more vulnerable to illnesses such as pneumonia and tuberculosis, and HIV can lead to the development of AIDS.

Approximately 38 million people are presently infected with HIV, and tens of millions have died as a result of AIDS-related illnesses since the epidemic's inception.

The most prevalent routes for HIV to be transferred across the world are (i) sexual contact, (ii) exchanging contaminated blood products or needles, and (iii) vertical transmission from infected mothers to their babies during gestation, birth, or breastfeeding [19]. Vertical HIV transmission continues to be a substantial contributor to the HIV epidemic, accounting for 9% of new infections worldwide.

Although there is no cure for HIV, antiretroviral medication decreases the quantity of HIV in the blood, allowing the victim to live longer without experiencing HIV-related symptoms [19].

Mathematical modelling is a widely used tool for studying and investigating infectious disease dynamics, as well as recommending disease outbreak mitigation techniques. Many scholars have employed mathematical models extensively to explore epidemiology [3, 4, 6, 10, 11, 12, 13]. Recently, mathematical modelling has been used in HIV/AIDS epidemiology to assist increase our understanding of the primary contributing causes to the epidemic. Furthermore, the HIV/AIDS dynamics pose a slew of new challenges for mathematics, biologists, and epidemiologists since it differs from classical infectious disease in a variety of ways. May and Anderson were the first to present the models [1, 15]. The existence of a threshold expressible in terms of epidemic and demographic characteristics (birth, death, transmission, and recovery rates) that distinguishes conditions in which the disease finally dies out from those in which the disease becomes endemic is of apparent epidemiological relevance.

Samanta et al. [18] developed a five compartmental HIV model using ordinary differential equations to investigate the influence of infection delay.

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - (\beta_1 I_1 + \beta_2 I_2)S - \mu S \\ \dfrac{dI_1}{dt} = (\beta_1 I_1 + \beta_2 I_2)S + \eta(I_1 + I_2) - (\alpha + \mu)I_1 \\ \dfrac{dI_2}{dt} = \alpha I_1 - (\sigma + \rho + \mu)I_2 \\ \dfrac{dT}{dt} = \rho I_2 - (\gamma + \mu)T \\ \dfrac{dA}{dt} = \gamma T + \sigma I_2 - (d + \mu)A \end{cases} \tag{1.1}$$

Here, the entire population is split up into five groups: Susceptible population $S(t)$, infective population without symptoms $I_1(t)$, infective population with symptoms $I_2(t)$, infected population under treatment $T(t)$ and full-blown AIDS group $A(t)$; the key parameters are: Λ, the recruitment rate of S; β_1, horizontal transmission rate of I_1; β_2, horizontal transmission rate of I_2; η, recruitment rate of new borne infected children into I_2; α, progression rate from I_1 to I_2; ρ, the proportion of I_2 who enter into T; σ, progression rate to A from I_2; γ, transfer rate from T to A; d, death rate of A due to the disease; μ, the natural death rate.

In our study we account for the vertical transmission of HIV infection and therapy that reduces infection transmission in the HIV epidemic model [18].

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - (\beta_1 I_1 + \beta_2 I_2)S - \mu S \\ \dfrac{dI_1}{dt} = (\beta_1 I_1 + \beta_2 I_2)S + \eta(I_1 + I_2) - (\alpha + \mu)I_1 \\ \dfrac{dI_2}{dt} = \alpha I_1 - (\sigma + \rho + \mu)I_2 \\ \dfrac{dT}{dt} = \rho I_2 - (\gamma + \mu)T \end{cases} \tag{1.2}$$

with initial conditions:

$$S(0) > 0,\ I_1(0) > 0,\ I_2(0) > 0,\ T(0) > 0.$$

The above deterministic model of system (1.2) has a disease-free equilibrium $E_0 = \left(\dfrac{\Lambda}{\mu}, 0, 0, 0\right)$ with basic reproduction number $R_0 = \dfrac{\Lambda}{\mu} \dfrac{\beta_1(\rho + \sigma + \mu) + \alpha\beta_2}{(\alpha + \mu - \eta)(\rho + \sigma + \mu) - \alpha\eta}$ which represents the average new

infected produced by one infected individual during his life time when the population at the disease-free equilibrium. Further we found the system (1.2) has a positive endemic equilibrium $E^* = \left(\frac{\Lambda}{\mu R_0}, \frac{\Lambda(\rho + \sigma + \mu)}{S^*(\beta_1(\rho + \sigma + \mu) + \alpha\beta_2)} \right.$

$\left. \left(1 - \frac{1}{R_0}\right), \frac{\alpha I_1^*}{\gamma + \mu}, \frac{\alpha\mu\rho(R_0 - 1)}{(\gamma + \mu)(\beta_1(\rho + \sigma + \mu) + \alpha\beta_2)} \right)$.

The diversity and unpredictability of the environment are intrinsically tied to the character of an epidemic. The deterministic technique has substantial shortcomings in mathematical modelling of infectious disease transmission, and precisely projecting the future dynamics of the system is challenging. This occurs when deterministic models fail to account for the impact of changing environmental conditions. Stochastic differential equation models, which provide more realism than deterministic counterparts, are used in a wide range of applications, including infectious disease dynamics. In fact, due to continual environmental fluctuation, the parameters employed in ecological system modeling are never absolute constants and always oscillate around prescribed average values. As a result, several researchers have studied the dynamics of epidemic models with parameter perturbation [5, 8, 16, 17, 20, 21].

To account for the influence of a randomly changing environment in system (1.2), we assumed stochastic disturbances of white noise type that are directly proportional to $S(t)$, $I_1(t)$, $I_2(t)$ and $T(t)$.

The stochastic model corresponding to sub model of system (1.1) is:

$$\begin{cases} dS = [\Lambda - (\beta_1 I_1 + \beta_2 I_2)S - \mu S]dt + \theta_1 S dB_1(t) \\ DI_1 = [(\beta_1 I_1 + \beta_2 I_2)S + \eta(I_1 + I_2) - (\alpha + \mu)I_1]dt + \theta_2 I_1 dB_2(t) \\ dI_2 = [\alpha I_1 - (\sigma + \rho + \mu)I_2]dt + \theta_3 I_2 dB_3(t) \\ dT = [\rho I_2 - (\gamma + \mu)T]dt + \theta_4 T dB_4(t) \end{cases} \tag{1.3}$$

where θ_i are the white noise intensities and $B_i(t)$ are independent Wiener processes.

The following is how this article is structured. Section 2 demonstrates that the system (1.3) has a unique positive solution by the method indicated

in [5]. In Section 3, we establish the threshold value $\widetilde{R}^S$ and we show that when $\widetilde{R}^S < 1$, the solution of the system (1.3) oscillates around the disease-free equilibrium E_0. In Section 4, we investigate the asymptotic behaviour of the stochastic model (1.3) around the endemic equilibrium of the deterministic model. Finally, we carry out numerical simulations to verify our theoretical conclusions in Section 5.

Throughout the article, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ that meets the standard criteria (i.e., it is right continuous and $\mathcal{F}_0$ includes all P-null sets).

Denote $R_+^n = \{x \in R^n : x_i > 0, \text{ for all } 1 \leq i \leq n\}$, $\overline{R_+^n} = \{x \in R^n : x_i \geq 0, \text{ for all } 1 \leq i \leq n\}$.

A general d-dimensional stochastic differential equation takes the form [14]

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t) \text{ on } t \geq t_0 \tag{1.4}$$

with initial value $x(t_0) = x_0 \in R^d$, where $B(t)$ denotes d-dimensional standard Brownian motion defined on the above probability space.

2. Existence and Uniqueness of Global Positive Solution

In this section, we show that the system (1.3) owns a unique, positive and global solution using Lyapunov analysis approach.

Theorem 2.1. *For any initial value* $(S(0), I_1(0), I_2(0), T(0)) \in R_+^4$, *the system* (1.3) *admits a unique solution* $(S(t), I_1(t), I_2(t), T(t))$ *on* $t \geq 0$, *and the solution will remain in* R_+^4 *with probability* 1, (i.e.), $(S(t), I_1(t), I_2(t), T(t)) \in R_+^4$ *for all* $t \geq 0$ *almost surely*.

Proof. Note that the coefficients of the equations in system (1.3) are locally Lipschitz continuous for any given initial value $(S(0), I_1(0), I_2(0), T(0)) \in R_+^4$. As a result, the system possesses a unique local solution

$(S(t), I_1(t), I_2(t), T(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time [2]. To demonstrate that this solution is global, just we only need to show that $\tau_e = \infty$ a.s. Allow $k' \geq 1$ to be sufficiently large enough that all $(S(0), I_1(0), I_2(0), T(0))$ lie inside the interval $[1/k', k']$. For each integer $n \geq k'$, let us define the stopping time as

$$\tau_n =$$

$$\inf\left\{t \in [0, \tau_e) : \min\{S(t), I_1(t), I_2(t), T(t)\} \leq \frac{1}{n} \text{ or } \max\{S(t), I_1(t), I_2(t), T(t)\} \geq n\right\},$$

where, throughout the paper, we assume $\inf \oslash = \infty$ (as usual $\oslash$ denotes the empty set). Obviously, τ_n is increasing when $n \to \infty$. Denote $\tau_\infty = \lim_{n\to\infty} \tau_n$, when $\tau_\infty \leq \tau_e$ a.s. If $\tau_\infty = \infty$ a.s., consequently $\tau_e = \infty$ and $(S(0), I_1(0), I_2(0), T(0)) \in R_+^4$ a.s. for all $t \geq 0$. To put it in other words, we just need to claim that $\tau_\infty = \infty$ a.s.

If it is not, we may find some $U > 0$ and $\zeta \in (0, 1)$ such that

$$P\{\tau_\infty \leq U\} > \zeta.$$

Thus there is an integer $k_1 \geq k'$ such that

$$P\{\tau_\infty \leq U\} > \zeta \text{ for all } n \geq k_1. \tag{2.1}$$

Meanwhile, for $t \leq \tau_n$,

$$d(S + I_1 + I_2, T) = [\Lambda - \mu S + \eta(I_1 + I_2) - \mu I_1 - \sigma I_2 - \mu I_2 - \gamma T - \mu T]dt$$

$$\leq [\Lambda - (\mu - \eta)(S + I_1 + I_2 + T)]dt$$

and

$$N(t) \leq \begin{cases} \dfrac{\Lambda}{\mu - \eta}, & \text{when } N(0) \leq \dfrac{\Lambda}{\mu - \eta} \\ N(0), & \text{when } N(0) > \dfrac{\Lambda}{\mu - \eta} \end{cases} : K$$

Now a C^2 function $V_1 : R_+^4 \to \overline{R}_+$ is defined by

$$V_1(S, I_1, I_2, T) = (S - 1 - \log S) + (I_1 - 1 - \log I_1) + (I_2 - 1 - \log I_2)$$

$$+ (T - 1 - \log T),$$

which is non-negative, since $v - 1 - \log v \geq 0, \forall v \geq 0$.

Itô's formula yields,

$$dV_1 = \left(1 - \frac{1}{S}\right)[(\Lambda - \beta_1 I_1 S - \beta_2 I_2 S - \mu S)]dt$$

$$+ \left(1 - \frac{1}{I_1}\right)[(\beta_1 I_1 S + \beta_2 I_2 S + \eta(I_1 + I_2) - (\alpha + \mu)I_1)]dt$$

$$+ \left(1 - \frac{1}{I_2}\right)[\alpha I_1 (\sigma + \rho + \mu)I_2]dt + \left(1 - \frac{1}{T}\right)[\rho I_2 - (\gamma + \mu)T]dt$$

$$+ \frac{1}{2}[\theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2]dt + (S - 1)\theta_1 dB_1(t) + (I_1 - 1)\theta_2 dB_2(t)$$

$$+ (I_2 - 1)\theta_3 dB_3(t) + (T - 1)\theta_4 dB_4(t)$$

$$\leq \left[\Lambda + 4\mu + \alpha + \sigma + \rho + \gamma + (\eta + \beta_1 + \beta_2)K + \frac{1}{2}[\theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2]\right]dt$$

$$+ (S - 1)\theta_1 dB_1(t) + (I_1 - 1)\theta_2 dB_2(t) + (I_2 - 1)\theta_3 dB_3(t) + (T - 1)\theta_4 dB_4(t)$$

$$= K^* dt + (S - 1)\theta_1 dB_1(t) + (I_1 - 1)\theta_2 dB_2(t) + (I_2 - 1)\theta_3 dB_3(t)$$

$$+ (T - 1)\theta_4 dB_4(t)$$

Therefore, if $t_1 \leq U$,

$$\int_0^{\tau_n \wedge t_1} dV_1(S(t), I_1(t), I_2(t), T(t)) \leq \int_0^{\tau_n \wedge t_1} K^* dt + \int_0^{\tau_n \wedge t_1} \theta_1 (S - 1) dB_1(t)$$

$$+ \int_0^{\tau_n \wedge t_1} \theta_2 (I_1 - 1) dB_2(t) + \int_0^{\tau_n \wedge t_1} \theta_3 (I_2 - 1) dB_3(t)$$

$$+ \int_0^{\tau_n \wedge t_1} \theta_4 (T - 1) dB_4(t).$$

Taking expectations,

$$E[V_1(S(\tau_n \wedge t_1),\ I_1(\tau_n \wedge t_1),\ I_2(\tau_n \wedge t_1),\ T(\tau_n \wedge t_1))]$$

$$\leq V_1(S(0),\ I_1(0),\ I_2(0),\ T(0)) + E\int_0^{\tau_n \wedge t_1} K^* dt$$

$$\leq V_1(S(0),\ I_1(0),\ I_2(0),\ T(0)) + K^* U \tag{2.2}$$

Set $\Omega_n = \{\tau_n \leq U\}$ for $n \geq k_1$ and by (2.1), $P(\Omega_n) \geq \zeta$. Notice that, for any $\nu \in \Omega_n$, there is at least one of $S(\tau_n, \nu)$, $I_1(\tau_n, \nu)$, $I_2(\tau_n, \nu)$ and $T(\tau_n, \nu)$ takes the value either n or $1/n$, and hence

$$V_1(S(\tau_n, \nu),\ I_1(\tau_n, \nu),\ I_2(\tau_n, \nu),\ T(\tau_n, \nu)) \geq (n - 1 - \log n) \wedge \left(\frac{1}{n} - 1 + \log n\right).$$

It then follows from (2.1) and (2.2) that

$$V_1(S(0),\ I_1(0),\ I_2(0),\ T(0)) + K^* U$$

$$\geq E[1_{\Omega_n(\nu)} V_1(S(\tau_n, \nu),\ I_1(\tau_n, \nu),\ I_2(\tau_n, \nu),\ T(\tau_n, \nu))]$$

$$\geq P\{1_{\Omega_n(\nu)}\} E[V_1(S(\tau_n, \nu),\ I_1(\tau_n, \nu),\ I_2(\tau_n, \nu),\ T(\tau_n, \nu))]$$

$$\geq \zeta\left[(n - 1 - \log n) \wedge \left(\frac{1}{n} - 1 + \log n\right)\right],$$

where $1_{\Omega_n(\nu)}$ is the indicator function of Ω_n. Allowing $n \to \infty$ results in the contradiction $\infty > V_1(S(0),\ I_1(0),\ I_2(0),\ T(0)) + K^* U = \infty$. As a result, $\tau_\infty = \infty$ a.s. which completes the proof of Theorem 2.1.

In nature, the initial value $S(0)$, $I_1(0)$, $I_2(0)$, and $T(0)$ may be zero. It is both interesting and practically important to consider what happens when $(S(0),\ I_1(0),\ I_2(0),\ T(0)) \in \overline{R}_+^4$.

Theorem 2.2. *For any initial value* $(S(0),\ I_1(0),\ I_2(0),\ T(0)) \in \overline{R}_+^4$, *the solution of system* (1.3) *will remain in* $\overline{R}_+^4$ *with probability* 1.

Proof. Clearly, first equation of (1.3) gives

$$S(t) = e^{-\mu t - \frac{1}{2}\theta_1^2 t - \int_0^t [\beta_1 I_1(u) + \beta_2 I_2(u)]du + \int_0^t \theta_1 dB_1(u)}$$

$$\left[S(0) + \Lambda \int_0^t e^{\mu u + \frac{1}{2}\theta_1^2 u + \int_0^t [\beta_1 I_1(u) + \beta_2 I_2(u)]du - \int_0^t \theta_1 dB(u)} du \right]$$

Then $S(t) > 0$ if $S(0) > 0$ or $S(t) = 0$.

Next, we consider the infective population without symptoms $I_1(t)$.

$$I_1(t) = e^{-\left[(\alpha+\mu-\eta)+\frac{\theta_2^2}{2}\right]t + \int_0^t \beta_1 S(u)du + \int_0^t \theta_2 dB_2(u)}$$

$$\left[I_1(0) + \int_0^t (\beta_2 I_2(u) S(u) + \eta I_2(u)) e^{\left[(\alpha+\mu-\eta)+\frac{\theta_2^2}{2}\right]u + \int_0^t \beta_1 S(u)du + \int_0^t \theta_2 dB_2(u)} \right]$$

Obviously, $I_1(t) > 0$ no matter $I_1(t) > 0$ or $I_1(t) = 0$.

Third equation of (1.3) yields,

$$I_2(t) =$$

$$e^{-\left[(\rho+\sigma+\mu)+\frac{\theta_3^2}{2}\right]t + \int_0^t \theta_3 dB_3(u)} \left[I_2(0) + \alpha \int_0^t I_1(u) e^{\left[(\rho+\sigma+\mu)+\frac{\theta_3^2}{2}\right]u - \int_0^t \theta_3 dB_3(u)} du \right]$$

Clearly $I_2(0) > 0$ no matter if $I_2(0) > 0$ or $I_2(0) = 0$.

From (1.3),

$$T(t) = e^{-\left[(\gamma+\mu)+\frac{\theta_4^2}{2}\right]t + \int_0^t \theta_4 dB_4(u)} \left[T(0) + \rho \int_0^t I_2(u) e^{\left[(\gamma+\mu)+\frac{\theta_4^2}{2}\right]u - \int_0^t \theta_4 dB_4(u)} du \right]$$

Clearly $T > 0$.

(i.e.) we can conclude that the variables $S(t) > 0$, $I_1(t) > 0$, $I_2(t) > 0$ and $T(0) > 0$.

Remark 2.3. From Theorems 2.1 and 2.2, we see that that for any initial value $(S(0), I_1(0), I_2(0), T(0)) \in \overline{R}_+^4$, the system (1.3) admits a unique, global solution $(S(t), I_1(t), I_2(t), T(t)) \in \overline{R}_+^4$ almost surely. Therefore,

$$d(S + I_1 + I_2 + T) \leq [\Lambda - (\mu - \eta)(S + I_1 + I_2 + T)]dt$$

and

$$S(t) + I_1(t) + I_2(t) + T(t) \leq \frac{\Lambda}{\mu - \mu} + e^{-(\mu-\eta)t}\left[S(0) + I_1(0) + I_2(0) + T(0) - \frac{\Lambda}{\mu - \eta}\right]$$

Obviously, $S(t) + I_1(t) + I_2(t) + T(t) \leq \frac{\Lambda}{\mu - \eta}$, when $S(0) + I_1(0) + I_2(0) + T(0) \leq \frac{\Lambda}{\mu - \eta}$.

$\therefore$ The region defined by

$$\Pi = \left\{(S, I_1, I_2, T) : S > 0, I_1 > 0, I_2 > 0, T > 0, S + I_1 + I_2 + T \leq \frac{\Lambda}{\mu + \eta}\right\}$$

is a positively invariant set of the stochastic system (1.3).

Hereafter, we assume that any initial solution $(S(0) + I_1(0) + I_2(0) + T(0)) \in \Pi$.

3. Asymptotic Behavior around Disease-Free Equilibrium

In this section, we use the stochastic Lyapunov function [14] to examine the stability of the disease free equilibrium $E_0 = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right)$ of the deterministic model.

Lemma 3.1[14] (Strong Law of Large Numbers)**.** *Let $M = \{M_t\}_{t \geq 0}$ denote a real-valued continuous local martingale that vanishes at $t = 0$. Then*

$$\lim_{t \to \infty} \langle M, M \rangle_t = \infty \ a.s. \Rightarrow \lim_{t \to \infty} \frac{M_t}{\langle M, M \rangle_t} = 0 \ a.s.$$

and also

$$\limsup_{t\to\infty} \frac{\langle M, M\rangle_t}{t} < \infty \ a.s. \Rightarrow \lim_{t\to\infty} \frac{M_t}{t} = 0 \ a.s.$$

Based on this lemma, we give the main theorem in this section.

Define

$$\widetilde{R}^S = \frac{\frac{\Lambda}{\mu}[\beta_1(\rho+\sigma+\mu)+\beta_2(\alpha+\eta)]+\eta^2+\frac{1}{2}\theta_2^2(\sigma+\rho+\mu)}{(\alpha+\mu-\eta)(\alpha+\pi+\mu)-\alpha\eta}.$$

Theorem 3.2. *Let* $(S(t)+I_1(t)+I_2(t)+T(t))$ *be the solution of the system* (1.3) *with any initial value* $S(0)+I_1(0)+I_2(0)+T(0)\in R_+^4$. *If* $\widetilde{R}^S < 1$, $(\rho+\sigma+\mu)-\frac{\rho^2}{\gamma+\mu} > \theta_3^2$, *and* $\gamma+\mu > \theta_4^2$,

$$\limsup_{t\to\infty} \frac{1}{t} E\int_0^t \left[r_1\left(S(s)-\frac{\Lambda}{\mu}\right)^2 + r_2 I_1^2(s) + r_3 I_2^2(s) + r_4 T^2(s)\right] ds \le \theta_1^2\left(\frac{\Lambda}{\mu}\right)^2$$

where $r_1 = \mu$, $r_2 = 2(\alpha+\mu-\eta)$, $r_3 = (\rho+\sigma+\mu)-\frac{\rho^2}{\gamma+\mu}-\theta_3^2$ *and* $r_4 = (\gamma+\mu)-\theta_4^2$.

Proof. Let $u = S-\frac{\Lambda}{\mu}$, $v = I_1$, $w = I_2$, $z = T$. Then $u \le 0$, $v > 0$, $w > 0$, $z > 0$.

The system (1.3) deduces to the system of equations:

$$du(t) = \left[\Lambda - \beta_1 v\left(u+\frac{\Lambda}{\mu}\right) - \beta_2 w\left(u+\frac{\Lambda}{\mu}\right) - \mu\left(u+\frac{\Lambda}{\mu}\right)\right]dt + \theta_1\left(u+\frac{\Lambda}{\mu}\right)dB_1(t)$$

$$dv(t) = \left[\beta_1 v\left(u+\frac{\Lambda}{\mu}\right) - \beta_2 w\left(u+\frac{\Lambda}{\mu}\right) - \eta(v+w) - (\alpha+\mu)v\right]dt + \theta_2 v dB_2(t)$$

$$dw(t) = [\alpha v - (\rho+\sigma+\mu)w]dt + \theta_3 w dB_3(t)$$

$$dz(t) = [\rho w - (\gamma+\mu)]dt + \theta_4 z dB_4(t)$$

Define the stochastic Lyapunov function $R^4 \to \overline{R}_+$:

$$V_2(u, v, w, z) = (u + v)^2 + av^2 + w^2 + z^2,$$

where $a > 0$, to be found later. Then

$$LV_2 \le 2(u + v)\left[\Lambda - \mu\left(u + \frac{\Lambda}{\mu}\right) + \eta(v + w) - (\alpha + \mu)v\right]$$

$$+ 2av\left[\beta_1 v\left(u + \frac{\Lambda}{\mu}\right) + \beta_2 w\left(u + \frac{\Lambda}{\mu}\right) + \eta(v + w) - (\alpha + \mu)\right]v + 2w[\alpha v - (\rho + \rho + \mu)w]$$

$$+ 2z[\rho w - (\gamma + \mu)z] + \theta_1^2\left(u + \frac{\Lambda}{\mu}\right)^2 + (1 + a)\theta_2^2 v^2 + \theta_3^2 w^2 + \theta_4^2 z^2.$$

Note that $u \le 0$. Then

$$LV_2 \le 2(u + v)[-\mu u + \eta w - (\alpha + \mu - \eta)v]$$

$$+ 2av\left[\beta_1 v\left(\frac{\Lambda}{\mu}\right) + \beta_2 w\left(\frac{\Lambda}{\mu}\right) + \eta w - (\alpha + \mu - \eta)v\right]$$

$$+ 2w[\alpha v - (\rho + \sigma + \mu)w] + 2z[\rho w - (\gamma + \mu)z] + \theta_1^2\left(\frac{\Lambda}{\mu}\right)^2 + (1 + a)\theta_2^2 v^2$$

$$+ \theta_3^2 w^2 + \theta_4^2 z^2$$

$$\le -2\mu u^2 - 2\left[(\alpha + \mu - \eta) - \alpha\left(\beta_1 \frac{\Lambda}{\mu} - (\alpha + \mu - \eta)\right)\right]v^2 - 2(\rho + \sigma + \mu)w^2$$

$$- 2(\gamma + \mu)z^2$$

$$- 2(\alpha + 2\mu - \eta)uv + 2\left[(\alpha + \eta) + a\left(\eta + \beta_2 \frac{\Lambda}{\mu}\right)\right]vw + 2\rho wz + \theta_1^2\left(\frac{\Lambda}{\mu}\right)^2 + (1 + \alpha)$$

$$\theta_2^2 v^2 + \theta_3^2 w^2 + \theta_4^2 z^2$$

$$\le -2\mu u^2 - 2\left[(\alpha + \mu - \eta) - \alpha\left(\beta_1 \frac{\Lambda}{\mu} - (\alpha + \mu - \eta)\right)\right]v^2 - 2(\rho + \sigma + \mu)w^2$$

$$- 2(\gamma + \mu)z^2$$

$$+\frac{(\alpha+2\mu-\eta)^2}{\mu}v^2+\mu u^2+\frac{\left[(\alpha+\eta)+a\left(\eta+\beta_2\frac{\Lambda}{\mu}\right)\right]^2}{\rho+\sigma+\mu}v^2+(\rho+\sigma+\mu)w^2$$

$$+\frac{\rho^2}{\gamma+\mu}w^2+(\gamma+\mu)z^2+\theta_1^2\left(\frac{\Lambda}{\mu}\right)^2+(1+a)\theta_2^2v^2+\theta_3^2w^2+\theta_4^2z^2.$$

After some simple calculations, we get

$$LV_2\le-\mu u^2-2\Bigg[(\alpha+\mu-\eta)-a\left(\beta_1\frac{\Lambda}{\mu}-(\alpha+\mu-\eta)\right)-\frac{1}{2}\frac{(\alpha+2\mu-\eta)^2}{\mu}$$

$$-\frac{1}{2}\frac{[(\alpha+\eta)+a(\eta+\beta_2\frac{\Lambda}{\mu})]^2}{\rho+\sigma+\mu}-\frac{(1+a)}{2}\theta_2^2\Bigg]v^2-\left[(\rho+\sigma+\mu)-\frac{\rho^2}{\gamma+\mu}\theta_3^3\right]w^2$$

$$-[(\gamma+\mu)-\theta_4^2]z^2+\theta_1^2\left(\frac{\Lambda}{\mu}\right)^2$$

$$\le-\mu u^2-\frac{2}{\rho+\sigma+\mu}\Bigg[(\alpha+\mu-\eta)(\rho+\sigma+\mu)-\frac{1}{2}\left(a^2\left(\eta+\beta_2\frac{\Lambda}{\mu}\right)\right)^2$$

$$+2a(\tilde{R}^S-1)((\alpha+\mu-\eta)(\rho+\sigma+\mu)-\alpha\eta)+(\alpha+\eta)^2$$

$$+\frac{(\alpha+2\mu-\eta)^2}{\mu}(\rho+\sigma+\mu)+\theta_2^2(\rho+\sigma+\mu)\Bigg]v^2$$

$$-\left[(\rho+\sigma+\mu)-\frac{\rho^2}{\gamma+\mu}-\theta_2^2\right]w^2-[(\gamma+\mu)-\theta_4^2]z^2+\theta_1^2\left(\frac{\Lambda}{\mu}\right)^2$$

Choose a such that

$$a^2\left(\eta+\beta_2\frac{\Lambda}{\mu}\right)^2+2a(\tilde{R}^S-1)((\alpha+\mu-\eta)(\rho+\sigma+\mu)-\alpha\eta)+(\alpha+\eta)^2$$

$$+\left[\frac{(\alpha+2\mu-\eta)^2}{\mu}+\theta_2^2\right](\rho+\sigma+\mu)=0$$

Thus

$$LV_2 \le -\mu u^2 - 2(\alpha + \mu - \eta)v^2 - \left[(\rho + \sigma + \mu) - \frac{\rho^2}{\gamma + \mu} - \theta_2^2\right]w^2$$

$$- [(\gamma + \mu) - \theta_4^2]z^2 + \theta_1^2\left(\frac{\Lambda}{\mu}\right)^2$$

$$\text{(i.e.) } LV_2 \le -r_1\left(S - \frac{\Lambda}{\mu}\right)^2 - r_2 I_1^2 - r_3 I_2^2 - r_4 T^2 + \theta_1^2\left(\frac{\Lambda}{\mu}\right)^2$$

Integrating from 0 to t and taking expectations,

$$EV_2(t) \le V_2(0) - E\int_0^t r_1\left(S(s) - \frac{\Lambda}{\mu}\right)^2 ds - E\int_0^t r_2 I_1^2(s) - E\int_0^t r_3 I_2^2(s) ds$$

$$- E\int_0^t r_4 T^2(s) ds + E\int_0^t \theta_1^2\left(\frac{\Lambda}{\mu}\right)^2 ds$$

$$\le V_2(0) - E\int_0^t \left[r_1\left(S(s) - \frac{\Lambda}{\mu}\right)^2 - r_2 I_1^2(s) - r_3 I_2^2(s) - r_4 I_2^2(s) - r_4 T^2(s)\right] ds$$

$$+ \theta_1^2\left(\frac{\Lambda}{\mu}\right)^2 t$$

Thus

$$\limsup_{t\to\infty} \frac{1}{t} E\int_0^t \left[r_1\left(S(s) - \frac{\Lambda}{\mu}\right)^2 + r_2 I_1^2(s) + r_3 I_2^2(s) + r_4 T^2(s)\right] ds \le \theta_1^2\left(\frac{\Lambda}{\mu}\right)^2.$$

Remark 3.3. Theorem 3.2 shows that the solution of the system (1.3) oscillates near the disease free equilibrium $E_0 = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right)$ in the time mean sense if $\widetilde{R}^S < 1$ and the magnitude of the oscillation is proportional to the intensity of the noise. From the biological point of view, the disease will be controlled in a small.

Besides, if $\theta_1 = 0$, from the proof of Theorem 3.2, we have

$$LV_2 \le -r_1\left(S - \frac{\Lambda}{\mu}\right)^2 - r_2 I_1^2 - r_3 I_2^2 - r_4 T^2,$$

which is negative definite.

Thus the solution of system (1.3) is stochastically asymptotically stable in the large.

4. Asymptotic Behavior near the Endemic Equilibrium of the Deterministic System

In this section, we look at how the solution of the stochastic system (1.3) behaves towards the endemic equilibrium of the deterministic system E^*, to see whether the disease will prevail.

Theorem 4.1. *Let* $(S(t), I_1(t), I_2(t), T(t))$ *be the solution of system* (1.3) *with any initial value* $(S(0), I_1(0), I_2(0), T(0)) \in \overline{R_+^4}$. *If* $\widetilde{R}^S > 1$ *and, then we have*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \widetilde{d}[(S(s) - S^*)^2 + (I_1(s) - I_1^*)^2 + (I_2(s) - I_2^*)^2 + (T(s) - T^*)^2]ds \le H\theta^2$$

a.s.,

where (S^*, I_1^*, I_2^*, T^*) *is the endemic equilibrium of system* (1.2),

$$H = \frac{\left(\frac{\Lambda}{\mu - \eta}\right)^2}{\widetilde{d}},\ \theta^2 = \theta_1^2 \vee (1 + a_1)\theta_2^2 \vee \theta_3^2 \vee \theta_4^2,$$

$$\widetilde{d} = \min\{\mu - \eta, \alpha + \mu - \eta, 2a_2(\rho + \sigma + \mu) - (\eta(1 + a_1) + \alpha a_2) - 2\eta - \rho, \gamma + \mu - \frac{\rho}{2}\},$$

$$a_1 = \frac{(\alpha + 2\mu - \eta)(\mu - \eta)}{\Lambda(\beta_1 + \beta_2)}; a_2$$

$$= \frac{2\alpha_1[(\alpha + \mu - 1.5\eta) - S^*[(\beta_1 + \beta_2)\frac{\Lambda}{\mu - \eta} - (\beta_1 I_1^* + \beta_2 I_2^*)]] + (\alpha + \mu - \eta)}{\alpha}.$$

Proof. Define

$$V_3 = (S - S^* + I_1 - I_1^*)^2 + a_1(I_1 - I_1^*)^2 + a_2(I_2 - I_2^*)^2 + (T - T^*)$$

where a_1 and a_2 are positive constants to be found later.

By Itô's formula,

$$dV_3 \le 2(S - S^* + I_1 - I_1^*)\theta_1 S dB_1(t) + [2(S - S^* + I_1 - I_1^*) + 2a_1(I_1 - I_1^*)]$$

$$\theta_2 I_1 dB_2(t) + 2a_2(I_2 - I_2^*)\theta_3 I_2 dB_3(t) + 2(T - T^*)\theta_4 T dB_4(t)$$

where

$$LV_3 \le$$

$$2(S - S^* + I_1 - I_1^*)[(\alpha + \mu)I_1^* - \eta(I_1^* + I_2^*) + \mu S^* - \mu S + \eta(I_1 + I_2) - (\alpha + \mu)I_1]$$

$$+ 2a_1(I_1 - I_1^*)\{(\beta_1 + \beta_2)\frac{\Lambda}{\mu - \eta}(S - S^*) + (\beta_1 + \beta_2)\frac{\Lambda}{\mu - \eta}S^* + \eta(I_1 + I_2)$$

$$- (\alpha + \mu)(I_1 - I_1^*) - \eta(I_1^* + I_2^*) - (\beta_1 I_1^* + \beta_2 I_2^*)S^*\}$$

$$+ 2a_2(I_2 - I_2^*)[\alpha(I_1 - I_1^*) - (\rho + \sigma + \mu)(I_2 - I_2^*)]$$

$$+ 2(T - T^*)[\rho(I_2 - I_2^*) - (\gamma + \mu)(T - T^*)] + \theta_1^2 S^2 + (1 + a_1)\theta_2^2 I_1^2 + \theta_3^2 I_2^2 + \theta_4^2 T^2$$

$$\le -2\mu(S - S^*)^2$$

$$- 2\left[(1 + a_1)(\alpha + \mu - \eta) - a_1 S^*\left[(\beta_1 + \beta_2)\frac{\Lambda}{\mu - \eta} - (\beta_1 I_1^* + \beta_2 I_2^*)\right]\right](I_1 - I_1^*)^2$$

$$- 2a_2(\rho + \sigma + \mu)(I_2 - I_2^*)^2 - 2(\gamma + \mu)(T - T^*)$$

$$+ (S - S^*)(I_1 - I_1^*)\left\{12(\alpha + 2\mu - \eta) + 2a_1(\beta_1 + \beta_2)\frac{\Lambda}{\mu - \eta}\right\}$$

$$+ (I_1 - I_1^*)(I_2 - I_2^*)\{2(1 + a_1)\eta + 2a_2\alpha\} + 2\eta(I_2 - I_2^*)(S - S^*)$$

$$+ 2\rho(I_2 - I_2^*)(T - T^*) + \theta_1^2 S^2 + (1 + a_1)\theta_2^2 I_1^2 + \theta_3^2 I_2^2 + \theta_4^2 T^2$$

Choose a_1 such that

$$2a_1(\beta_1 + \beta_2)\frac{\Lambda}{\mu - \eta} - 2(\alpha + 2\mu - \eta) = 0.$$

Thus

$$a_1 = \frac{(\alpha + 2\mu - \eta)(\mu - \eta)}{\Lambda(\beta_1 + \beta_2)}.$$

$$\therefore LV_3 \le -2(\mu - \eta)(S - S^*)^2 - \{2(1 + a_1)(\alpha + \mu - \eta) - (\eta(1 + a_1) + \alpha a_2)$$

$$- 2a_1 S^*[(\beta_1 + \beta_2)\frac{\Lambda}{\mu - \eta} - (\beta_1 I_1^* + \beta_2 I_2^*)]\}(I_1 - I_1^*)^2$$

$$- \{2a_2(\rho + \sigma + \mu) - (\eta(1 + a_1) + \alpha a_2) - 2\eta - \rho\}(I_2 - I_2^*)^2$$

$$- 2\left[(\gamma + \mu) - \frac{\rho}{2}\right](T - T^*)^2 + \theta_1^2 S^2 + (1 + a_1)\theta_2^2 I_1^2 + \theta_3^2 I_2^2 + \theta_4^2 T^2$$

Choose a_2 such that

$$2(1 + a_1)(\alpha + \mu - \eta) - (\eta(1 + a_1) + \alpha a_2) - 2a_1 S^*\left[(\beta_1 + \beta_2)\frac{\Lambda}{\mu - \eta} - (\beta_1 I_1^* + \beta_2 I_2^*)\right]$$

$$- (\alpha + \mu - \eta) = 0.$$

$$\therefore a_2 = \frac{2a_1\left[(\alpha + \mu - 1.5\eta) - S^*\left[(\beta_1 + \beta_2)\frac{\Lambda}{\mu - \eta} - (\beta_1 I_1^* + \beta_2 I_2^*)\right]\right] + (\alpha + \mu - \eta)}{\alpha}$$

$$\therefore LV_3 \le -2(\mu - \eta)(S - S^*)^2 - (\alpha + \mu - \eta)(I_1 - I_1^*)^2$$

$$- \{2a_2(\rho + \sigma + \mu) - (\eta(1 + a_1) + \alpha a_2) - 2\eta - \rho\}(I_2 - I_2^*)^2$$

$$- 2\left[(\gamma + \mu) - \frac{\rho}{2}\right](T - T^*)^2 + \theta_1^2 S^2 + (1 + a_1)\theta_2^2 I_1^2 + \theta_3^2 I_2^2 + \theta_4^2 T^2$$

$$\le -d_1(S - S^*)^2 - d_2(I_1 - I_1^*)^2 - d_3(I_2 - I_2^*)^2 - d_4(T - T^*)^2$$

$$+ \theta_1^2 S^2 + (1 + a_1)\theta_2^2 I_1^2 + \theta_3^2 I_2^2 + \theta_4^2 T^2$$

$$\le -d[(S - S^*)^2 + (I_1 - I_1^*)^2 + (I_2 - I_2^*)^2 + (T - T^*)^2]$$

$$+ \theta_1^2 S^2 + (1 + a_1)\theta_2^2 I_1^2 + \theta_3^2 I_2^2 + \theta_4^2 T^2$$

$$\therefore dV_3 \leq \{- d[(S - S^*)^2 + (I_1 - I_1^*)^2 + (I_2 - I_2^*)^2 + (T - T^*)^2]$$

$$+ \theta^2\left(\frac{\Lambda}{\mu - \eta}\right)^2\}dt + 2(S - S^* + I_1 - I_1^*)S\theta_1 dB_1(t)$$

$$+ 2[S - S^* + (1 + a_1)(I_1 - I_1^*)]\theta_2 I_1 dB_2(t) + 2a_2(I_2 - I_2^*)\theta_3 I_2 dB_3(t)$$

$$+ 2(T - T^*)\theta_4 T dB_4(t)$$

Integrating it from 0 to t gives,

$$V_3(t) - V_3(0) \leq -\tilde{d}\int_0^t [(S(s) - S^*)^2 + (I_1(s) - I_1^*)^2 + (I_2(s) - I_2^*)^2 + (T(s) - T^*)^2]ds$$

$$+ \theta^2\left(\frac{\Lambda}{\mu - \eta}\right)^2 t + 2\theta_1\int_0^t S(s)(S(s) - S^* + I_1(s) - I_1^*)dB_1(s)$$

$$+ 2\theta_2\int_0^t I_1(s)[(S(s) - S^*) + (1 + a_1)(I_1(s) - I_1^*)]dB_2(s)$$

$$+ 2a_2\theta_3\int_0^t I_2(s)(I_2(s) - I_2^*)dB_3(s) + 2\theta_4\int_0^t T(s)(T(s) - T^*)dB_4(s)$$

Let

$$M_1(t) = \int_0^t S(s)(S(s) - S^* + I_1(s) - I_1^*)dB_1(s)$$

$$M_2(t) = \int_0^t I_1(s)[(S(s) - S^*) + (1 + a_1)(I_1(s) - I_1^*)]dB_2(s)$$

$$M_3(t) = \int_0^t I_3(s)(I_2(s) - I_2^*)dB_3(s)$$

$$M_4(t) = \int_0^t T(s)(T(s) - T^*)dB_4(s)$$

Clearly $M_i(t)$, $(i = 1, 2, 3, 4)$ are continuous, local martingale and also $M_i(0) = 0$ for $i = 1, 2, 3, 4$.

It is easy to check that

$$\lim_{t\to\infty} \frac{M_i(t)}{t} = 0 \text{ a.s. for } i = 1, 2, 3, 4.$$

Thus

$$\frac{V_3(t) - V_3(0)}{t} \le -\frac{\widetilde{d}}{t}\int_0^t [(S(s) - S^*)^2 + (I_1(s) - I_1^*)^2 + (T(s) - T^*)^2]ds$$

$$+ \theta^2\left(\frac{\Lambda}{\mu - \eta}\right)^2 + 2\theta_1 \frac{M_1(t)}{t} + 2\theta_2 \frac{M_2(t)}{t} + 2a_3\theta_3 \frac{M_3(t)}{t} + 2\theta_4 \frac{M_4(t)}{t}$$

It therefore follows from (4.1) that

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t [(S(s) - S^*)^2 + (I_1(s) - I_1^*)^2 + (I_2(s) - I_2^*)^2 + (T(s) - T^*)^2)]ds$$

$$\le \frac{\theta^2}{\widetilde{d}}\left(\frac{\Lambda}{\mu - \eta}\right)^2 \text{ a.s.}$$

This completes the proof of Theorem 4.1.

Remark 4.2. Theorem 4.1 shows that, if $\widetilde{R}^S > 1$, the solution of the system (1.3) oscillates around the endemic equilibrium E^*, for a long time while the intensity of the white noise is weak.

The parameter values used in the simulation of the stochastic model (1.3) are given in the following tables.

Table 1. Existence for E_0.

Parameter	Values	Source
Λ	0.3	[18]
μ	0.2	Assumed
α	0.2	[18]
β_1	0.03	[18]
β_2	0.04	[18]
σ	0.07	[18]
ρ	0.09	[18]
η	0.05	[18]

Table 2. Existence for E^*.

Parameter	Values	Source
Λ	0.1	[18]
μ	0.1	[18]
α	0.2	Assumed
β_1	0.1	Assumed
β_2	0.8	[18]
σ	0.007	[18]
ρ	0.009	[18]
η	0.05	[18]

5. Numerical Simulation

In this section, we present some examples to illustrate the obtained theoretical results and give a brief discussion. For the numerical simulation, we use Milstein's higher order method in [7] to obtain the following discretization of system (1.3).

$$S_{k+1} = S_k + (\Lambda - \beta_1 S_k I_{1,k} - \beta_2 S_k I_{2,k} - \mu S_k)\Delta t + \theta_1 S_k \sqrt{\Delta t}\xi_{1,k} + \frac{\theta_1^2}{2} S_k^2 \Delta t(\xi_{1,k}^2 - 1)$$

$$I_{1,k+1} = I_{1,k} + (\beta_1 S_k I_{1,k} + \beta_2 S_k I_{2,k} + \eta(I_{1,k} + I_{2,k}) - (\alpha + \mu) I_{1,k})\Delta t + \theta_2 I_{1,k}\sqrt{\Delta t}\xi_{2,k}$$

$$+ \frac{\theta_2^2}{2} I_{1,k}^2 \Delta t(\xi_{2,k}^2 - 1)$$

$$I_{2,k+1} = I_{2,k} + (\alpha I_{1,k} - (\rho + \sigma + \mu) I_{2,k})\Delta t + \theta_3 I_{2,k}\sqrt{\Delta t}\xi_{3,k} + \frac{\theta_3^2}{2} I_{2,k}^2 \Delta t(\xi_{3,k}^2 - 1)$$

$$T_{k+1} = T_k + (\rho I_{2,k} - (\gamma + \mu) T_k)\Delta t + \theta_4 T_k \sqrt{\Delta t}\,\xi_{4,k} + \frac{\theta_4^2}{2} T_k^2 \Delta t(\xi_{4,k}^2 - 1)$$

where the increment $\Delta t > 0$, $\theta_i^2 > 0$ denote the intensities of white noise, $\xi_{i,k}\,(i = 1, 2, 3, 4)$ are the Gaussian random variables which follow the

distribution $N(0, 1)$. Choosing different values of parameters, we give its simulation with initial value $(S(0), I_1(0), I_2(0), T(0)) = (1, 1, 0.5, 0.4)$.

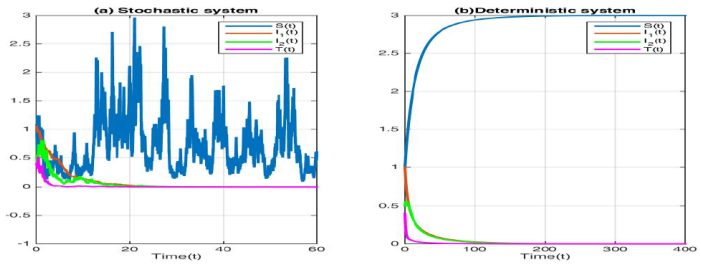

Figure 1. Simulation of trajectories of $S(t)$, $I_1(t)$, $I_2(t)$ and $T(t)$ of system (1.3) and system (1.2) with $\tilde{R}^S < 1$.

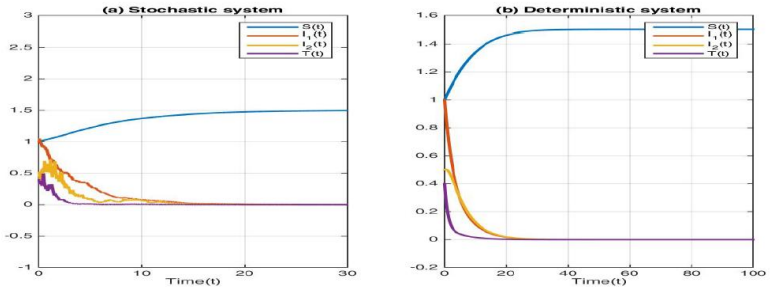

Figure 2. Simulation of trajectories of $S(t)$, $I_1(t)$, $I_2(t)$ and $T(t)$ of system (1.3) and system (1.2) with $\tilde{R}^S < 1$ and $\theta_1 = 0$.

Example 5.1. We have taken the parameter values $\Lambda = 0.3$, $\mu = 0.2$, $\alpha = 0.2$, $\beta_1 = 0.03$, $\beta_2 = 0.04$, $\sigma = 0.07$, $\rho = 0.09$, $\eta = 0.05$, as in Table 1, with $\gamma = 0.7$, white noise intensities $\theta_1 = 0.8$, $\theta_2 = 0.1$, $\theta_3 = 0.5$, $\theta_4 = 0.5$ such that $R_0 = 0.2431 < 1$, $\tilde{R}^S = 0.3060 < 1$ and $0.369 = (\rho + \sigma + \mu) - \frac{\rho^2}{\gamma + \mu}$ $> \theta_3^2 = 0.25$, $0.9 = \gamma + \mu > \theta_4^2 = 0.25$. Thus the conditions in Theorem 3.1 are satisfied. Therefore by Theorem 3.1, the paths of SDE system (1.3) oscillates around the paths of ODE system (1.2). See Figure 1(a).

Moreover as mentioned in Remark 3.3, if $\theta_1 = 0$, then the solution of system (1.3) is asymptotically stable in the large. See Figure 2(a).

From Figures 1 and 2, it is clear that the solution of both systems converges to the disease-free equilibrium $E_0 = (1.5, 0, 0, 0)$. That is, both infections $I_1(t)$ and $I_2(t)$ tend to zero very quickly and the disease will die out in the population. Also the susceptible population $S(t)$ will approach $\frac{\Lambda}{\mu} = 1.5$ in time average.

Example 5.2. Now we give a numerical simulation to explain Theorem 4.1 using the parameter values given in Table 2. $\Lambda = 0.1$, $\mu = 0.1$, $\alpha = 0.2$, $\gamma = 0.7$, $\beta_1 = 0.1$, $\beta_2 = 0.8$, $\sigma = 0.007$, $\rho = 0.009$, $\eta = 0.05$, $\theta_1 = 0.9$, $\theta_2 = 0.8$, $\theta_3 = 0.8$, $\theta_4 = 0.8$ with $\widetilde{R}^S = 12.7642 > 1$. In this case, the disease will persists and the trajectories of stochastic system (1.3) oscillate around the trajectories of deterministic system (1.2) which supports Theorem 4.1. See Figure 3. We noticed that $R_0 > 1$ and the solution of deterministic system (1.2) converges to the positive endemic equilibrium $E^* = (0.1107, 0.5429, 0.99361, 0.0085)$.

From Figure 4, it is evident that when the intensity of noises decreases, the fluctuation become small.

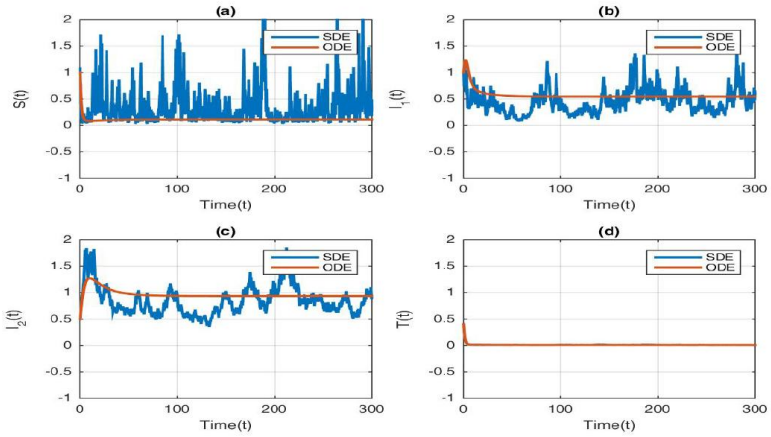

Figure 3. Simulation of trajectories of $S(t)$, $I_1(t)$, $I_2(t)$ and $T(t)$ of system (1.3) and system (1.2).

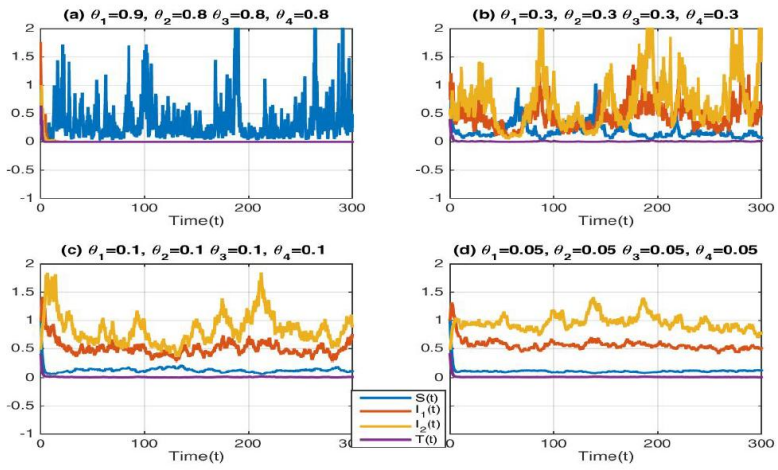

Figure 4. Simulation of trajectories of $S(t)$, $I_1(t)$, $I_2(t)$ and $T(t)$ of system (1.3) for different white noise intensities.

6. Conclusion

In this work we proposed an HIV/AIDS epidemic model with vertical transmission and analyzed it with stochastic disturbances of white noise type. The importance of study may be divided into two categories. First, it contained the existence and uniqueness of positive global solution of the stochastic system. Second, it examines the stability of the stochastic system by establishing a threshold parameter $\widetilde{R}^S$. When $\widetilde{R}^S < 1$, the solution of stochastic system (1.3) oscillating around the disease free equilibrium of deterministic system (1.2). Furthermore we showed that when $\widetilde{R}^S > 1$ with increasing the noise intensities, the solution of stochastic system (1.3) is oscillating substantially around the endemic equilibrium of deterministic system (1.2). From our analytical and numerical results, we conclude that the main factor that affects the stability of the stochastic model (1.3) is the intensities of white noise. Finally, numerical simulations are used to validate our derived conclusions.

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