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VIZING'S CONJECTURE ON CORPORATE DOMINATION OF $P_{3k+1} \square P_n$

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Abstract

This paper discuss about the concept of corporate domination in cartesian product of two paths. We find the exact value of the corporate domination number of the cartesian product of two paths $P_{3k+1}(k \geq 1)$ and $P_n(n \geq 2)$. Also we resolve the Vizings conjecture for $G \cong P_{3k+1}$ and $H \cong P_n$.

1. Introduction

Let $G = (V, E)$ be a graph with vertex set V of order n and edge set E of size m. All graphs are finite, simple, and undirected. For a set $S \subset V$, let $\langle S \rangle$ be the subgraph of G induced by S. For $v \in V$ and $x = ab \in E$ we define the open neighborhood of a vertex v as $N(v) = \{u / uv \in E(G)\}$, the closed neighborhood of a vertex v as $N[v] = N(v) \cup \{v\}$ and $N[x] = N(a) \cup N(b)$. For graph theoretic terminology, we refer Chartrand, Lesinak [4] and Harray [3]. The elements $x, y \in V \cup E$ are said to be associated if they are adjacent or incident in G. Consider two sets $X, Y \in \{V, E, V \cup E\}$. A subset $D \subseteq X$ dominates Y if every element of $Y \setminus (D \cap Y)$ is associated with an element of D. The minimum cardinality among all the subsets of X is denoted by

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$\gamma_{(X,Y)}(G)$. This concept is suitable for the domination parameters such as $\gamma_{(V,V)}(G), \gamma_{(V,E)}(G), \gamma_{(V,VUE)}(G), \gamma_{(E,E)}(G), \gamma_{(E,V)}(G), \gamma_{(E,VUE)}(G), \gamma_{(VUE,V)}(G)$, $\gamma_{(VUE,E)}(G), \gamma_{(VUE,VUE)}(G)$. Various domination parameters have been focused to dominate the vertices, edges, mixing the vertices and edges. The detailed study of numerous domination parameters were established in [1, 8]. A dominating set $S \subseteq V$ of a graph G is perfect if each vertex in $V - S$ is dominated by exactly one vertex of S. The minimum cardinality of S, denoted by $\gamma_p(G)$, is the perfect domination number of G. Let $S \subseteq V(G)$. Then S is independent if no two vertices in S are adjacent. A dominating set S is said to be an efficient dominating set, if S is both perfect and independent. The graph G is an efficient open domination graph if there exists an efficient open dominating set S, for which $\bigcup_{v \in S} N(v) = V(G)$ and $N(u) \cap N(v) = \phi$ for every pair u and v of distinct vertices of S. Moreover, the efficient open domination graphs among graph products were characterized in [2] and the detailed study was discussed in [9]. The Cartesian product $G_1 \square G_2$ of graphs G_1 and G_2 is a graph with $V(G_1 \square G_2) = V(G_1) \square V(G_2)$ and $((x_1, y_1), (x_2, y_2)) \in E(G_1 \square G_2)$ if and only if either $x_1 = y_1$ and x_2 adjacent to y_2 in G_2 or $x_2 = y_2$ and x_1 adjacent y_1 in G_1. A detailed study of dominating set and its algorithm of Cartesian product of paths and cycles have been discussed by Polana Palvic, Janez Zerovnik [5]. We initiated the study of corporate domination and found the exact value of the corporate domination number for some classes of graphs in [6] and also established the corporate domination number of the Cartesian product of cycle $C_m, n \geq 2$ and path $P_n (n \geq 2)$ in [7]. In this paper, we determine the corporate domination number of the Cartesian product of two paths $P_{3k+1}, k \geq 1$ and $P_n, n \geq 2$ and we conclude the paper along with Vizing's conjecture for any two graphs $G \cong P_{3k+1}$ and $H \cong P_n$.

2. Corporate Dominating Set

Definition 2.1. Let $G = (V, E)$ be a graph and S, the set of all vertices V and edges E of G. A subset $C \subseteq S$ is said to be a corporate dominating set if every vertex v in $(P \cup Q)^c$ is adjacent to exactly one vertex of $P \cup Q$, where $P = \{u \in V(G[E_1])/| N(u) \cap N(w) | \leq 1$ for all $w(\neq u) \in V(G[E_1])\}$, $V(G[E_1])$ denotes the vertex set of an edge induced subgraph $G[E_1]$ and $Q = \{v \in V_1 / N(v) \cap N(s) = \phi$ for all $s(\neq v) \in V_1\}$. The corporate domination number of G, denoted by $\gamma_{cor}(G)$, is the minimum cardinality of elements in C.

Example 2.2. In the following tree G, let $C = \{v_2 v_3, v_6\}$. Here $E_1 = \{v_2 v_3\}$, $P = \{v_2, v_3\}$ and $Q = \{v_6\}$. Then $\gamma_{cor}(G) = 2$.

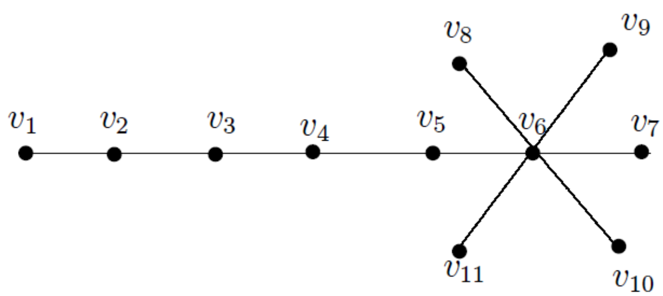

Figure 1.

Observation 2.3[6]**.** Let G be a graph of order n. Then $\gamma_{cor}(G) = 1$ if and only if either there exists an edge $xy \in G$ such that $d(x) + d(y) = n - 2$ and xy is not an edge of any triangle in G or there exists a vertex v such that $d(v) = n - 1$.

Observation 2.4[6]**.** Corporate dominating set need not exist for all graphs.

Example 2.5[6]**.** The following 4-regular graph G does not have a corporate dominating set.

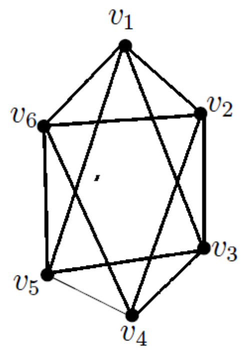

Figure 2.

If we take $C = \{v_i\}, 1 \leq i \leq 6$, or $C = \{v_i v_j\}, 1 \leq i \leq 6, i \neq j$, then by using observation 2.3, the graph G does not have a corporate dominating set and $|C| \geq 2$. Therefore, $|P \cup Q| \geq 2$. But any two vertices in this graph have a common neighbour which is a contradiction.

Observation 2.6[6]**.** Let C be a corporate dominating set with $C = V_1$. Then

(i) Every corporate dominating set is the dominating set.

(ii) Every corporate dominating set is the perfect dominating set. But the converse need not be true.

Proposition 2.7[6]**.** (a) *For any cycle* $C_m (m \geq 3)$ *and path* $P_m (m \geq 3)$, $\gamma_{cor}(C_m) = \gamma_{cor}(P_m) = \left\lceil \frac{m}{4} \right\rceil$.

(b) *For any wheel graph* $W_p (p > 3)$, $\gamma_{cor}(W_p) = 1$.

(c) *For any complete graph* $K_p (p \geq 3)$, $\gamma_{cor}(K_p) = 1$.

(d) *For any star graph* $K_{1,m} (m \geq 2)$, $\gamma_{cor}(K_{1,m}) = 1$.

3. Main Results

In this section, we found the corporate domination number of the cartesian product of two paths $P_{3k+1}, (k \geq 1)$ and $P_n, (n \geq 2)$.

Theorem 3.1. *Let* $P_{3k+1}(k \geq 1)$ *and* $P_{3s}(s \geq 1)$ *be any two paths. Then* $\gamma_{cor}(P_{3k+1} \square P_{3s}) = \left\lceil \frac{3k+1}{2} \right\rceil s$.

Proof. Let $P_{3k+1} \square P_{3s}$ be a graph. Let $m = 3k+1$ and $n = 3s$. Here $P = \{v_{tm+1}, v_{tm+2}, v_{tm+3}, \ldots, v_{(t+1)m-1}\}$ and $Q = \{v_{(t+1)m}\}$. Consider the following cases.

Case 1. Let $m \equiv 1 (\text{mod } 2)$.

For $1 \leq t \leq n-2$, $t \equiv 1 (\text{mod } 3)$, let

$$C = \{v_{tm+1}v_{tm+2}, v_{tm+3}v_{tm+4}, \ldots, v_{(t+1)m-2}v_{(t+1)m-1}, v_{(t+1)m}\}.$$

Clearly, $|Q| = \frac{n}{3} (= s)$.

Since every vertex in $V - (P \cup Q)$ is adjacent to a vertex in $P \cup Q$, C is the corporate dominating set.

Since every vertex in $P \cup Q$ is adjacent to exactly two vertices in $V - (P \cup Q)$, $|P \cup Q| = \left(\frac{mn}{3}\right)$. Therefore $|P| = \frac{(m-1)n}{3}$, as $|Q| = \frac{n}{3}$.

Hence C contains $\frac{(m-1)n}{6}$ edges and $\frac{n}{3}$ vertices. Thus $|C| = \left\lceil \frac{m}{2} \right\rceil \left(\frac{n}{3}\right)$ $\left(= \left\lceil \frac{3k+1}{2} \right\rceil s\right)$.

Now we show that C is minimum. Let $C' \neq C$ be a corporate dominating set and the two sets P', Q' corresponding to C' such that for any $x \in (P' \cup Q')^c$, $N(x) \cap (P' \cup Q') = \{u\}$, and the possible forms of C' are given below. (i) C' contains only vertices $(C' = V_1')$. (ii) C' contains only edges $(C' = E_1')$. (iii) $C' = V_1' \cup E_1'$.

If $C' = V_1'$ holds, then $P' = \phi$ and $Q' \neq \phi$. Since for any $u \in V_1'$, $N(u) \cap N(w) \neq \phi$ for some $w \in V_1'$ which is a contradiction.

If $C' = E_1'$, then $Q' = \phi$ and $P' \neq \phi$. Let $|Q'| = 0$ and $|P'| > |P|$ with

$2 < |P'| \leq n\left\lceil \frac{m}{3} \right\rceil$. Then $2 < |E_1'| \leq n\left\lceil \frac{m}{3} \right\rceil - \frac{n}{3}$. Thus $|C'| \geq |C|$.

Suppose $C' = V_1' \cup E_1'$. Then $P' \neq \phi$ and $Q' \neq \phi$.

If $|P' \cup Q'| < |P \cup Q|$, then there exists at least one vertex $v_i \notin (P' \cup Q')$ which is not adjacent to the vertices in $P' \cup Q'$, a contradiction. Hence $|P' \cup Q'| < |P \cup Q|$.

(a) For $m < n$, let $|Q'| \geq |Q|$ and $|P'| < |P|$ with $\frac{n}{3} \leq |Q'| \leq \frac{2n}{3}$ and $3 \leq |P'| \leq \frac{n}{3}(m-1)$. Clearly, $|E_1'| \leq \frac{(m-2)n}{3}$ and $|V_1'| \leq \frac{2n}{3}$. Hence $|C'| \leq \frac{mn}{3}$ and $|C| \leq |C'|$.

(b) For $m \geq n$, let $|Q'| \geq |Q|$ and $|P'| < |P|$ with $\frac{n}{3} \leq |Q'| \leq 2\left\lceil \frac{m}{3} \right\rceil$ and $3 \leq |P'| \leq \frac{n}{3}(m-1)$. Then $|C'| \leq \frac{(m-2)n}{3} + 2\left\lceil \frac{m}{3} \right\rceil$.

(c) Suppose $|P'| > |P|$ and $|Q'| \geq |Q|$ with $\frac{mn - n + 3}{3} \leq |P'| \leq n\left\lceil \frac{m}{3} \right\rceil - \frac{n}{3}$ and $\frac{n}{3} \leq |Q'| \leq \frac{2n}{3} - 1$. Then C' has at most $(n-1)\left\lceil \frac{m}{3} \right\rceil - \frac{n}{3}$ edges and $\frac{2n}{3} - 1$ vertices. Clearly, $|C'| \leq (n-1)\left\lceil \frac{m}{3} \right\rceil + \frac{2n}{3} - 1$.

(d) Let $|P'| > |P|$ and $|Q'| < |Q|$ with $\frac{mn - n + 3}{3} \leq |P'| \leq n\left\lceil \frac{m}{3} \right\rceil - 1$ and $1 \leq |Q'| \leq \frac{n-3}{3}$. Then C' has at most $\frac{n-3}{3}$ vertices and $(n-1)\left\lceil \frac{m}{3} \right\rceil - 1$ edges and hence $|C'| \geq |C|$.

Case 2. Let $m \equiv 0 \pmod 2$.

For $1 \leq t \leq n-2$, $t \equiv 1 \pmod 3$, let

$$C = \{v_{tm+1}v_{tm+2}, v_{tm+3}v_{tm+4}, \ldots, v_{(t+1)m-3}v_{(t+1)m-2}, v_{(t+1)m-1}v_{(t+1)m}\}.$$

Proceed as in Case 1, C is the corporate dominating set and $|C| = \frac{mn}{6}$. To prove C is minimum, let C' be another corporate dominating set. Proceed as before, we can prove that C is minimum for if $C' = V_1'$ and $C' = E_1'$. If $C' = V_1' \cup E_1'$, then $P' \neq \phi$ and $Q' \neq \phi$.

(a) For $m > n$, let $|Q'| > |Q|$ and $|P'| < |P|$ with $2 \leq |P'| \leq \frac{mn}{3} - 1$ and $1 \leq |Q'| \leq \frac{2n}{3}$. Then C' contains at most $\frac{n(m-1)}{3} - 1$ edges and $\frac{2n}{3}$ vertices. Hence $|C'| \leq \frac{n(m-1)}{3} - 1 + \frac{2n}{3} = \frac{mn + n - 3}{3}$. Therefore, $|C'| \geq |C|$.

(b) For $m \leq n$, let $|Q'| > |Q|$ and $|P'| < |P|$ with $2 \leq |P'| \leq \frac{mn-3}{3}$ and $1 \leq |Q'| \leq 2\left\lceil \frac{m}{3} \right\rceil$. Then C' has at most $\frac{(m-1)n}{3}$ edges and $\frac{2n}{3}$ vertices.

(c) Let $|P'| > |P|$ and $|Q'| > |Q|$ with $1 + \frac{mn}{3} \leq |P'| \leq \left\lceil \frac{m}{3} \right\rceil n - 1$ and $1 \leq |Q'| \leq \frac{n}{3} - 1$. Therefore, $|C'| \leq \left\lceil \frac{m}{3} \right\rceil (n+1) - 2 + \frac{n}{3}$. From all the above cases, C is the minimum corporate dominating set.

Theorem 3.2. *Let $P_{3k+1} (k \geq 1)$ and $P_{3s+2} (s \geq 0)$ be any two paths. Then* $\gamma_{cor}(P_{3k+1} \square P_{3s+2})$

$$= \begin{cases} \left\lceil \frac{3k+1}{3} \right\rceil \left(\frac{3s+2}{2} \right) & \text{if } 3k+1 \text{ is odd and } 3s+2 \text{ is even}, 3k+1 > 3s+2 \\ \left\lceil \frac{3k+1}{2} \right\rceil \left\lceil \frac{3s+2}{3} \right\rceil & \text{otherwise} \end{cases}.$$

Proof. Let $(P_{3k+1} \square P_{3s+2})$ be a graph and let $m = 3k+1$ and $n = 3s+2$. Consider the following cases.

Case 1. Let $m \equiv 1 (\text{mod } 2)$ and $n \equiv 0 (\text{mod } 2)$

Subcase 1.1. Let $m > n$.

For $2 \leq i \leq m-1$, $i \equiv 2 (\bmod 3)$, let

$$C = \{v_i v_{m+i},\ v_{2m+i} v_{3m+i},\ \ldots,\ v_{mn-4m+i} v_{mn-3m+i},\ v_{mn-2m+i} v_{mn-m+i}\}.$$

Here $P = \{v_i,\ v_{m+i},\ v_{2m+i},\ v_{3m+i},\ \ldots,\ v_{mn-2m+i},\ v_{mn-m+i}\}$ and $Q = \phi$.

Since every vertex in $(P \cup Q)^c$ is adjacent to a vertex in $P \cup Q$, C is the corporate dominating set.

As $Q = \phi$, $|P| = |P \cup Q| = \left\lceil \frac{m}{3} \right\rceil n$. Hence C contains $\left\lceil \frac{m}{3} \right\rceil \left(\frac{n}{2} \right)$ edges. Therefore, $|C| = \left\lceil \frac{m}{3} \right\rceil \left(\frac{n}{2} \right)$.

We shall claim that C is minimum. Let $C' \neq C$ be a corporate dominating set. If $C' = V_1'$ holds, then $P' = \phi$ and $Q' \neq \phi$. This is true when $n = 2$, $(s = 0)$. Hence $|C'| = \left\lceil \frac{m}{3} \right\rceil n$. Thus $|C| \leq |C'|$.

If $C' = E_1'$ holds, then $P' \neq \phi$ and $Q' = \phi$.

Let $|P'| \leq |P|$ with $|P'| \leq n \left\lceil \frac{m}{3} \right\rceil$. Then $|C'| \leq \left\lceil \frac{m}{3} \right\rceil (n-1)$.

Suppose $|P'| > |P|$ with $|P'| \leq n \left\lceil \frac{m}{3} \right\rceil$. Then $|C'| \leq (m-1) \left\lceil \frac{n}{3} \right\rceil$. Therefore, $|C| \leq |C'|$.

If $C' = V_1' \cup E_1'$ holds, then $P' \neq \phi$ and $Q' \neq \phi$.

Suppose $|P'| \leq |P|$ and $|Q'| > |Q|$ with $6 \leq |P'| \leq n \left\lceil \frac{m}{3} \right\rceil - 1$ and $1 \leq |Q'| \leq \left\lceil \frac{m}{3} \right\rceil$. Then $|C'| \leq (n-1) \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{m}{3} \right\rceil - 1$ and hence $|C'| \geq |C|$.

Subcase1.2. Let $m < n$.

For $1 \leq t \leq n-2$ and $t \equiv 1 (\bmod 3)$, let

$$C = \{v_{tm+1} v_{tm+2},\ v_{tm+3} v_{tm+4},\ \ldots,\ v_{(t+1)m-2} v_{(t+1)m-1},\ v_{(t+1)m}\}.$$

Here $P = \{v_{tm+1}, v_{tm+2}, v_{tm+3}, v_{tm+4}, \ldots, v_{(t+1)m-2}, v_{(t+1)m-1}\}$.

$Q = \{v_{(t+1)m}\}$. Clearly, $|Q| = \left(\frac{n-2}{3} + 1\right) = \left\lceil \frac{n}{3} \right\rceil$.

Then C is the corporate dominating set, as for any $w \in (P \cup Q)^c$, $|N(w) \cap (P \cup Q)| = 1$.

As $|P \cup Q| = m\left\lceil \frac{n}{3} \right\rceil$ and $|Q| = \left\lceil \frac{n}{3} \right\rceil$, $|P| = (m-1)\left\lceil \frac{n}{3} \right\rceil$.

Hence C contains $\left(\frac{m-1}{2}\right)\left\lceil \frac{n}{3} \right\rceil$ edges and $\left\lceil \frac{n}{3} \right\rceil$ vertices. Thus $|C| = \left\lceil \frac{m}{2} \right\rceil \left\lceil \frac{n}{3} \right\rceil$.

We shall prove that C is minimum. Let C' be any other corporate dominating set. Then C' will be in one of the following forms. (i) $C' = V_1'$ (ii) $C' = E_1'$ (iii) $C' = V_1' \cup E_1'$

If (i) holds, then $P' = \phi$ and $Q' \neq \phi$.

Since for any $u \epsilon V_1'$, $N(u) \cap N(w) \neq \phi$, for some $w \in V_1'$, this is a contradiction.

If (ii) holds, then $P' \neq \phi$ and $Q' = \phi$. Let $|P'| \leq |P|$ with $|P'| \leq m\left\lceil \frac{n}{3} \right\rceil$ and $|Q'| = 0$. Then $|C'| = |E_1'| \leq (m-1)\left\lceil \frac{n}{3} \right\rceil$ and hence $|C'| \geq |C|$.

Suppose $|P'| > |P|$ with $|P'| \leq \left\lceil \frac{m}{3} \right\rceil$. Then $|C'| \leq \left\lceil \frac{m}{3} \right\rceil (n-1)$. Therefore, $|C'| \geq |C|$. If (iii) holds, then $P' \neq \phi$ and $Q' \neq \phi$.

Suppose $|P' \cup Q'| < |P \cup Q|$. Then there exists at least one vertex $v_i \epsilon (P' \cup Q')^c$ which is adjacent to none of the vertices in $P' \cup Q'$ which is a contradiction. Hence $|P' \cup Q'| \geq |P \cup Q|$.

(a) Let $|P'| \leq |P|$ and $|Q'| \geq |Q|$ with $|P'| \leq (m-1)\left\lceil \frac{n}{3} \right\rceil$ and

$\left\lceil \frac{n}{3} \right\rceil \le |Q'| \le 2\left\lceil \frac{m}{3} \right\rceil$. Clearly, $|E_1'| \le (m-2)\left\lceil \frac{n}{3} \right\rceil$ and $|V_1'| \le 2\left\lceil \frac{m}{3} \right\rceil$. Therefore, $|C'| \ge |C|$.

(b) Suppose $|P'| > |P|$ and $|Q'| \ge |Q|$ with $(m-1)\left\lceil \frac{n}{3} \right\rceil + 1 \le |P'| \le \left\lceil \frac{m}{3} \right\rceil n - \left\lceil \frac{n}{3} \right\rceil$ and $\left\lceil \frac{m}{3} \right\rceil \le |Q'| \le 2\left\lceil \frac{n}{3} \right\rceil$. Then C' has at most $(n-1)\left\lceil \frac{m}{3} \right\rceil - \left\lceil \frac{n}{3} \right\rceil$ edges and $2\left\lceil \frac{n}{3} \right\rceil$ vertices.

Hence $|C'| \le (n-1)\left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil$.

(c) Let $|P'| > |P|$ and $|Q'| < |Q|$ with $(m-1)\left\lceil \frac{m}{3} \right\rceil + 1 \le |P'| \le m\left\lceil \frac{n}{3} \right\rceil - 1$ and $1 \le |Q'| \le \left\lceil \frac{n}{3} \right\rceil - 1$. Then C' contains at most $m\left\lceil \frac{n}{3} \right\rceil - 2$ elements. Thus $|C'| \ge |C|$.

Case 2. Let $m \equiv 0 (\text{mod } 2)$ and $n \equiv 1 (\text{mod } 2)$.

For $1 \le t \le n-1$ and $t \equiv 1 (\text{mod } 3)$, let

$C = \{v_{tm+1}v_{tm+2}, v_{tm+3}v_{tm+4}, \ldots, v_{(t+1)m-3}v_{(t+1)m-2}, v_{(t+1)m-1}v_{(t+1)m}\}$.

Here $P = \{v_{tm+1}, v_{tm+2}, v_{tm+3}, v_{tm+4}, \ldots, v_{(t+1)m-1}, v_{(t+1)m}\}$, $Q = \phi$.

Proceed as in Subcase 1.1, replace m by n and n by m, we can prove that C is the minimum corporate dominating set and $|C| = \left(\frac{m}{2}\right)\left\lceil \frac{n}{3} \right\rceil$.

Case 3. Let $m \equiv 0 (\text{mod } 2)$ and $n \equiv 0 (\text{mod } 2)$.

Similar to proof of Case 2, $|C| = \left(\frac{m}{2}\right)\left\lceil \frac{n}{3} \right\rceil$.

Case 4. Let $m \equiv 1 (\text{mod } 2)$ and $n \equiv 1 (\text{mod } 2)$.

As in Subcase1.2, $|C| = \left\lceil \frac{m}{2} \right\rceil \left\lceil \frac{n}{3} \right\rceil$.

Theorem 3.3. *Let* P_{3k+1} *and* $P_{3s+1}(k, s \geq 1)$ *be any two paths.*

(i) *If* $3k + 1 > 3s + 1$, *then* $\gamma_{cor}(P_{3k+1} \square P_{3s+1})$

$$= \begin{cases} \left(\frac{3k+1}{2}\right)\left\lceil \frac{3s+1}{3} \right\rceil & if\ k \equiv 1 (\mathrm{mod}\ 2)\ and\ s \equiv 0 (\mathrm{mod}\ 2) \\ \left\lceil \frac{3k+1}{3} \right\rceil \left\lceil \frac{3s+1}{2} \right\rceil & otherwise \end{cases}$$

(ii) *If* $3k + 1 < 3s + 1$, *then* $\gamma_{cor}(P_{3k+1} \square P_{3s+1})$

$$= \begin{cases} \left(\frac{3k+1}{2}\right)\left\lceil \frac{3s+1}{3} \right\rceil & if\ k \equiv 1 (\mathrm{mod}\ 2)\ and\ s \equiv 0 (\mathrm{mod}\ 2) \\ \left\lceil \frac{3k+1}{2} \right\rceil \left\lceil \frac{3s+1}{3} \right\rceil & otherwise \end{cases}$$

(iii) *If* $3k + 1 = 3s + 1$, *then* $\gamma_{cor}(P_{3k+1} \square P_{3s+1})$

$$= \begin{cases} \frac{3k^2 + 5k + 2}{2} & if\ 3k+1\ is\ odd \\ \frac{3k^2 + 4k + 1}{2} & if\ 3k+1\ is\ even \end{cases}.$$

Proof. (i) Let $P_{3k+1} \square P_{3s+1}$ be a graph with $k, s \geq 1$ and $3k + 1 > 3s + 1$. Let $m = 3k + 1$ and $n = 3s + 1$. Consider the following cases.

Case 1. Let $m \equiv 0 (\mathrm{mod}\ 2)$ and $n \equiv 1 (\mathrm{mod}\ 2)$.

For $0 \leq t \leq n - 1$ and $t \equiv 0 (\mathrm{mod}\ 3)$, let

$$C = \{v_{tm+1}v_{tm+2}, v_{tm+3}v_{tm+4}, \ldots, v_{(t+1)m-3}v_{(t+1)m-2}, v_{(t+1)m-1}v_{(t+1)m}\}.$$

Here $P = \{v_{tm+1}, v_{tm+2}, v_{tm+3}, v_{tm+4}, \ldots, v_{(t+1)m-1}, v_{(t+1)m}\}$, $Q = \phi$.

Since every vertex in $(P \cup Q)^c$ is adjacent to exactly one vertex in $P \cup Q$, C is the corporate dominating set. As $|Q| = 0, |P| = |P \cup Q|$ $= m\left\lceil \frac{n}{3} \right\rceil$. Hence C contains $\left(\frac{m}{2}\right)\left\lceil \frac{n}{3} \right\rceil \left(= \left(\frac{3k+1}{2}\right)\left\lceil \frac{3s+1}{3} \right\rceil\right)$ edges.

To prove C is minimum, let C' be any other corporate dominating set and it will be in one of the following forms. (i) $C' = V_1'$ (ii) $C' = E_1'$ (iii)

$C' = V_1' \cup E_1'$. Proceed as in Subcase1.2 of Theorem 3.2, we can prove that C is minimum for if $C' = V_1$ and $C' = E_1'$.

If $C' = V_1' \cup E_1'$, then $P' \neq \phi$ and $Q' \neq \phi$.

If $|P' \cup Q'| > |P \cup Q|$, then $|N(w) \cap (P' \cup Q')| > 1$ for some $w \in (P' \cup Q')^c$ which is a contradiction. Hence $|P' \cup Q'| > |P \cup Q|$.

Let $|P'| < |P|$ and $|Q'| > |Q|$ with $(n-2)\left\lceil \frac{m}{3} \right\rceil \leq |P'| \leq m\left\lceil \frac{n}{3} \right\rceil - 1$ and $1 \leq |Q'| \leq 2\left\lceil \frac{m}{3} \right\rceil$. Then C' contains at most $(m-1)\left\lceil \frac{n}{3} \right\rceil - 1$ edges and $2\left\lceil \frac{m}{3} \right\rceil$ vertices and hence $|C'| \geq |C|$.

Case 2. Let $m \equiv 1 (\bmod 2)$ and $n \equiv 0 (\bmod 2)$.

For $1 \leq i \leq m$ and $i \equiv 1 (\bmod 3)$, let $C = \{v_i v_{m+i}, v_{2m+i} v_{3m+i}, \ldots, v_{mn-2m+i} v_{mn-m+i}\}$.

Here $P = \{v_i, v_{m+i}, v_{2m+i}, v_{3m+i}, \ldots, v_{mn-2m+i}, v_{mn-m+i}\}$ and $Q = \phi$.

As every vertex not in $P \cup Q$ is adjacent to exactly one vertex in $P \cup Q$, C is the corporate dominating set.

Since $|Q| = 0$, $|P| = |P \cup Q| = \left\lceil \frac{m}{3} \right\rceil n$. Hence $|C| = \left\lceil \frac{m}{3} \right\rceil \left(\frac{n}{2} \right)$.

We shall prove that C is minimum. Proceed as in Case 1, we can prove that C is minimum for if $C' = V_1'$ and $C' = E_1'$. If $C' = V_1' \cup E_1'$, then $P' \neq \phi$ and $Q' \neq \phi$. If $|P' \cup Q'| < |P \cup Q|$. Then there exists at least one vertex $u \in (P' \cup Q')^c$ which is not adjacent to the vertices in $P \cup Q$, a contradiction. Hence $|P' \cup Q'| \geq |P \cup Q|$. Let $|P'| \geq |P|$ and $|Q'| > |Q|$ with $\left\lceil \frac{m}{3} \right\rceil n \leq |P'| \leq \left\lceil \frac{n}{3} \right\rceil m - 1$ and $1 \leq |Q'| \leq \left\lceil \frac{n}{3} \right\rceil$. Hence C' contains at most $\left\lceil \frac{n}{3} \right\rceil (m-1) - 1$ edges and $\left\lceil \frac{n}{3} \right\rceil$ vertices. Therefore $|C'| \geq |C|$.

Let $|P'| < |P|$ and $|Q'| > |Q|$ with $6 \leq |P'| \leq \left\lceil \frac{m}{3} \right\rceil n - 1$ and $1 \leq |Q'| \leq 2\left\lceil \frac{m}{3} \right\rceil$. Hence C' contains at most $\left\lceil \frac{m}{3} \right\rceil (n-1) - 1$ edges and $2\left\lceil \frac{m}{3} \right\rceil$ vertices. Thus $|C'| \geq |C|$.

Case 3. Let $m \equiv 1 (\text{mod } 2)$ and $n \equiv 1 (\text{mod } 2)$.

For $1 \leq i \leq m$ and $i \equiv 1 (\text{mod } 3)$, let $C = \{v_i v_{m+i}, v_{2m+i} v_{3m+i}, \ldots, v_{mn-3m+i} v_{mn-2m+i}, v_{mn-m+i}\}$.

Here $P = \{v_i, v_{m+i}, v_{2m+i}, \ldots, v_{mn-3m+i}, v_{mn-2m+i}\}$, $Q = \{v_{mn-m+i}\}$. Clearly, $|Q| = \left\lceil \frac{m}{3} \right\rceil$. Since for any $u \epsilon (P \cup Q)^c$, $N(u) \cap (P \cup Q) = \{w\}$ where $w \epsilon P \cup Q$, C is the corporate dominating set. Since $|Q| = \left\lceil \frac{m}{3} \right\rceil$, $|P \cup Q| = \left\lceil \frac{m}{3} \right\rceil n$. Therefore $|P| = (n-1)\left\lceil \frac{m}{3} \right\rceil$. Hence C contains $\left(\frac{n-1}{2}\right)\left\lceil \frac{m}{3} \right\rceil$ edges and $\left\lceil \frac{m}{3} \right\rceil$ vertices. Thus $|C| = \left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{m}{3} \right\rceil$. We shall prove that C is minimum. Proceed as in Case 2, we can prove that C is minimum for if $C' = V_1'$ and $C' = E_1'$. If $C' = V_1' \cup E_1'$, then $P' \neq \phi$ and $Q' \neq \phi$. Since $|P' \cup Q'| \geq |P \cup Q|$, let $|P'| \geq |P|$ and $|Q'| \leq |Q|$ with $(n-1)\left\lceil \frac{m}{3} \right\rceil \leq |P'| \leq n\left\lceil \frac{m}{3} \right\rceil - 1$ and $1 \leq |Q'| \leq \left\lceil \frac{m}{3} \right\rceil$. Then C' contains at most $(n-1)\left\lceil \frac{m}{3} \right\rceil$ edges and $\left\lceil \frac{m}{3} \right\rceil$ vertices. Hence $|C'| \geq |C|$.

Suppose $|P'| \geq |P|$ and $|Q'| > |Q|$ with $(n-1)\left\lceil \frac{m}{3} \right\rceil \leq |P'| \leq n\left\lceil \frac{m}{3} \right\rceil - 1$ and $\left\lceil \frac{m}{3} \right\rceil + 1 \leq |Q'| \leq 2\left\lceil \frac{m}{3} \right\rceil$. Hence C' contains at most $(n-1)\left\lceil \frac{m}{3} \right\rceil - 1$ edges and $2\left\lceil \frac{m}{3} \right\rceil$ vertices. Thus $|C'| \geq |C|$.

If $|P'| < |P|$ and $|Q'| > |Q|$ with $29 \leq |P'| \leq (n-1)\left\lceil \frac{m}{3} \right\rceil - 1$ and

$\left\lceil \frac{m}{3} \right\rceil + 1 \leq |Q'| \leq 2\left\lceil \frac{m}{3} \right\rceil$. Then $|C'| \leq n\left\lceil \frac{m}{3} \right\rceil - 1$ and hence $|C'| \geq |C|$.

Case 4. Let $m \equiv 0 (\bmod 2)$ and $n \equiv 0 (\bmod 2)$.

Proof is similar to Case 2.

(ii) Let $P_{3k+1} \square P_{3s+1}$ be a graph with $k, s \geq 1$ and $3k + 1 < 3s + 1$. Let $m = 3k + 1$ and $n = 3s + 1$. Consider the following cases.

Case 1. Let $m \equiv 0 (\bmod 2)$ and $n \equiv 1 (\bmod 2)$.

Proceed as in Case 1 of (i), C is the minimum corporate dominating set and $|C| = \left(\frac{m}{2}\right)\left\lceil \frac{n}{3} \right\rceil$.

Case 2. Let $m \equiv 1 (\bmod 2)$ and $n \equiv 0 (\bmod 2)$.

Proceed as in Case 2 of (i), C is the minimum corporate dominating set.

Case 3. Let $m \equiv 1 (\bmod 2)$ and $n \equiv 1 (\bmod 2)$.

For $0 \leq t \leq n - 1$ and $t \equiv 0 (\bmod 3)$, let $C = \{v_{tm+1}v_{tm+2}, v_{tm+3}v_{tm+4}, \ldots, v_{(t+1)m-2}v_{(t+1)m-1}, v_{(t+1)m}\}$. Here $P = \{v_{tm+1}, v_{tm+2}, v_{tm+3}, v_{tm+4}, \ldots, v_{(t+1)m-2}, v_{(t+1)m-1}\}$, $Q = \{v_{(t+1)m}\}$. Clearly, $|Q| = \left\lceil \frac{n}{3} \right\rceil$.

Proceed as in Case 3 of (i), by replacing m by n and n by m we can prove that C is the minimum corporate dominating set and $|C| = \left\lceil \frac{m}{2} \right\rceil \left\lceil \frac{n}{3} \right\rceil$.

Case 4. Let $m \equiv 1 (\bmod 2)$ and $n \equiv 1 (\bmod 2)$.

Proof is similar to Case 1.

(iii) Let $P_{3k+1} \square P_{3k+1}$ be a graph and let $m = 3k + 1$. Consider the following cases.

Case 1. Let m be odd.

For $0 \leq t \leq m - 1$ and $t \equiv 0 (\bmod 3)$, let

$$C = \{v_{tm+1}v_{tm+2}, v_{tm+3}v_{tm+4}, \ldots, v_{(t+1)m-2}v_{(t+1)m-1}, v_{(t+1)m}\}.$$

Proceed as in Case 3 of (iii), by replacing n by m, we can prove that C is the minimum corporate dominating set and $|C| = \left\lceil \frac{m}{2} \right\rceil \left\lceil \frac{m}{3} \right\rceil = \left(\frac{(3k+3)(3k+2)}{6} \right)$.

Case 2. Let m be even.

For $1 \le i \le m$ and $i \equiv 1 (\bmod 3)$, let

$$C = \{v_i v_{m+i}, v_{2m+i} v_{3m+i}, \ldots, v_{m^2-4m+i} v_{m^2-3m+i}, v_{m^2-2m+i} v_{m^2-m+i}\}.$$

Here $P = \{v_i, v_{m+i}, v_{2m+i}, v_{3m+i}, \ldots, v_{m^2-4m+i}, v_{m^2-3m+i}, v_{m^2-2m+i}, v_{m^2-m+i}\}$ and $Q = \phi$.

Proceed as in Case 2 of (i), by replacing n by m, C is the minimum corporate dominating set and $|C| = \left(\frac{m}{2} \right) \left\lceil \frac{m}{3} \right\rceil = \left(\frac{(3k+1)(3k+3)}{6} \right)$.

Theorem 3.4. *For any two graphs* $G \cong P_{3k+1}$ *and* $H \cong P_n$,

$$\gamma_{cor}(P_{3k+1} \square P_n) \ge \gamma_{cor}(P_{3k+1}) \gamma_{cor}(P_n).$$

Proof. Let $G \square H$ be the cartesian product of two graphs, where $G \equiv P_{3k+1}$ and $H \equiv P_n$. As $\gamma_{cor}(P_n) = \left\lceil \frac{n}{4} \right\rceil$, we have $\gamma_{cor}(P_{3k+1}) \gamma_{cor}(P_n) = \left\lceil \frac{3k+1}{4} \right\rceil \left\lceil \frac{n}{4} \right\rceil$. From the above theorems, we conclude that $\gamma_{cor}(P_{3k+1} \square P_n) = \min \left\{ \left\lceil \frac{3k+1}{3} \right\rceil \left\lceil \frac{n}{2} \right\rceil, \left\lceil \frac{3k+1}{2} \right\rceil \left\lceil \frac{n}{3} \right\rceil \right\}$. It is simple to show that $\min \left\{ \left\lceil \frac{3k+1}{3} \right\rceil \left\lceil \frac{n}{2} \right\rceil, \left\lceil \frac{3k+1}{2} \right\rceil \left\lceil \frac{n}{3} \right\rceil \right\} \ge \left\lceil \frac{3k+1}{4} \right\rceil \left\lceil \frac{n}{4} \right\rceil$. Hence $\gamma_{cor}(P_{3k+1} \square P_n) \ge \gamma_{cor}(P_{3k+1}) \gamma_{cor}(P_n)$.

4. Conclusion

We determined the corporate domination number for $C_m \square C_n (m, n \ge 3)$, $C_m \square P_n (m \ge 3, n \ge 2)$, $P_m \square P_n (m, n \ge 2)$ and proved the Vizing's conjecture for $G \cong P_{3k+1}$ and $H \cong P_n$. The Vizing's Conjecture has yet to be proven for the corporate domination number of cartesian product of various graphs.

Conflicts of interest

The authors confirm that there is no conflict of interest to declare for this publication.

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