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A NOTE ON COMPATIBLE NORM OF CIRCULANT FUZZY MATRICES

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Abstract

Circulant matrix is a square matrix whose rows are obtained by cyclically rotating by its first row. The purpose of this paper is to introduce some algebraic properties of circulant fuzzy matrices on compatible norm CFM_c. Some idea of reflexive, symmetric, transitive, idempotent determinant and adjoint of circulant fuzzy matrices CFM_c are discussed. A new type of semiring properties has been studied.

1. Introduction

The concept of fuzzy set was introduced by Zadeh [11]. In 1995 Ragab M. Z. and Eman E. G. [9] introduced the determinant and adjoint of square fuzzy matrix. Nagoorgani A. and Kalyani G. [6] introduced the bi normed sequences in fuzzy matrices. Nagoorgani A. and Manikandan A. R. [7] defined fuzzy det-norm matrices. Some concept of matrix theory and applications in fuzzy matrices, Meenakshi A.R [3]. Dennis and Bernstein [1] introduced compatible norm in matrix mathematics theory, facts and formulas. Monoranjan Bhowmik, Madhumangal Pal and Anita Pal [4] presented some properties on circulant triangular fuzzy number matrices. Nagoorgani A. and Pappa A. [8]

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defined determinant for non-square fuzzy matrices with compatible norm. Muruganantham P. Nagoorgani A. and Pappa A. [5] introduced some important results on adjoint of non-square fuzzy matrices with compatible norm. Loganathan C. and Puspalatha V. [2] obtained the concept of circulant inter valued fuzzy matrices. Sriram S. and Murugadas P. [10] introduced semiring of fuzzy matrices.

In this paper our main intension is to introduced a new concept of circulant fuzzy matrices on compatible norm CFM_c. In Section 2 generalized circulant matrix and we recall the definition explained. In Section 3 determinant and adjoint of CFM_c and some of its properties are presented. In Section 4 some binary operations on CFM_c are given. In Section 5 semiring properties on CFM_c explained and some important theorems are proved.

2. Preliminaries and Definitions

In this section, we prove generalized circulant matrices and basic definitions have been studied.

Definition 2.1. Generalized circulant matrix

Converting Non-square (rectangular) matrix to a circulant matrix.

In the rectangular matrix, the elements are arranged in rows and columns $(m \times n)$. The arrangement of row (column) is in the $m \times n$ order and the same order which is less than the other eliminated. So that it would become of contains elements of same order is square matrix $n \times n\,(m \times m)$ in which all row (column) composed of the same elements and each row (column) vector is rotated one element to the right relative to the preceding row (column) vector is called circulant matrix.

Remark 2.1. The first row of the circulant matrices play important role.

Definition 2.2. An $m \times n$ matrix $A = [a_{ij}]$ whose components are in the unit interval [0, 1] is called fuzzy matrix.

Definition 2.3. The determinant $|A|$ of a $n \times n$ fuzzy matrix A is defined as follows: $|A| = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)}$. Where S_n denotes the symmetric group of all permutations of the indices $(1, 2, \ldots, n)$.

Definition 2.4 (minor). Let $A = [a_{ij}]$ be square of order of $n \times n$ over $\mathcal{F}_{nn}$. The minor of an element a_{ij} in $\det|A|$ is the order $(n-1) \times (n-1)$ which is obtained by deleting i-th row and the j-th column from $A = (a_{ij})$ denoted by M_{ij}.

Definition 2.5. (Cofactor). Let $A = (a_{ij})$ be square of order of $n \times n$ over $\mathcal{F}_{nn}$. The cofactor of an element a_{ij} in A is denoted by A_{ij} and is defined as $A_{ij} = (1)^{i+j} M_{ij}$.

Definition 2.6 (Adjoint). The adjoint matrix of an $n \times n$ fuzzy matrix over $\mathcal{F}_{nn} A$ is denoted by the (i, j^{th}) entry adj A and is defined as

$$b_{ij} = |A_{ji}|,$$

where $|A_{ji}|$ is the determinant of the $(n-1) \times (n-1)$ fuzzy matrix formed by deleting row j and column i from A and $B = adjA$.

Remark 2.2. We can rewrite b_{ij} of $adj\ A = B = [b_{ij}]$ as follows

$$b_{ij} = \sum_{\sigma \in S_{n_j n_i}} \prod_{t \in n_j n_i} a_{t\sigma(t)}, \text{ where } n_j = \{1, 2, \ldots, n\} \setminus \{j\}$$

and $S_{n_j n_i}$ is the set of all permutations of set n_j over the set n_i.

Definition 2.7 (**Compatible fuzzy matrix**) FM_c. Compatible fuzzy matrices which can be multiplied for this to be possible, the number of columns in the first fuzzy matrix must be equal to the number of rows in the second fuzzy matrix.

Definition 2.8 (Compatible Norm $\|\cdot\|_c$). Let $\mathcal{F}_{nn}$ is the set of all $n \times n$ over $\mathcal{F} = [0, 1]$. Define the norms $\|\cdot\|_c$.

3. Determinant of Circulant Fuzzy Matrix

In this section, circulant fuzzy matrix, some algebraic properties, determinant and adjoint of circulant fuzzy matrix, some result of CFM_c over $\mathcal{F}_{nn}$ are studied.

Definition 3.1. Circulant Matrix. An $n \times n$ circulant matrix has the form

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ a_n & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ \vdots & \vdots & \dots & \dots & \vdots & \vdots \\ a_3 & a_4 & a_5 & \dots & a_1 & a_2 \\ a_2 & a_3 & a_4 & \dots & a_n & a_1 \end{bmatrix}.$$

Definition 3.2 Circulant Fuzzy Matrix (CFM). For any given $A_1, A_2, A_3, \dots, A_n \in \mathcal{F}_{nn}$ be the circulant matrix $A = (a_{ij})_{n \times n}$ is defined as $(a_{ij}) = a_{j-1(\text{mod } n)}$.

A circulant fuzzy matrix is the form

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ a_n & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ \vdots & \vdots & \dots & \dots & \vdots & \vdots \\ a_3 & a_4 & a_5 & \dots & a_1 & a_2 \\ a_2 & a_3 & a_4 & \dots & a_n & a_1 \end{bmatrix}$$

with entries in $[0, 1]$.

Example 3.3.

$$A = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.2 & 0.3 & 0.5 \\ 0.5 & 0.2 & 0.3 \end{bmatrix}.$$

Remark 3.1. Here all the diagonal elements are equal.

Definition 3.4. A fuzzy matrix A is said to be circulant fuzzy matrix of order $n \times n$ if all the elements of A can be determined completed by its first row. Suppose the first row of $|A|$ is $(a_1, a_2, \dots, a_n)$. The determinant of a

CFM of order $n \times n$ is defined by $|A|$ and is defined as then $|A| = \sum_{\sigma \in s_n} \operatorname{sgn} \sigma \pi_{i=1}^{n} a_{i\sigma i}$ is the fuzzy matrix and s_n denotes the symmetric group of permutation of the indices $(1, 2, \ldots, n)$ and $\sigma = 1$.

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix} \text{ is even or odd respectively.}$$

The computation $\det(A)$ involves several products CFM. Since A is circulant fuzzy matrix the value $a_{ij} = a_{1(n-i+j+1)}$ with $a_{1(n+k)} = a_{1k}$.

Remark 3.2. It is noted that the matrix CFM A is circulant if and only if $a_{ij} = a_{(k \oplus i)(k \oplus j)}$ for every $i, j, k \in \{1, 2, \ldots, n\}$ where $\oplus$ sum modulo n. This supply that the elements of the diagonal are all equals.

Remark 3.3. For a matrix CFM A we notice that $a_{in} = a_{(i \oplus 1)1}$ and $a_{in} = a_{1(j \oplus 1)}$ for every $i, j \in \{1, 2, \ldots, n\}$.

Remark 3.4. For a matrix CFM A we notice that $a_{(i \oplus (n-1))j} = a_{i(j \oplus 1)}$ for every $i, j\{1, 2, \ldots, n\}$.

Theorem 3.5. *If* $\| A \|_c = [a_{ij}]_{n \times n}$ *be a circulant fuzzy matrix. Then determinant of* $\| A \|_c$ *is the largest element in* $\| A \|_c$.

Proof. Let $[a_{1m}] \geq [a_{1i}]$ for every $i \in \{1, 2, \ldots, n\}$ i.e. $[a_{1m}]$ is the largest element of A. Then by definition of $\| A \|_c$, we have

$$\| A \|_c = \sum_{\sigma \in s_n} \operatorname{sgn} \sigma \pi_{i=1}^{n} a_{i\sigma i}$$

$$= \sum_{\sigma \in s_n} \operatorname{sgn} \sigma \pi_{i=1}^{n} a_{i\sigma i} = \pi_{i=1}^{n} [a_{i\pi(i)}] \text{ for some } \pi \in s_n$$

$$= [a_{1\pi(1)}][a_{2\pi(2)}] \ldots [a_{n\pi(n)}]$$

Let $\pi(1) = 1$ since A is circulant, we get

$$[a_{1m}] = [a_{2(m \oplus 1)}] = [a_{3(m \oplus 2)}] = \ldots = [a_{n(m \oplus (n-1))}]$$

Let the permutation π defined as

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ m & m \oplus 1 & m \oplus 2 & \dots & m \oplus (n-1) \end{pmatrix}$$

Therefore $\| A \|_c = [a_{1m}] \cdot [a_{2(m \oplus 1)}] \cdot [a_{3(m \oplus 2)}] \dots [a_{n(m \oplus (n-1))}]$

$\| A \|_c = [a_{1m}]$. From now $\| A \|_c$ is largest element in A.

Example 3.6.

$$\| A \|_c = \begin{bmatrix} 0.3 & 0.9 & 0.7 \\ 0.7 & 0.3 & 0.9 \\ 0.9 & 0.7 & 0.3 \end{bmatrix} \text{ then } \| A \|_c = 0.9.$$

Hence $\| A \|_c$ is the largest element in A.

Algebraic properties of circulant fuzzy matrices with compatible norm 3.7.

For the circulant fuzzy matrix compatible norm is taken into consideration then the following properties will hold.

Let A, B, C circulant fuzzy matrices then CFM_c over $\mathcal{F}_{nn}$

(i) $\| A + B \|_c = \| B + A \|_c$

(ii) $\| A \|_c = \| A \|_c^T$

(iii) $\| A + (B + C) \|_c = \| (A + B) + C \|_c$

(iv) $\| A + B \|_c = \| A \|_c + \| B \|_c$

(v) $\| A + B \|_c^T = \| A \|_c^T + \| B \|_c^T$

(vi) $\| A + A \|_c = \| A \|_c$

(vii) $\| A + O \|_c = \| A \|_c$

(viii) $\| A + J \|_c = \| J \|_c$

(ix) $\| \alpha A \|_c = \alpha \| A \|_c$ for any α in $[0, 1]$

(x) $\| \alpha A \|_c^T = \alpha \| A \|_c^T$ for any α in $[0, 1]$

(xi) $\| \alpha(A + B) \|_c = \alpha \| A \|_c + \alpha \| B \|_c$ for any α in $[0, 1]$

(xii) $\| (\alpha + \beta) A \|_c = \alpha \| A \|_c + \beta \| B \|_c$ for all $\alpha + \beta$ in $[0, 1]$

(xiii) $\alpha \| \beta A \|_c = \alpha\beta \| A \|_c$ for any α, β in $[0, 1]$.

Remark 3.5. (i) $A + B = \{\max(a_{ij}, b_{ij})\}$

(ii) $AB = \{\max\{\min(a_{ij}, b_{ij})\}\}$.

Definition 3.8. A circulant fuzzy matrix A in $\mathcal{F}_{nn}$ is called idempotent if $A^2 = A$ (or) $\| A \|_c^2 = \| A \|_c$ where $A = [a_{ij}]$.

Example 3.9. If

$$\| A \|_c = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.2 & 0.3 & 0.5 \\ 0.5 & 0.2 & 0.3 \end{bmatrix}$$

then $\| A \|_c = 0.5$

Note: From now $\| A \|_c$ is largest element in A

$$\| A \|_c^2 = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.2 & 0.3 & 0.5 \\ 0.5 & 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.2 & 0.3 & 0.5 \\ 0.5 & 0.2 & 0.2 \end{bmatrix}.$$

$$= \begin{bmatrix} 0.3 & 0.3 & 0.5 \\ 0.5 & 0.3 & 0.3 \\ 0.3 & 0.5 & 0.3 \end{bmatrix} = 0.5$$

Therefore $\| A \|_c^2 = \| A \|_c = 0.5$.

Theorem 3.10. *An $n \times n$ CFM$_c$ $\| A \|_c$ is circulant if and only if $\| AC_n \|_c = \| C_n A \|_c$ where $\| C_n \|_c$ is the permutation matrix*

$$\| C_n \|_c = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{bmatrix}.$$

Proof. Let $\| A \|_c$ be a CFM_c and $U = \| AC_n \|_c$ then $= u_{ij} = \sum_{h=1}^{n} a_{ih} c_{hj}$ (compatible).

In first row only $C_{1n} = 1$ and all the other elements are 0 therefore we get $u_{ij} = a_{1(j \oplus 1)}$

Let $V = \| C_n A \|_c$, then $v_{ij} = \sum_{h=1}^{n} a_{ih} c_{hj} = a_{(i \oplus (n-1) j)}$

By remark 3.4, $u_{ij} = v_{ij}$ for all $i, j \in \{1, 2, \dots, n\}$

Hence $\| AC_n \|_c = \| C_n A \|_c$. Therefore we get $\| A \|_c$ is CFM_c

The converse is straightforward.

Example 3.11. Let $\| A \|_c$ and $\| C \|_c$ are compatible of two CFM_c of order 3×3 where

$$\| A \|_c = \begin{bmatrix} 0.2 & 0.4 & 0.7 \\ 0.7 & 0.2 & 0.4 \\ 0.4 & 0.7 & 0.2 \end{bmatrix} \text{ and } \| C \|_c = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Then $\| CA \|_c = \begin{bmatrix} 0.4 & 0.7 & 0.2 \\ 0.2 & 0.4 & 0.7 \\ 0.7 & 0.2 & 0.4 \end{bmatrix}$ and

$$\| AC \|_c = \begin{bmatrix} 0.4 & 0.7 & 0.2 \\ 0.2 & 0.4 & 0.7 \\ 0.7 & 0.2 & 0.4 \end{bmatrix}$$

Therefore $\| CA \|_c = \| AC \|_c$.

Theorem 3.12. *For the circulant fuzzy matrix* CFM_c *then* $\| A \|_c$ *and* $\| B \|_c$

(i) $\| A + B \|_c$ *is a* CFM_c, (ii) $\| A' \|_c$ *is a* CFM_c, (iii) $\| AB \|_c$ *is also a* CFM_c.

Proof. (i) Proof is straightforward

(ii) Since A is CFM_c then $\| A \|_c$ commutes with $\| C_n \|_c$.

Therefore, we get $\| AC_n \|_c = \| C_n A \|_c$ transposing both sides, we get $\| C'_n A' \|_c = \| A' C'_n \|_c$

Pre multiply by C_n, we get $\| C_n C'_n A' \|_c = \| C_n A' C'_n \|_c = \| A' \|_c = \| C_n A' C'_n \|_c$ [since $C'_n C_n = I = C_n C'_n$]

Post multiply by C_n, we get $\| A' C'_n \|_c = \| C_n A' C'_n C_n \|_c = \| C_n A' \|_c$

Hence $\| A' C_n \|_c = \| C_n A' \|_c$

So, $\| A' \|_c$ is circulant fizzy matrix CFM_c.

(iii) Since A and B are CFM_c each of $\| A \|_c$ and $\| B \|_c$ commutes with $\| C_n \|_c$.

From now $\| AB \|_c$ commutes with $\| C_n \|_c$ by Remark 3.4 and theorem 3.9, we get $\| AB \|_c$ is CFM_c.

Similarly, $\| A^k \|_c$ and A^k is also CFM_c.

Since A and A' are CFM_c by Remark 3.4 $\| AA' \|_c$ commutes with $\| C_n \|_c$.

Hence $\| AA' \|_c$ is CFM_c.

Remark 3.6. $A' = [a_{ji}]$ (the transpose of A).

Theorem 3.13. *If* $\| A \|_c$ *and* $\| B \|_c$ *are* CFM_c, *then* $\| AB \|_c = \| BA \|_c$.

Proof. Let $\| AB \|_c = U$ and u_{ij} for $i, j \in \{1, 2, \ldots, n\}$.

Let $\| BA \|_c = V$ and v_{ij} for $i, j \in \{1, 2, \ldots, n\}$.

Then both the u and v are circulant by Theorem 3.12 (iii) and their first rows are $[u_1, u_2, u_3, \ldots, u_n]$ and $[v_1, v_2, \ldots, u_n]$ respectively.

Then k^{th} element of the first of u is

$$U_k = \left[\sum_{p=1}^{k}(a_p b_{(k-p+1)})\right] + \left[\sum_{p=k+1}^{n}(a_p b_{(n-p+k+1)})\right]$$

$$= (a_1 b_k) + (a_2 b_{(k-1)}) + \ldots + (a_{(k-1)} b_2) + (a_k b_1) + (a_{(k+1)} b_n)$$

$$+ (a_{(k+2)} b_{(k-1)}) + \ldots + (a_{(n-1)} b_{(k+1)}) + (a_n b_{(k+1)})$$

K^{th} element of first row of v is

$$V_k = \left[\sum_{p=1}^{k}(b_p a_{(k-p+1)})\right] + \left[\sum_{p=k+1}^{n}(b_p a_{(n-p+k+1)})\right]$$

$$= (b_1 a_k) + (b_2 a_{(k-1)}) + \ldots + (b_{(k-1)} a_2) + (b_k a_1) + (b_{(k+1)} a_n)$$

$$+ (b_{(k+2)} a_{(k-1)}) + \ldots + (b_{(n-1)} a_{(k+1)}) + (b_n a_{(k+1)})$$

From now, we get $U_k = V_k$ i.e. $u_{ij} = v_{ij}$

i.e. $U = V$ therefore we have $\| AB \|_c = \| BA \|_c$.

Example 3.14. $\| A \|_c = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.2 & 0.3 & 0.5 \\ 0.5 & 0.2 & 0.3 \end{bmatrix}$ and $\| B \|_c = \begin{bmatrix} 0.4 & 0.2 & 0.7 \\ 0.7 & 0.4 & 0.2 \\ 0.2 & 0.7 & 0.4 \end{bmatrix}$

Then $\| AB \|_c = \begin{bmatrix} 0.5 & 0.4 & 0.3 \\ 0.3 & 0.5 & 0.4 \\ 0.4 & 0.3 & 0.5 \end{bmatrix} = 0.5$

Then $\| BA \|_c = \begin{bmatrix} 0.5 & 0.4 & 0.3 \\ 0.3 & 0.5 & 0.4 \\ 0.4 & 0.3 & 0.5 \end{bmatrix} = 0.5$

$\| AB \|_c = \| BA \|_c$.

Remark 3.7. The determinant value is unchanged.

Remark 3.8. Commutative properties satisfied.

Theorem 3.15. *If a circulant fuzzy matrices* $\| A \|_c$ *is circulant then* $\| EA \|_c$ *is symmetric, when* $\| E \|_c$ *is a permutation matrix of unit circulant fuzzy matrices.*

$$\| E \|_c = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & \dots & 0 & 0 \end{bmatrix}.$$

Proof. Let $\| G \|_c = \| EA \|_c$ then $g_{ij} = \sum_{h=1}^{n} e_{ih} a_{hj}$ (compatible)

$$= \sum_{h=1}^{n} [e_{ik}][a_{hj}] \text{ for all } i, j \in \{1, 2, \dots, n\}$$

Now E is a permutation matrix of unit circulant fuzzy matrix and only the elements $e_{1n}, e_{2(n-1)}, e_{3(n-2)} \dots e_{n1}$ are 1 and are 0 then elements are 0.

Then $g_{ij} = [a_{(n-i+1)j}]$ since $\| A \|_c$ is CFM

We get $g_{ij} = [a_{(n-i+1)j}] = a_{((n-i+1)\oplus k)((k\oplus j))}$ for all $i, j, k \in \{1, 2, \dots, n\}$

When $k = 1$, then

$$g_{ij} = a_{(n-i+1)j} = a_{((n-i+1)\oplus k)(i\oplus j)} = a_{(n\oplus 1)(i\oplus j)} = a_{1(i\oplus j)}$$

and $g_{ij} = a_{(n-j+1)i} = a_{(n-j+1)\oplus k(k\oplus i)}$ for all $i, j, k \in \{1, 2, \dots, n\}$

When $k = j$, then

$$g_{ji} = a_{(n-j+1)i} = a_{(n-j+1\oplus j)(j\oplus 1)} = a_{(n\oplus 1)(i\oplus j)} = a_{1(i\oplus j)}$$

Therefore, we get $g_{ij} = g_{ji}$

From now G is symmetric.

Example 3.16. $\| E \|_c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $\| A \|_c = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.2 & 0.3 & 0.5 \\ 0.5 & 0.2 & 0.3 \end{bmatrix}$

$\| EA \|_c = \| G \|_c = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \end{bmatrix}$

$g_{ij} = g_{ji}$. Hence G is symmetric.

Theorem 3.17. *If* $\| A \|_c$ *be circulant fuzzy matrix, then* $\| adjA \|_c$ *is circulant.*

Proof. By definition 2.6, we have to prove co factor of the elements $a_{i(j\oplus 1)}$ and $a_{(i\oplus(n-1))j}$ are same by remark 3.4, we have $a_{i(j\oplus 1)}$ and $a_{i(j\oplus 1)} = a_{(i\oplus(n-1))j}$. So the minor of $a_{i(j\oplus 1)}$ and $a_{(i\oplus(n-1))j}$ will be same.

Co-factor of $a_{i(j\oplus 1)} = (1)^{i+(j\oplus 1)} \sum_{\sigma \in s_n} \pi^{n}_{k=1, k\neq i,\ k\neq j\oplus 1} [a_{k\sigma(k)}]$

Co-factor of $a_{(i\oplus(n-1))j} = (1)^{i\oplus(n-1)+j} \sum_{\sigma \in s_n} \pi^{n}_{k=1, k\neq i,\ k\neq(i\oplus(n-1))} [a_{k\sigma(k)}]$

Now the sign of $(1)^{i+(j\oplus 1)}$ = the sign of $(1)^{i\oplus(n-1)+j}$ (since n is fixed)

So, the co-factor of $a_{i(j\oplus 1)}$ and $a_{(i\oplus(n-1))j}$ are same.

From now $\| adj\ A \|_c$ is circulant.

Example 3.18. $\| A \|_c = \begin{bmatrix} 0.3 & 0.5 & 0.6 \\ 0.6 & 0.3 & 0.5 \\ 0.5 & 0.6 & 0.3 \end{bmatrix}$

$\| adj\ A \|_c = \| b_{ij} \|$

$b_{11} = 0.5 \quad b_{12} = 0.5 \quad b_{13} = 0.6$

$b_{21} = 0.6 \quad b_{22} = 0.5 \quad b_{23} = 0.5$

$b_{31} = 0.5 \quad b_{32} = 0.6 \quad b_{33} = 0.5$

$$\| b_{ij} \| = \| adj\ A \|_c = \begin{bmatrix} 0.5 & 0.5 & 0.6 \\ 0.6 & 0.5 & 0.5 \\ 0.5 & 0.6 & 0.5 \end{bmatrix}^T$$

$$\| b_{ij} \| = \begin{bmatrix} 0.5 & 0.6 & 0.5 \\ 0.5 & 0.5 & 0.6 \\ 0.6 & 0.5 & 0.5 \end{bmatrix} = \| A_{ji} \|_c.$$

Theorem 3.19. *For a circulant fuzzy matrix* $\| A \|_c$. *We have* $\| A(adj\ A) \|_c$ *is weakly reflexive.*

Proof. Let $\| A \|_c$ be a circulant fuzzy matrix, let $\| A \|_c = (a_{ij})_{n \times n}$ $a_{1l} \geq a_{1i}$ for every $i \in \{1, 2, \ldots, n\}$.

Let $C = \| A(adj\ A) \|_c$. Then C is circulant, since $\| A \|_c$ and $\| adjA \|_c$ are circulant. Now, we have $C_{11} = \sum_{k=1}^{n} [a_{1k}][A_{1k}] \geq C_{1l} | A_{1l} |$ for every $l \in \{1, 2, \ldots, n\}$, now $| A_{1l} | = \sum_{\sigma \in s_n 1nl} \operatorname{sgn} \sigma\pi = a_{t\sigma(t)l} = [a_{2\pi(2)l}][a_{3\pi(3)l}]$ $[a_{4\pi(4)l}] \ldots [a_{n\pi(n)l}]$ for some $\pi \in S_n$.

Since $\| A \|_c$ is circulant, we get

$$a_{1l} = a_{2(l \oplus 1)} = a_{3(l \oplus 2)} = \ldots = a_{n(l \oplus (n-1))}$$

Suppose $\pi \in s_{nl}$ be defined as

$$\pi = \begin{bmatrix} 2 & 3 & 4 & n \\ l \oplus 1 & l \oplus 2 & l \oplus 3 & l \oplus (n-1) \end{bmatrix}$$

Then $| A_{1l} | = [a_{2(l \oplus 1)}][a_{3(l \oplus 2)}][a_{n(l \oplus (n-1))}] = a_{1l}$. Therefore $C_{11} \geq a_{1l}$.

But $[a_{1l}] \geq [a_{1i}]$ for every $i \in \{1, 2, \ldots, n\}$. Then $C_{11} \geq C_{1i}$ for every $i \in \{1, 2, \ldots, n\}$. Again since C is circulant, the elements of its diagonal are all equal.

From now $[C_{ii}] \geq [C_{ij}]$ for every $i, j \in \{1, 2, \ldots, n\}$.

Therefore $C = \| A(adj\ A) \|_c$ is weakly reflexive.

Theorem 3.20. *If* $\| A \|_c = (a_{ij})_{n \times n}$ *be a* CFM_c, *then* $\| A(adj\ A) \|_c$ *is transitive.*

Proof. Let $C = \| A(adj\ A) \|_c$, then $C_{ij} = \sum_{k=1}^{n} [a_{ik}] \cdot [A_{jk}] = [a_{it}][A_{jt}]$

$[C_{ij}]^2 = \sum_{k=1}^{n} [C_{is}][C_{si}]$ (compatible)

$$= \sum_{s=1}^{n} [\sum_{p=1}^{n} [a_{ip}][A_{sp}] \sum_{q=1}^{n} [a_{sq}][A_{jq}]]$$

$$= \sum_{s=1}^{n} [a_{ih}][A_{sh}][a_{sk}][A_{jk}]$$

$$\leq [a_{ih}][A_{jk}] \leq [a_{it}][A_{jt}]$$

From now $\| A(adj\ A) \|^2 \leq \| A(adj\ A) \|_c$.

Therefore $\| A(adj\ A) \|_c$ is transitive.

Theorem 3.21. *If* $\| A \|_c = (a_{ij})_{n \times n}$ *be a* CFM_c, *then* $\| A(adj\ A) \|_c$ *is idempotent.*

Proof. Let $C = \| A(adj\ A) \|_c$ by Theorem 3.20 $[C_{ij}]^2 \leq [C_{ij}]$ for every $i, j \in \{1, 2, \ldots, n\}$.

But $[C_{ij}]^2 = \sum_{k=1}^{n} [C_{ik}][C_{kj}] \geq [C_{ii}][C_{ij}] = [C_{ij}]$

From now we get $[C_{ij}]^2 = [C_{ij}]$

Therefore $\| A(adj\ A) \|_c$ is idempotent.

Example 3.22. Let $\| A \|_c = \begin{bmatrix} 0.3 & 0.5 & 0.6 \\ 0.6 & 0.3 & 0.5 \\ 0.5 & 0.6 & 0.3 \end{bmatrix}$ then

$$\| adj\ A \|_c = \begin{bmatrix} 0.5 & 0.6 & 0.5 \\ 0.5 & 0.5 & 0.6 \\ 0.6 & 0.5 & 0.5 \end{bmatrix}$$

Let $C = \| A(adj\ A) \|_c = \begin{bmatrix} 0.6 & 0.5 & 0.5 \\ 0.5 & 0.6 & 0.5 \\ 0.5 & 0.5 & 0.6 \end{bmatrix}$

Now $[C_{ii}] \geq [C_{ij}]$

From now $\| A(adj\ A) \|_c$ is weakly reflexive.

$$\| A(adj\ A) \|_c^2 = \begin{bmatrix} 0.6 & 0.5 & 0.5 \\ 0.5 & 0.6 & 0.5 \\ 0.5 & 0.5 & 0.6 \end{bmatrix}$$

From now $\| A(adj\ A) \|_c^2 \leq \| A(adj\ A) \|_c$. Therefore $\| A(adj\ A) \|_c$ is transitive. From now $[C_{ij}]^2 = [C_{ij}]$ therefore $C = \| A(adj\ A) \|_c$ is idempotent.

4. Operators on Circulant Fuzzy Matrices with Compatible Norm

In this section, some binary operators viz. $\cap, \cup, (\wedge, \vee)$ are defined and explained. De Morgan's law properties satisfied.

Definition 4.1. Let $\| A \|_c = (a_{ij})_{n \times n}$ and $\| B \|_c = (b_{ij})_{n \times n}$ are CFM_c then over $\mathcal{F}_{nn}$

$$\| A \|_c \cup \| B \|_c = [a_{ij}] \cup [b_{ij}] = \max(a_{ij}, b_{ij}).$$

Theorem 4.2. *If* $\| A \|_c$ *and* $\| B \|_c$ *are two circulant fuzzy matrix, then* $\| A \|_c \cup \| B \|_c$ *is also circulant fuzzy matrix.*

Proof. Proof is straightforward.

Definition 4.3. The $\cap$ operation is similar to $\cup$ operation.

Let $\| A \|_c = [a_{ij}]_{n \times n}$ and $\| B \|_c = [b_{ij}]_{n \times n}$ are two circulant fuzzy matrix, then $\| A \|_c \cap \| B \|_c = [a_{ij}] \cap [b_{ij}] = \min(a_{ij}, b_{ij})$.

Theorem 4.4. *If* $\| A \|_c$ *and* $\| B \|_c$ *are two circulant fuzzy matrix, then* $\| A \|_c \cap \| B \|_c$ *is also circulant fuzzy matrix.*

Proof. Proof is straightforward.

Definition 4.5. The complement of CFM_c $\| A \|_c = [a_{ij}]_{n \times n}$ is defined $\| (A)^c \|_c = [1 - a_{ij}]_{n \times n}$.

Definition 4.6. An CFM_c is called **self-complement** $\| (A^c)^c \|_c = \| A \|_c$.

Theorem 4.7. *If a* CFM_c $\| A \|_c = [a_{ij}]_{n \times n}$ *is self-complement then* $\| (A^c)^c \|_c = \| A \|_c$.

Proof. Let $B = \| (A)^c \|_c$. Then $[b_{ij}] = [1 - a_{ij}]$. If $D = \| (B)^c \|_c$ $= \| (A^c)^c \|_c$, then $d_{ij} = 1 - b_{ij}$

$= [1 - (1 - a_{ij})]$

$= [a_{ij}] = \| A \|_C$

Therefore $\| (A^c)^c \|_c = \| A \|_c$.

Theorem 4.8. De Morgan's Law:

Let $\| A \|_c = [a_{ij}]_{n \times n}$ *and* $\| B \|_c = [b_{ij}]_{n \times n}$ *are two* CFM_c, *then over* $\mathcal{F}_{nn}$

(i) $\| ((A \cup B))^c \|_c = \| (A)^c \|_c \cap \| (B)^c \|_c$

(ii) $\| ((A \cap B))^c \|_c = \| (A)^c \|_c \cup \| (B)^c \|_c$.

Proof. (i) Let $P = \| A \cup B \|_c$ then $P_{ij} = [a_{ij} \cup b_{ij}]$

$Q = P^c$ then $Q_{ij} = R_{ij} = [1 - (a_{ij} \cup b_{ij})]$

Let $R = \| (A)^c \|_c \cap \| (B)^c \|_c$, then

$R_{ij} = [(1 - a_{ij}) \cap (1 - b_{ij})] = [1 - (a_{ij} \cup b_{ij})] = Q_{ij}$

From now $\| ((A \cup B))^c \|_c = \| (A)^c \|_c \cap \| (B)^c \|_c$

(ii) Proof is similar to (i).

Theorem 4.9. *If A, B, C are three CFM_c, then over $\mathcal{F}_{nn}$ more properties satisfied.*

(i) $\| ((A \cap B) \cap C) \|_c = \| A \cap (B \cap C) \|_c$

(ii) $\| (A \cap B) \cup C \|_c = \| A \cup (B \cup C) \|_c$

(iii) $\| A \cap (B \cup C) \|_c = \| (A \cap B) \cup (A \cap C) \|_c$

(iv) $\| A \cup (B \cap C) \|_c = \| (A \cup B) \cap (A \cup C) \|_c$

The proof is obviously.

5. Semiring of Circulant Fuzzy Matrices with Compatible Norm

In this section, we prove that CFM_c in fuzzy algebra and form a fuzzy vector space under the componentwise addition, componentwise multiplication and scalar multiplication $(+, \odot)$ is associative and distributive in $\mathcal{F}_{nn}$. Also, by using the definition of comparability of CFM_c some properties are proved.

Theorem 5.1. *The set CFM_c is fuzzy algebra under componentwise addition and multiplication operation $(+, \odot)$ defined as follows, if A, B, C are CFM_c, then*

(i) $A + B = \{\max(a_{ij}, b_{ij})\}$

(ii) $A \odot B = \{\min(a_{ij}, b_{ij})\}$.

Proof. A semiring $(S, +, \odot)$ set S equipped with two binary operations $(+, \odot)$ and called addition and multiplication. Also, $\| A + O \|_c = \| A \|_c$ and $\| A \odot J \|_c = \| A \|_c$ for all $\mathcal{F}_{nn}$, hence the zero matrix O is the additive identity and the universal matrix J is the multiplicative identity. Like this identity element relative to the operations $+$ and $\odot$ exist. Also

$\| A + J \|_c = \| J \|_c$ and $\| A \odot O \|_c = \| O \|_c$. Hence universal bond exists for all $A \in \mathcal{F}_{nn}$.

If $\| A \|_c = [a_{ij}]_{n \times n}$, $\| B \|_c = [b_{ij}]_{n \times n}$ and $\| C \|_c = [c_{ij}]_{n \times n}$ over $\mathcal{F}_{nn}$

(1) $(S, +)$ is a commutative monoid with identity element O

(i) $\| A + (B + C) \|_c = \| (A + B) + C \|_c$ (Associativity)

(ii) $\| O + A \|_c = \| A \|_c = \| A + O \|_c$ (Additive identity)

(iii) $\| A + B \|_c = \| B + A \|_c$ (Commutativity)

(2) $(S, \odot)$ is a monoid with identity element J.

(i) $\| A \odot B \|_c \odot \| C \|_c = \| A \|_c \odot \| B \odot C \|_c$

(ii) $\| A \odot J \|_c = \| A \|_c = \| J \odot A \|_c$

(3) Multiplication left and right distributes over addition.

(i) $\| A \odot (B + C) \|_c = \| A \|_c \odot \| B + C \|_c = \| A \odot B \|_c + \| A \odot C \|_c$

(ii) $\| (A + B) \odot C \|_c = \| A + B \|_c \odot \| C \|_c = \| A \odot C \|_c + \| B \odot C \|_c$

(iii) $\| A + (B \odot C) \|_c = \| A + B \|_c \odot \| A + C \|_c$

(4) If A, B are CFM_c over $\mathcal{F}_{nn}$ and any scalar α, β in $[0, 1]$ we have

(i) $\alpha \| A + B \|_c = \alpha J \odot \| A + B \|_c$

$$= \| \alpha(J \odot A) \|_c + \| \alpha(J \odot B) \|_c$$

$$= \| \alpha A \|_c + \| \alpha B \|_c$$

(ii) $(\alpha + \beta) \| A \|_c = (\alpha + \beta) \| J \odot A \|_c$

$$= \| (\alpha J + \beta J) \|_c \odot \| A \|_c$$

$$= \| \alpha(J \odot A) \|_c + \| \beta(J \odot A) \|_c$$

$$= \| \alpha A \|_c + \| \beta A \|_c.$$

Remark 5.1. If CFM_c a semiring multiplication is a commutative, then it is called a commutative semiring.

Definition 5.2. Comparable CFM_c. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ CFM_c over $\mathcal{F}_{nn}$ A is defined greater than B it $\| B \|_c \leq \| A \|_c$ B is greater than A if $\| A \|_c \leq \| B \|_c$ A and B CFM_c are said to comparable it either $\| A \|_c \leq \| B \|_c$ or $\| B \|_c \leq \| A \|_c$.

Theorem 5.3. *Let* A, B CFM_c *over* $\mathcal{F}_{nn}$. *Then* $\| A \|_c \leq \| B \|_c$ $\Leftrightarrow \| A + B \|_c = \| B \|_c$.

Proof. $\| A \|_c \leq \| B \|_c$, then $\| A \|_c + \| B \|_c \max \{a_{ij}, b_{ij}\} = [b_{ij}] = \| B \|_c$. Conversely, if $\| A + B \|_c = \| B \|_c$ then $a_{ij} \leq b_{ij}$ that is $\| A \|_c \leq \| B \|_c$ thus $\| A \|_c \leq \| B \|_c \Leftrightarrow \| A + B \|_c = \| B \|_c$.

Theorem 5.4. *Let* A, B *be* CFM_c *over* $\mathcal{F}_{nn}$ *if* $\| A \|_c \leq \| B \|_c$, *then for any* $C \in CFM_c \| AC \|_C \leq \| BC \|_c$ *and for any* $D \in CFM_c \| DA \|_C \leq \| DB \|_c$.

Proof. If $\| A \|_c \leq \| B \|_c CFM_c$ for C is CFM_c, then $\| AC \|_c \leq \| BC \|_c$ $A = (a_{ij}), B = [b_{ij}], C = [c_{ij}]$ thus $\| AC \|_c \leq \| BC \|_c \cdot \| DA \|_c \leq \| DB \|_c$ can be proved in this same manner.

6. Conclusion

In this paper, some properties of circulant fuzzy matrices with compatible norm (CFM_C) are discussed with the examples. The concept and some properties of determinant circulant fuzzy matrices are also discussed. De Morgan's Law are proved using elementary operators. The elements of diagonal parallel to the main diagonal are the same. A linear combination of circulant fuzzy matrices is a circulant fuzzy matrices. The product of circulant fuzzy matrices is circulant matrices. Also, circulant fuzzy matrix form a commutative property in semiring satisfied.

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