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INTERVAL VALUED WEAK FUZZY GRAPHS AND FERMAT'S WEAK FUZZY GRAPHS

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Abstract

In this article the concept of interval valued weak fuzzy graphs and Fermat's weak fuzzy graphs are introduced. It is proved that the set of weak fuzzy graphs is an abelian semi group with respect to union. Some connectivity, complement properties of weak fuzzy graphs and some properties of Fermat's fuzzy graphs are derived.

1. Introduction

L A Zadeh in 1965 [1] introduced fuzzy sets to describe the vagueness phenomena in real world problems. In 1975 [2], Azriel Rosenfeld introduced fuzzy graphs. In 2006 [3], Gani A. N and Chandrasekharan V. T introduced μ complement of a fuzzy graph. Nishad T. M. discussed the importance of studies on fuzzy graphs having specific property in 2018 [11]. Prasanna A and Nishad T M introduced weak fuzzy graphs in 2021 [4]. Some applications of fuzzy graphs in simulating problems and networks are discussed by various authors [6, 7, 8]. In 2009, Hongmei and Lianhua defined Interval

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Valued Fuzzy Graph (IVFG) [9] and in 2013 Talebi and Rashmanlou studied properties of isomorphism and complement of an IVFG [10]. Gani A. N, Akram et al., invented fuzzy labeling graphs in 2014 [5]. They also introduced fuzzy magic graphs. The modeling of anonymous networks using fuzzy graphs is introduced in 2018 [7] by Vasishtduddu, Debasis Samanta and D. Vijay Rao. Fuzzy magic graphs is used to model facility sharing networks [5]. There is possibility to assign same percentage of facility to more than one node and the total percentage of facility to each node need not be constant. Weak fuzzy graphs overcome the flaws of fuzzy magic graphs in facility sharing networks. To simulate anonymous communication networks and signal strength variation in simulation of neural networks, the assumptions include uncertainty in varying traffic density, mitigation of traffic jams, the limits in variation of traffic density and the limits in variation in presence of nodes. Interval Valued Weak Fuzzy Graph (IVWFG) capture those assumptions. When the sum of membership and non membership of an object becomes greater than 1, the Intuitionistic Fuzzy Graph cannot address some problems. For example: $0.9 + 0.8 > 1.$ Also the square scale or cubic scale have the same deficiency. Example: $0.9^2 + 0.8^2 > 1$ and $0.9^3 + 0.8^3 > 1.$ So membership and non membership degrees should be determined in any power scale than square or cubic scale and in decision tools, the hesitancy degree should be based on general scale. Muhammed Akram, Anna Habib, etc discussed specific types of Pythagorean Fuzzy Graphs and applications to decision making in 2018 [12]. When $n > 2,$ in the generalization of intuitionistic fuzzy graph of nth type, the general in equation $x^n + y^n \leq z^n$ is used by keeping RHS as $1^n.$ Positive solutions to this inequation has applications by representing the measure of acceptance and rejections of two nodes of a network in comparison. The French mathematician Pierre de Fermat (1607-1665), conjectured that the equation $x^n + y^n = z^n$ has no solution in positive integers x, y and z if n is a positive integer $\geq 3.$ So the name of Fermat is used in generalization. In decision tools the square root in measuring hesitancy degree cannot address general scale. In this article the concept of interval valued weak fuzzy graphs and Fermat's weak fuzzy graphs, Fermat's Fuzzy Preference Relation are introduced. Hesitancy degree is measured in general scale. To mention the expansion of networks which are governed by properties of weak fuzzy graphs, it is proved that the set of weak fuzzy graphs is an abelian semi group with respect to union. Some connectivity, complement properties of weak fuzzy graphs and some properties of Fermat's fuzzy graphs are derived.

2. Preliminaries

A mapping $m : N \to [0, 1]$ from a non empty set N is a fuzzy subset of N. A fuzzy relation r on the fuzzy subset m, is a fuzzy subset of $N \times N$. The fuzzy relation r is assumed as symmetric and N is assumed as finite non empty set.

Definition 2.1. Suppose N is the underlying set. A fuzzy graph is a pair of functions $F : (m, r)$ where $m : N \to [0, 1]$ is a fuzzy subset and the fuzzy relation r on m is denoted by $r : N \times N \to [0, 1]$, such that for all $a, b \in N$, we have $r(a, b) \leq m(a) \wedge m(b)$ where $\wedge$ stands minimum. $F^* : (m^*, r^*)$ denotes the underlying crisp graph of a fuzzy graph $F : (m, r)$ where $m^* = \{a \in N / m(a) > 0\}$ and $r^* = \{(a, b) \in N \times N / r(a, b) > 0\}$. The nodes a and b are known as neighbours if $r(a, b) > 0$.

Definition 2.2. A fuzzy graph $F : (m, r)$ is a strong fuzzy graph if $r(a, b) = m(a) \wedge m(b), \forall (a, b) \in r^*$.

Definition 2.3. The μ-complement of a fuzzy graph $F : (m, r)$ is a fuzzy graph $F^{\mu} : (m, r^{\mu})$ where r^{μ} is defined as $r^{\mu}(a, b) = 0$ if $r(a, b) = 0$ and $r^{\mu}(a, b) = m(a) \wedge m(b) - r(a, b)$ if $r(a, b) > 0$ for all $a, b \in m^*$.

Definition 2.4. A fuzzy graph $F : (m, r)$ is a weak fuzzy graph if $r(a, b) < m(a) \wedge m(b)$ for all $(a, b) \in r^*$.

Definition 2.5. A fuzzy graph $F : (m, r)$ is said to be a fuzzy labeling graph if m and r are bijective and $r(a, b) < m(a) \wedge m(b)$ for all $(a, b) \in r^*$.

Definition 2.6. An interval valued fuzzy graph (IVFG) is defined as $G = (\sigma, \mu)$ where $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$, where

$\sigma(u) = [\sigma_u^-, \sigma_u^+], 0 \le \sigma_u^- \le \sigma_u^+ \le 1$ and $\mu(u, v) = [\mu_{uv}^-, \mu_{uv}^+], 0 \le \mu_{uv}^- \le \mu_{uv}^+ \le 1$ represent the interval number of the vertex u and the interval number of the edge uv in G respectively satisfying $\mu_{uv}^- \le \sigma_u^- \wedge \sigma_v^-, \mu_{uv}^+ \le \sigma_u^+ \wedge \sigma_v^+$, for all $u, v \in \sigma^*$.

Definition 2.7. An interval valued weak fuzzy graph (IVWFG) is defined as $G = (\sigma, \mu)$ where $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$, where $\sigma(u) = [\sigma_u^-, \sigma_u^+], 0 \le \sigma_u^- \le \sigma_u^+ \le 1$ and $\mu(u, v) = [\mu_{uv}^-, \mu_{uv}^+], 0 \le \mu_{uv}^- \le \mu_{uv}^+ \le 1$ represent the interval number of the vertex u and the interval number of the edge uv in G respectively satisfying $\mu_{uv}^- < \sigma_u^- \wedge \sigma_v^-, \mu_{uv}^+ < \sigma_u^+ \wedge \sigma_v^+$ and $\mu_{uv}^+ - \mu_{uv}^+ < \sigma_u^+ \wedge \sigma_v^+ - \sigma_u^- \wedge \sigma_v^-$ for all $u, v \in \sigma^*$.

Definition 2.8. A Fermat's Fuzzy Set (FFS) on a universal set A is a set of 3 tuples of the form $F = \{(u, I_F(u), O_F(u))\}$ where $I_F(u)$ and $O_F(u)$ represents the membership and non membership degrees of $u \in A$ and $I_F(u), O_F(u)$ satisfy $0 \le I_F^n(u) + O_F^n(u) \le 1$ for all $u \in A, n \in N = \{1, 2, 3, \ldots\}$.

Definition 2.9. A Fermat's fuzzy relation (FFR) R on $A \times A$ is a set of 3 tuples of the form $R = \{(uv, I_R(uv), O_R(uv))\}$ where $I_R(uv)$, and $O_R(uv)$ represents the membership degree and non membership degree of uv in R and $I_R(uv), O_R(uv)$ satisfy $0 \le I_R^n(uv) + O_R^n(un) \le 1$ for all $uv \in A \times A$. FFR need not be symmetric. Hence $I_R(uv)$ need not be equal to $I_R(vu)$.

Definition 2.10. A Fermat's fuzzy graph $(FFG(n))$ on a non empty set A is a pair $G : (\sigma, \mu)$ with σ as FFS on A and μ as FFR on A such that

$I_\mu(uv) \le I_\sigma(u) \wedge I_\sigma(v), O_\mu(uv) \ge O_\sigma(u) \vee O_\sigma(v)$ and

$0 \le I_\mu^n(uv) + O_\mu^n(uv) \le 1$ for all $u, v \in A, n \in N = \{1, 2, 3, \ldots\}$ where $I_\mu : A \times A \to [0, 1]$ and $O_\mu : A \times A \to [0, 1]$ represents the membership and non membership functions of μ respectively.

Definition 2.11. A Fermat's fuzzy preference relation (FFPR) on the set of nodes $N = \{x_1, x_2, \ldots, x_n\}$ is represented by a matrix $M = (m_{ij})_{n\times n}$, where $m_{ij} = (x_i x_j, I(x_i x_j), O(x_i x_j))$ for all $i, j = 1, 2, 3, \ldots, n$. Let $m_{ij} = (I_{ij}, O_{ij})$ where I_{ij} indicates the degree to which the node x_i is preferred to node x_j and O_{ij} denotes the degree to which the node x_i is not preferred to the node x_j and $\pi_{ij} = \sqrt[n]{1 - I_{ij}^n - O_{ij}^n}$ is interpreted as hesitancy degree, with the conditions, $I_{ij}, O_{ij} \in [0, 1], 0 \le I_{ij}^n + O_{ij}^n \le 1, I_{ij} = O_{ji}, I_{ii} = O_{ii} = 0.5$ for all $i, j = 1, 2, 3, \ldots, n$.

Definition 2.12. A Fermat's fuzzy graph $G : (\sigma, \mu)$ is said to be Fermat's Strong fuzzy Graph $FSFG(n)$ with underlying crisp graph $G^* : (\sigma^*, \mu^*)$ if $I_\mu(uv) = I_\sigma(u) \wedge I_\sigma(v), O_\mu(uv) = O_\sigma(u) \vee O_\sigma(v)$ for all $uv \in \mu^*$.

Definition 2.13. A Fermat's fuzzy graph $G : (\sigma, \mu)$ is said to be Fermat's Weak Fuzzy Graph $FWFG(n)$ with underlying crisp graph $G^* : (\sigma^*, \mu^*)$ if $I_\mu(uv) < I_\sigma(u) \wedge I_\sigma(v), O_\mu(uv) > O_\sigma(u) \vee O_\sigma(v)$ for all $uv \in \mu^*$.

Definition 2.14. A Fermat's fuzzy graph $G : (\sigma, \mu)$ is said to be complete FFG with underlying crisp graph $G^* : (\sigma^*, \mu^*)$ if $I_\mu(uv) = I_\sigma(u) \wedge I_\sigma(v)$, $O_\mu(uv) = O_\sigma(u) \vee O_\sigma(v)$ for all $uv \in \sigma^*$.

Definition 2.15. The μ-complement of an IVFG $G : (\sigma, \mu)$ is an IVFG $G^\mu : (\sigma, \mu^\mu)$ where μ^μ is defined as $\mu^\mu(u, v) = [\mu_{uv}^{\mu-}, \mu_{uv}^{\mu+}]$ where $\mu^\mu(u, v) = [0, 0]$ if $\mu(u, v) = [0, 0]$ otherwise $\mu_{uv}^{\mu-} = \sigma_u^- \wedge \sigma_v^- - \mu_{uv}^-$ and $\mu_{uv}^{\mu+} = \sigma_u^+ \wedge \sigma_v^+ - \mu_{uv}^+$ for all $u, v \in \sigma^*$.

Definition 2.16. A path P of length n is a sequence of distinct nodes $u_0, u_1, \ldots, u_n$ such that $r(u_i, u_{i-1}) > 0$ and degree of membership of weakest link is defined as its strength. If $u_0 = u_n$ and $n \ge 3$ then P is called a cycle and it is a fuzzy cycle if there is more than one weak link. A fuzzy graph is called connected fuzzy graph if there exists at least one path between every pair of its nodes.

Definition 2.17. The strength of connectedness between two nodes a, b is defined as the maximum strengths of all paths between a and b and is denoted by CONNF (a, b). A link (a, b) is called a fuzzy bridge in F if the removal of (a, b) reduces the strength of connectedness between some pair of nodes in F. A connected fuzzy graph is called a fuzzy tree if it contains a spanning subgraph H which is a tree such that for all link (a, b) not in H, $r(a, b) < CONNH(a, b)$. A link (a, b) in F is called m-strong if $r(a, b) = m(a) \wedge m(b)$.

Example 2.1.

IVWFG G:

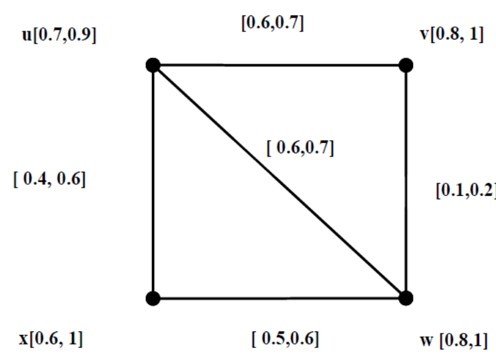

Figure 2.1.

Example 2.2.

G:

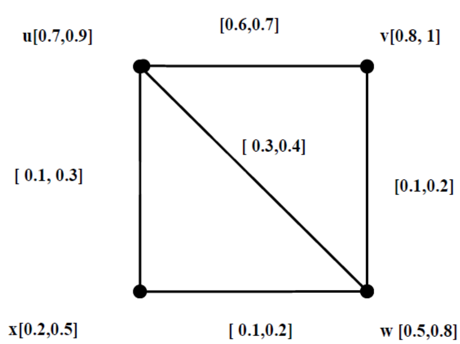

Figure 2.2.

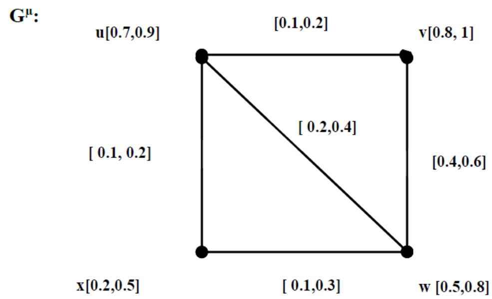

Figure 2.3

3. Some Connectivity Properties of Weak Fuzzy Graphs

Theorem 3.1. *Let F be a connected weak fuzzy graph, then F^c is a connected fuzzy graph.*

Proof. Since $F:(m, r)$ is a connected weak fuzzy graph there exists a path between every pair of nodes in F and $r(a, b) < m(a) \wedge m(b)$ for all $(a, b) \in r^*$.

To prove that there exists a path between every pair of nodes in $F^c:(m^c, r^c)$, where $m^c = m$ and r^c is defined as $r^c(a, b) = m(a) \wedge m(b)$ $-r(a, b)$.

Since $r(a, b) < m(a) \wedge m(b)$ for all $(a, b) \in r^*$, $r^c(a, b) > 0$ for all $(a, b) \in r^*$ and so all $(a, b) \in r^*$ is also belong to r^{c*}. The case $r(a, b) = 0$ for any $(a, b) \in r^*$ is trivial.

Therefore there exists a path between every pair of nodes in r^{c*}. Therefore F^c is a connected fuzzy graph.

Theorem 3.2. *Let F be a connected weak fuzzy graph, then F^{μ} is a connected weak fuzzy graph.*

Proof. Since $F:(m, r)$ is a connected weak fuzzy graph there exists a path between every pair of nodes in F and $r(a, b) < m(a) \wedge m(b)$ for all $(a, b) \in r^*$.

To prove that there exists a path between every pair of nodes in $F^{\mu} : (m^{\mu}, r^{\mu})$, where $m^{\mu} = m$ and r^{μ} is defined as $r^{\mu}(a, b) = m(a) \wedge m(b) - r(a, b)$ for all $(a, b) \in r^{*}$.

Since $r(a, b) < m(a) \wedge m(b)$ for all $(a, b) \in r^{*}, r^{\mu}(a, b) > 0$ for all $(a, b) \in r^{*}$. By the definition of μ complement, $F^{\mu} : (m^{\mu}, r^{\mu})$ is a weak fuzzy graph having all active links (i.e., links (a, b) such that $r(a, b) > 0$) of F. i.e., $r^{\mu}(a, b) \neq 0$ for any $(a, b) \in r^{*}$. Hence all $(a, b) \in r^{*}$ is also belonging to $r^{\mu *}$. Hence for all (a, b) in $r^{\mu *}$, there exists a path between a and b. Therefore F^{μ} is a connected fuzzy graph.

4. Some Algebraic Properties of Weak Fuzzy Graphs

Theorem 4.1. *The set of weak fuzzy graphs is an abelian semi group with respect to union of fuzzy graphs.*

Proof. In [4], it is proved that the set of weak fuzzy graphs is closed under union. Let $F_1 : (m_1, r_1), F_2 : (m_2, r_2)$ and $F_3 : (m_3, r_3)$ be three weak fuzzy graphs with the underlying crisp graphs $F_1^{*} : (N_1, X_1), F_2^{*} : (N_2, X_2)$ and $F_3^{*} : (N_3, X_3)$ respectively where $N_1 = m_1^{*}, N_2 = m_2^{*}, N_3 = m_3^{*}$, $X_1 = r_1^{*}, X_2 = r_2^{*}$ and $X_3 = r_3^{*}$. The union $F_i \cup F_j = G_{i,j} : (m_i \cup m_j, r_i \cup r_j)$ is defined by

$(m_i \cup m_j)(a) = m_i(a)$ if $a \in N_i - N_j$.

$(m_i \cup m_j)(a) = m_j(a)$ if $a \in N_j - N_i$, and

$(m_i \cup m_j)(a) = \max\{m_i(a), m_j(a)\}$ if $a \in N_i \cap N_j$.

$(r_i \cup r_j)(a, b) = r_i(a, b)$ if $(a, b) \in X_i - X_j$.

$(r_i \cup r_j)(a, b) = r_j(a, b)$ if $(a, b) \in X_j - X_i$, and

$(r_i \cup r_j)(a, b) = \max\{r_i(a, b), r_j(a, b)\}$ if $(a, b) \in X_i \cap X_j$. Where $i \neq j, i, j = 1, 2, 3$.

If $a \in N_1 - N_2$ then $(m_1 \cup m_2)(a) = m_1(a) = (m_2 \cup m_1)(a)$

If $a \in N_2 - N_1$ then $(m_1 \cup m_2)(a) = m_2(a) = (m_2 \cup m_1)(a)$

If $a \in N_1 \cap N_2$ then

$(m_1 \cup m_2)(a) = \max\{m_1(a), m_2(a)\} = \max\{m_2(a), m_1(a)\}$

$= (m_2 \cup m_1)(a)$

If $(a, b) \in X_1 - X_2$ then $(r_1 \cup r_2)(a, b) = r_1(a, b) = (r_2 \cup r_1)(a, b)$

If $(a, b) \in X_2 - X_1$ then $(r_1 \cup r_2)(a, b) = r_2(a, b) = (r_2 \cup r_1)(a, b)$

If $(a, b) \in X_1 \cap X_2$ then

$(r_1 \cup r_2)(a, b) = \max\{r_1(a, b), r_2(a, b)\} = \max\{r_2(a, b), r_1(a, b)\}$

$= (r_2 \cup r_1)(a, b).$

Therefore the union of weak fuzzy graphs is commutative.

To prove $(F_1 \cup F_2) \cup F_3 = F_1 \cup (F_2 \cup F_3)$

If $a \in N_1 \cap N_2 \cap N_3$

Then $((m_1 \cup m_2) \cup m_3)(a) = \max\{\max\{m_1(a), m_2(a)\}, m_3(a)\}$

$= \max\{m_1(a), \max\{m_2(a), m_3(a)\}\} = (m_1 \cup (m_2 \cup m_3))(a)$

If $(a, b) \in X_1 \cap X_2 \cap X_3$ then

$((r_1 \cup r_2) \cup r_3)(a, b) = \max\{\max\{r_1(a, b), r_2(a, b)\}, r_3(a, b)\}$

$= \max\{r_1(a, b), \max\{r_2(a, b), r_3(a, b)\}\} = (r_1 \cup (r_2 \cup r_3))(a, b)$

If $a \in N_1 - (F_2 \cup F_3)$

Then $((m_1 \cup m_2) \cup m_3)(a) = m_1(a) = (m_1 \cup (m_2 \cup m_3))(a)$

If $(a, b) \in X_1 - (X_2 \cup X_3)$

Then $((r_1 \cup r_2) \cup r_3)(a, b) = r_1(a, b) = (r_1 \cup (r_2 \cup r_3))(a, b)$

Similarly other cases shall be proved.

Therefore the union of weak fuzzy graphs is associative.

Theorem 4.2. *All fuzzy labeling graphs are weak fuzzy graphs but the converse is not true.*

Proof. In every fuzzy labeling graph, the membership value given to each link is less than the minimum of the membership values of the corresponding nodes. Hence every fuzzy labeling graph is a weak fuzzy graph. In weak fuzzy graph the labels assigned to links and vertices need not be distinct and so the membership functions are not restricted to bijective. Hence the proof.

Corollary 4.2.1. *Fuzzy magic graphs are fuzzy labeling graph and so all fuzzy magic graphs are weak fuzzy graphs.*

Theorem 4.3. *IVFG is not closed under* μ*-complementation but IVFG holds law of double complementation with respect to* μ*-complement.*

Proof.

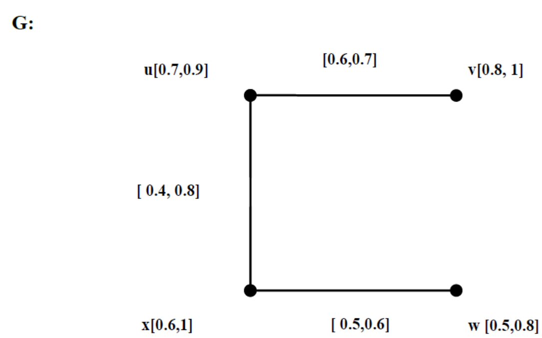

Figure 4.1.

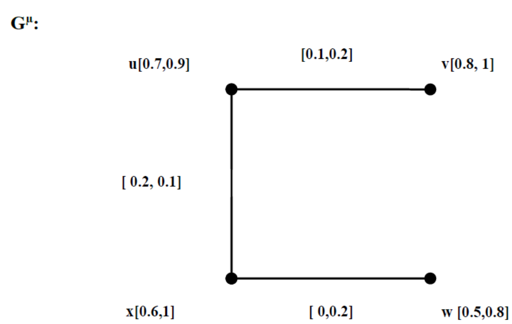

Figure 4.2.

In Figure 4.2, the link ux violates the requirement $\mu(u, v) = [\mu^{-}_{uv}, \mu^{+}_{uv}]$, $0 \leq \mu^{-}_{uv} \leq \mu^{+}_{uv} \leq 1$ for IVFG, Hence G^{μ} is not an IVFG.

To prove that IVFG holds the law of double complementation, we shall consider the following cases.

Case 1. $\mu^{\mu}(u, v) = [0, 0]$ then as per definition $(\mu^{\mu})\mu(u, v) = [0, 0]$.

Case 2. If $\mu^{\mu}(u, v) \neq [0, 0]$, $\mu^{\mu-}_{uv} = \sigma^{-}_{u} \wedge \sigma^{-}_{v} - \mu^{-}_{uv}$ and $\mu^{\mu+}_{uv} = \sigma^{+}_{u} \wedge \sigma^{+}_{v} - \mu^{+}_{uv}$ for all $u, v \in \sigma^{*}$

$(\mu^{\mu})^{\mu-}_{uv} = \sigma^{-}_{u} \wedge \sigma^{-}_{v} - \mu^{\mu-}_{uv}$ and $(\mu^{\mu})^{\mu+}_{uv} = \sigma^{+}_{u} \wedge \sigma^{+}_{v} - \mu^{\mu+}_{uv}$ for all $u, v \in \sigma^{*}$

$(\mu^{\mu})^{\mu-}_{uv} = \sigma^{-}_{u} \wedge \sigma^{-}_{v} - (\sigma^{-}_{u} \wedge \sigma^{-}_{v} - \mu^{-}_{uv}) = \mu^{-}_{uv}$ and

$(\mu^{\mu})^{\mu+}_{uv} = \sigma^{+}_{u} \wedge \sigma^{+}_{v} - (\sigma^{+}_{u} \wedge \sigma^{+}_{v} - \mu^{+}_{uv}) = \mu^{+}_{uv}$ for all $u, v \in \sigma^{*}$

Therefore $(\mu^{\mu})^{\mu}(u, v) = [\mu^{-}_{uv}, \mu^{+}_{uv}] = \mu(u, v)$. Hence $(G^{\mu})^{\mu} = G$.

Theorem 4.4. *IVWFG is closed under* μ*-complementation and holds law of double complementation with respect to* μ*-complement.*

Proof. To prove the closure property, it is enough to prove that the μ-complement of an IVWFG is an IVWFG. The proof of holding law of double complementation is similar to that of Theorem 4.3.

If $\mu(u, v) = [\mu^{-}_{uv}, \mu^{+}_{uv}] = [0, 0]$, it is considered as trivial case and $\mu^{\mu}(u, v) = [0, 0]$

Since $\mu^{+}_{uv} - \mu^{-}_{uv} < \sigma^{+}_{u} \wedge \sigma^{+}_{v} - \sigma^{-}_{u} \wedge \sigma^{-}_{v}$,

$$\sigma^{-}_{u} \wedge \sigma^{-}_{v} - \mu^{-}_{uv} < \sigma^{+}_{u} \wedge \sigma^{+}_{v} - \mu^{+}_{uv} \tag{1}$$

If $\mu^{\mu}(u, v) \neq [0, 0]$,

$$\mu^{\mu-}_{uv} = \sigma^{-}_{u} \wedge \sigma^{-}_{v} - \mu^{-}_{uv} \text{ and } \mu^{\mu+}_{uv} = \sigma^{+}_{u} \wedge \sigma^{+}_{v} - \mu^{+}_{uv} \text{ for all } u, v \in \sigma^{*} \tag{2}$$

Therefore equation (1) implies that in an IVWFG, $\mu^{\mu-}_{uv} < \mu^{\mu+}_{uv}$ for all

$u, v \in \sigma^*$. This shows that the μ-complement of IVWFG is IVWFG.

To prove that it holds the conditions $\mu_{uv}^{\mu-} < \sigma_u^- \wedge \sigma_v^-$ and $\mu_{uv}^{\mu+} < \sigma_u^+ \wedge \sigma_v^+$.

It is obvious from equation (2) since $\mu_{uv}^{\mu-}, \mu_{uv}^{+} > 0$ for all $(u, v) \in \mu^*$.

To prove the inequality $\mu_{uv}^{\mu+}, \mu_{uv}^{\mu-} < \sigma_u^+ \wedge \sigma_v^+ - \sigma_u^- \wedge \sigma_v^-$ for all $u, v \in \sigma^*$.

$$LHS = \sigma_u^+ \wedge \sigma_v^+ - \mu_{uv}^+ - (\sigma_u^- \wedge \sigma_v^- - \mu_{uv}^-) = \sigma_u^+ \wedge \sigma_v^+ - \sigma_u^- \wedge \sigma_v^- - (\mu_{uv}^+ - \mu_{uv}^-)$$

$$< \sigma_u^+ \wedge \sigma_v^+ - \sigma_u^- \wedge \sigma_v^- \text{ for all } u, v \in \sigma^*.$$

5. Some Applications of Weak Fuzzy Graphs

Fuzzy magic graphs is used to model facility sharing networks [5]. But fuzzy magic graphs cannot address all assumptions. There is possibility to assign same percentage of facility to more than one node and the total percentage of facility to each node need not be constant. To overcome the flaws of fuzzy magic graphs, weak fuzzy graphs is the best that covers all possible assumptions in facility sharing networks. To simulate anonymous communication networks and signal strength variation in simulation of neural networks, the assumptions include uncertainty in varying traffic density, mitigation of traffic jams, the limits in variation of traffic density and the limits in variation in presence of nodes [7]. IVWFG captures these assumptions.

Theorem 5.1. *$FWFG(n)$ implies but the converse is not true.*

Proof. Let $G : (\sigma, \mu)$ be a $FWFG(n-1)$ with σ as FFS on A and μ as FFR on A.

Since $I_\mu(uv) < 1, I_\mu^{n-1}(uv) < 1 \Rightarrow I_\mu^n(uv) < I_\mu^{n-1}(uv)$, for all $n \in N$ (1)

Similarly since $O_\mu(uv) < 1, O_\mu^{n-1}(uv) < 1 \Rightarrow O_\mu^n(uv) < O_\mu^{n-1}(uv)$

Therefore $I_\mu^{n-1}(uv) + O_\mu^{n-1}(uv) < 1 \Rightarrow I_\mu^n(uv) + O_\mu^n(uv) < 1$.

Hence $FWFG(n-1) \Rightarrow FWFG(n)$. It is obvious from equation (1) that the converse is not true.

Theorem 5.2. *There exists* $FFG(n-1)$ *such that* $FFG(n) \Leftrightarrow FFG(n-1),$ $n \geq 2.$

Proof. Let $G : (\sigma, \mu)$ be a $FFG(n-1)$ with σ as FFS on A and μ as FFR on A. Suppose

$$I_{\mu}(uv) = 1, \text{ for all } uv \in \mu^{*} \text{ then } I_{\mu}^{n-1}(uv) = 1$$

$$\Rightarrow I_{\mu}^{n}(uv) = I_{\mu}^{n-1}(uv), \text{ for all } n \in N$$

Similarly Suppose,

$$O_{\mu}(uv) = 0, \text{ for all } uv \in \mu^{*} \text{ then } O_{\mu}^{n-1}(uv) = 0$$

$$\Rightarrow O_{\mu}^{n}(uv) = O_{\mu}^{n-1}(uv), \text{ for all } n \in N$$

Therefore there exists $FFG(n-1)$ such that $FFG(n-1) \Leftrightarrow FFG(n).$

Theorem 5.3. *A* $FWF(n)$ *can be obtained from every* $FFG(n)$ *where alternatives are present i.e., when* $I_{\sigma}(u) = 1$ *and* $O_{\sigma}(u) = 0$ *for all u in* $\sigma^{*}.$

Proof. We shall construct a $FWF(n)$ from a given $FFG(n).$ in the following method.

Step 1. Let $I_{\mu}^{n}(st) = Min\{I_{\mu}^{n}(uv)/uv \in \mu^{*}\}$ in $FFG(n).$

Reduce $\varepsilon \in (0, \sqrt[n]{I_{\mu}^{n}(st)})$ from all $I_{\mu}(uv) \in \mu^{*}$ except from those $uv \in \mu^{*}$ where $I_{\mu}(uv) = 0.$

Step 2. Let $O_{\mu}^{n}(ab) = Max\{O_{\mu}^{n}(uv)/uv \in \mu^{*}\}$ in $FFG(n).$

Add $\delta \in (0, 1 - \sqrt[n]{O_{\mu}^{n}(ab)})$ to all $O_{\mu}(uv) \in \mu^{*}$

Now the resulting $FFG(n)$ satisfy

$I_{\mu}(uv) < I_{\sigma}(u) \wedge I_{\sigma}(v), O_{\mu}(uv) > O_{\sigma}(u) \vee O_{\sigma}(v)$ for all $uv \in \mu^{*}.$ Hence the proof.

Example 5.1. Mr. X wish to invest money in any of the following 5 schemes that helps him better financial security.

1. Public Provident Fund (S_1)

2. National Saving Certificate (S_2)

3. Atal Pension Yojana (S_3)

4. National Pension Scheme (S_4)

5. Sovereign Gold Bonds (S_5)

He consulted with 4 experts and they advised the merits and demerits of each particular scheme comparing with other. How can he select the best option?

Mathematical Modeling using $FFG(n)$: Consider the discrete set of alternatives $A = \{S_1, S_2, S_3, S_4, S_5\}$. Since the alternatives are present, assign the membership degree as 1 and non membership degree 0 to each alternatives. Consider the set of experts as $\{E_1, E_2, E_3, E_4\}$. Since each expert gives the acceptance and rejection reasons comparing every pair of alternatives, the aggregate of information FFPR can be represented as relation matrices. This data represents a $FFG(n)$. If in the given FFPRs, all the membership values are in (0, 1) and non membership values are greater than 0 then the given $FFG(n)$ will be $FWG(n)$. The best option can be obtained by using the generalized decision tools [12].

6. Conclusion

In this article the concept interval valued weak fuzzy graphs and Fermat's fuzzy graphs are introduced. It is proved that the set of weak fuzzy graphs is an abelian semi group with respect to union. Fuzzy labeling graphs and fuzzy magic graphs are found as subset of set of weak fuzzy graphs. Some connectivity, complement properties of weak fuzzy graphs and some properties of Fermat's Fuzzy graphs are derived. The generalization of decision tools [12] and applications of $FFG(n)$ in various fields are under investigation.

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