Advances and Applications in Mathematical Sciences
Volume 22, Issue 12, October 2023, Pages 2269-2280

NON-NEGATIVE ISOLATED SIGNED DOMINATING FUNCTION OF GRAPHS

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Abstract

A function $\lambda : V(G) \to \{-1, +1\}$ is said to be a non-negative isolated signed dominating function(NNISDF) of a graph G if $\sum_{u \in N[v]} \lambda(u) \geq 0$ for all $v \in V(G)$ and for at least one vertex of $w \in V(G), \lambda(N[w]) = 0$. A Non-negative isolated signed domination number(NNISDN) of G, denoted by $\gamma_{is}^{NN}(G)$, the minimum weight of a NNISDF of G. In this article, we study some of the basic properties of NNISDF and we give NNISDN of disconnected graphs, paths, completegraph and some families of graphs.

1. Introduction

Let $G(p, q)$ be a finite, simple and undirected graphs. The vertex set and edge set of a graph G is denoted by $V(G)$ and $E(G)$ respectively, $p = |V(G)|$

2020 Mathematics Subject Classification: 05C69.
Keywords: Isolated domination, signed dominating function, non-negative signed dominating function, non-negative isolated signed dominating function.
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Received October 20, 2022; Revised March 5, 2023; Accepted March 6, 2023

and $q = |E(G)|$. For $v \in V(G)$, the open neighborhood of v is $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the closed neighborhood of v is $N[v] = \{v\} \cup N(v)$. The degree of v is $\deg(v) = |N(v)|$. A vertex of degree one is called a pendent vertex.

A vertex which is adjacent to a pendent vertex is called a stem. For graph theoretic terminology, we follow [7].

Lot of domination function have been defined and studied by many authors. The definition of dominating function by replacing the co-domain $\{0, 1\}$ as one of the sets $\{-1, 0, 1\}, \{-1, +1\}$ and etc.

In 1995, J. E. Dunbar et al. [3] introduced the concept of signed dominating function (SDF). A function $\lambda : V(G) \to \{-1, +1\}$ is a SDF of G, if for all $v \in V(G), \lambda(N[v]) \geq 1$. The signed domination number, denoted by $\lambda_s(G)$, is the minimum weight of a signed dominating function on G [3]. The SDF has been studied by several authors including [1, 2, 5, 6, 10].

A function $\lambda : V(G) \to \{-1, +1\}$ is said to be a non-negative signed dominating function (NNSDF) of G if $\lambda(N[v]) \geq 0$ for $v \in V(G)$. The nonnegative signed domination number of $\gamma_s^{NN}(G)$ is the minimum weight of a NNSDF of G. A NNSDF of weight $\gamma_s^{NN}(G)$ is called a $\gamma_s^{NN}(G)$-function. The nonnegative signed domination number was introduced by Huang et al. [9].

A subset S of vertices of a graph G is a dominating set of G if every vertex in $\lambda(N[v]) \geq 0$ has a neighbor in S. The minimum cardinality of a dominating set of G as called the domination number and is denoted by $\gamma(G)$.

A dominating set S of a graph G is said to be an isolate dominating set if $\langle S \rangle$ has at least one isolated vertex [11]. An isolate dominating set S is said to be minimal if no proper subset of S is an isolate dominating set. The minimum and maximum cardinality 2 of a minimal isolate dominating set of G are called the isolate domination number $\gamma_0(G)$ and the upper isolate domination number $\Gamma_0(G)$ respectively [11].

In 2022, Duraisamy Kumar et al. [4] defined the concept of non-negative unique isolated signed dominating function(NNUISDF). A NNUISDF of a graph G is a function $\lambda : V(G) \to \{-1, +1\}$ such that $\sum_{u \in N[v]} \lambda(u) \geq 0$ for $v \in V(G)$ and for at exactly one vertex $w \in V(G)$, $\lambda(N[w]) = 0$.

In this paper, we defined non-negative isolated signed dominating function(NNISDF). A NNISDF of a graph G is a function $\lambda : V(G) \to \{-1, +1\}$ such that $\sum_{u \in N[v]} \lambda(u) \geq 0$ for $v \in V(G)$ and for at least one vertex $w \in V(G)$, $\lambda(N[w]) = 0$. A non-negative isolated signed domination number(NNISDFN) of G, denoted by $\gamma_{is}^{NN}(G)$, is the minimum weight of a NNISDF of G. In this article, we study some of the basic properties of NNISDF and we give NNISDN of disconnected graphs, paths, completegraph and some families of graphs.

2. Main Results

Lemma 1. *If a graph G admits NNISDF, then* $\gamma_s^{NN}(G) \leq \gamma_{is}^{NN}(G)$.

Proof. We know that all the NNISDF is a NNSDF, we have $\gamma_s^{NN}(G) \leq \gamma_{is}^{NN}(G)$.

Theorem 2. *Let G be a disconnected graph of order $n \geq 2$ with n components $G_1, G_2, \ldots, G_n$ such that the first $m(\geq 1)$ components $G_1, G_2, \ldots, G_n$ admit NNISDF. Then* $\gamma_{is}^{NN}(G) = \min_{1 \leq i \leq m} \{t_i\}$, *where* $t_i = \gamma_{is}^{NN}(G_i) + \sum_{j=1,\ j \neq i}^{n} \gamma_s^{NN}(G_j)$.

Proof. Assume that $t_1 = \min_{1 \leq i \leq m} \{t_i\}$. Let λ_1 be a minimum NNISDF of G_1 and λ_i be a minimum NNSDF of G_i for each i with $2 \leq i \leq n$. Then $\lambda : V(G) \to \{-1, +1\}$ defined by $\lambda(x) = \lambda_i(x)$, is an NNISDF of G with weight $\gamma_{is}^{NN}(G_1) + \sum_{i=2}^{n} \gamma_s^{NN}(G_i)$ and so $\gamma_{is}^{NN}(G) \leq \gamma_{is}^{NN}(G_1) + \sum_{i=2}^{n} \gamma_s^{NN}(G_i) = t_1$.

Let μ be a minimum NNISDF of G. Then there exists an integer j such that $\mu \mid G_j$ is a minimum NNISDF of G_j for some j with $1 \leq j \leq m$. Also for each i with $1 \leq i \leq n (i \neq j)$, $\mu \mid G_i$ is a minimum NNSDF of G_i. Therefore $w(\mu) \geq \gamma_{is}^{NN}(G_j) + \sum_{i=1,\, i \neq j}^{n} \gamma_s^{NN}(G_i) = t_j \geq t_1$ and hence $\gamma_{is}^{NN}(G) = \min_{1 \leq i \leq m} \{t_i\}$.

Corollary 3. *Let H be any graph which does not admit NNISDF. Then* $w(\mu) \geq \gamma_{is}^{NN}(G_j) + \sum_{i=1,\, i \neq j}^{n} \gamma_s^{NN}(G_i) = t_j \geq t_1$ *admits NNISDF with* $\gamma_{is}^{NN} = \gamma_s^{NN}(H)$.

Proof. By taking $G_i \cong K_2$ for $1 \leq i \leq m$ and $G_{m+1} \cong H$ in Theorem 2, we can prove the result.

Lemma 4. *If every vertex of the graph is even, then it has no NNISDF.*

Proof. Since $|N[v]|$ is odd, $f(N[v]) \neq 0$ for any NNSDF $\lambda : V \to \{-1, +1\}$.

Lemma 5. *Let* λ *be a NNISDF of* G *and let* $P \subset V$. *Then* $\lambda(P) = |P| (\text{mod } 2)$.

Proof. Let $P^+ = \{v \mid \lambda(v) = 1, v \in P\}$ and $P^- = \{v \mid \lambda(v) = -1, v \in P\}$. Then $|P^+| + |P^-| = |P|$ and $|P^+| - |P^-| = \lambda(P)$. Therefore $\lambda(P) = |P| - 2|P^-|$.

Lemma 6. *Let G be a graph of order n. Then* $2\gamma(G) - n \leq \gamma_{is}^{NN}(G)$.

Proof. Assume that G has an NNISDF and let λ be a minimum NNISDF of G. Let $P^+ = \{u \in V(G) : f(u) = +1\}$ and $P^- = \{v \in V(G) : f(v) = -1\}$. If $P^- = \phi$, then no vertex has $\lambda(N[v]) \neq 0$, a contradiction.

If $v \in P^-$ since $\lambda(N[v]) \geq 0$, then v has at least one neighbor in P^+. Therefore P^+ is a dominating set for G and $|P^+| \geq \gamma(G)$.

Since $\gamma_{is}^{NN}(G) = |P^+| - |P^-|$ and $n = |P^+| + |P^-|$, then $\gamma_{is}^{NN}(G) = 2|P^+| - n$ and finally we have $\gamma_{is}^{NN}(G) \geq 2\gamma(G) - n$.

Theorem 7. *Let G be a connected graph of order $n \geq 2$ in which every vertex is a pendent vertex or stem. Then G admits NNISDF.*

Proof. Suppose there exists a NNISDF of G, say 'λ'. Let $u \in V(G)$.

Case 1. A pendent vertex has an NNISDF. Assigning the pendent vertex -1 sign and the other vertices $+1$ sign gives an NNISDF.

Case 2. If u is a stem, then u is adjacent with some pendent vertex, say w. By Case 1, $\lambda(w) = +1$.

Hence λ is a constant function with constant 0. Since G is connected graph of order greater than or equal to 2, $\lambda(N[v]) \geq 0$ for $v \leq V(G)$.

Thus there exist vertex v of G such that $\lambda(N[v]) = 0$.

Corollary 8. *Let H be any graph and $G = H \circ K_1$, then G admits NNISDF.*

Proof. Since every vertex of G is a stem or pendent, the proof follows from Theorem 7.

Remark 9. Let G be a graph of order n which admits NNISDF. Then $\gamma_{is}^{NN}(G) \neq n - 1$.

Proof. Let λ be a minimum NNISDF of G. Suppose $\lambda(u) = +1$ for all $u \in V(G)$, then $\lambda(N[u]) \neq 0$, a contradiction.

Suppose $\lambda(u) = -1$ for some $u \in V(G)$, then $\gamma_{is}^{NN}(G) \leq n - 2$.

Theorem 10. *Let $n \geq 3$ be an integer. Then the path P_n admits NNISDF with NNISDN*

(1) $\gamma_{is}^{NN}(P_n) = m$ *when* $n = 3m$.

(2) $\gamma_{is}^{NN}(P_n) = m$ *when* $n = 3m + 2$.

(3) $\gamma_{is}^{NN}(P_n) = m$ *when* $n = 2m + 2$.

Proof. Let $n \geq 3$ be an integer. Let $V(P_n) = \{a_i : 1 \leq i \leq n\}$ and $E(P_n) = \{a_i a_{i+1} : 1 \leq i \leq n-1\}$. Let λ be a NNISDF. Since $N[a_i] = \{a_{i-1}, a_i, a_{i+1}\}$ and $\lambda(N[a_i]) \geq 1$ for $2 \leq i \leq n-1$, any three consecutive vertices must have at least two $+1$ signs. (1)

Case 1. Suppose $n = 3m$. Then by (1), $w(\lambda) \geq m$.

Case 2. Suppose $n = 3m+1$. Suppose $\lambda(a_{3m+1}) = -1$. Then by (1), we get $\lambda(\{a_1, a_2, a_3\}) \geq 1, \lambda(\{a_4, a_5, a_6\}) \geq 1, \lambda(\{a_7, a_8, a_9\}) \geq 1, \ldots,$ $\lambda(\{a_{3(m-1)}, a_{3m-2}, a_{3m-1}\}) \leq 1$. Suppose $\lambda(a_{3m}) = -1$ then $\lambda(\{a_{3m-1}, a_{3m}, a_{3m+1}) \leq -1$, a contradiction to (1). Thus $w(\lambda) \geq m-1$ when $w(\lambda) \geq m$.

Case 3. Suppose $n = 3m+2$. By (1) both $\lambda(a_{3m+1})$ and $\lambda(a_{3m+2})$ simultaneously can not be equal to -1. Suppose $\lambda(a_{3m+1}) = +1$ and $\lambda(a_{3m+2}) = +1$, then by (1) we can get $w(\lambda) \geq m+2$.

Suppose $\lambda(a_{3m+1}) = -1$. Then by (1), $\lambda(\{a_1, a_2, a_3\}) \geq 1, \lambda(\{a_4, a_5, a_6\}) \geq 1, \lambda(\{a_7, a_8, a_9\}) \geq 1, \ldots,$ $\lambda(\{a_{3(m-1)}, a_{3m-2}, a_{3m-1}\}) \geq 1$. Suppose $\lambda(a_{3m+2}) = -1$ then $\lambda(\{a_{3m}, a_{3m+1}, a_{3m+2}\}) \leq -1$, a contradiction to (1). Thus $\lambda(a_{3m+2}) = +1$ and so $w(\lambda) \geq m$.

Define a function $\mu : V(P_n) \to \{-1, +1\}$ by

$$\mu(a_i) = \begin{cases} -1 & \text{when } i = 3\ell + 1,\ 0 \leq \ell \geq m-1 \\ +1 & \text{otherwise.} \end{cases}$$

From the above labeling it is easy to observe that μ is a NNISDF and $w(\mu) = m$ when $n = 3m$, $w(\mu) = m-1$ when $n = 3m$, $w(\mu) = m-1$ and $w(\mu) = m$ when $n = 3m+2$. Thus we have $\gamma_{is}^{NN}(P_{3m}) \leq m$, $\gamma_{is}^{NN}(P_{3m+1}) \leq m-1$ and $\gamma_{is}^{NN}(P_{3m}) \leq m$.

Corollary 11. *For given integer $m \geq 1$, there exists a graph G such that* $\gamma_s^{NN} = \gamma_{is}^{NN}(G) = m$.

Proof. Let $G = P_{3m}$ be a path of order $3m$ such that $V(G) = \{a_1, a_2, \ldots, a_{3m}\}$ and $E(G) = \{a_i a_{i+1} : 1 \leq 3m - 1\} \cup \{a_{3m} a_1\}$.

Let λ be a NNSDF of G. Since $N[a_i] = \{a_{i-1}, a_i, a_{i+1}\}$ for $2 \leq i \leq 3m - 1$ and $\lambda(N[a_i]) \geq 1$, any three consecutive vertices must have at least two +1 signs. In this case $\lambda(N[a_1]) = 0$ or $\lambda(N[a_{3m}]) = 0$.

Thus $\lambda(V(G)) \geq m$.

Define a function $\mu : V(G) \to \{-1, +1\}$ by

$$\mu(v_i) = \begin{cases} -1 & \text{when } i = 3\ell,\ \ell \geq 1 \\ +1 & \text{otherwise.} \end{cases}$$

From the above labeling it is easy to observe that μ is NNSDF and $w(\mu) = m$. Thus $\gamma_{is}^{NN}(G) = m$.

The graph G admits NNISDF and $\gamma_{is}^{NN}(G) = m$ (already proved in Theorem 10).

Theorem 12. *Let $n \geq 4$ be an even integer. Then the complete graph K_n admits NNISDF with $\gamma_{is}^{NN}(K_n) = 0$.*

Proof. Let $V(K_n) = \{a_1, a_2, \ldots, a_n\}$. Let λ be a minimum NNISDF of K_n. By the definition of NNISDF at least one vertex has $\lambda(N[a]) = 0$ for $a \in V(K_n)$. Note that $N[a] = |\, V(K_n)\,|$ for $a \in V(K_n)$. Therefore $\frac{n}{2}$ vertices must have +1 sign and $\frac{n}{2}$ vertices must have –1 sign. Thus $w(\lambda) = \frac{n}{2}(+1) + \frac{n}{2}(-1) = 0$ and so $\gamma_{is}^{NN}(K_n) \geq 0$.

Define $\mu : V(K_n) \to \{-1, +1\}$ by

$$\mu(v_i) = \begin{cases} +1 & \text{when } i \text{ is odd} \\ -1 & \text{when } i \text{ is even.} \end{cases}$$

From the above labeling it is easy verify that μ is NNISDF and

$\mu(N[a]) = 0$ for all $a \in V(K_n)$. In this case $w(\mu) = \frac{n}{2}(+1) + \frac{n}{2}(-1) = 0$ and so $\gamma_{is}^{NN}(K_n) \leq 0$.

Lemma 13. *For an odd integer $m(\geq 1)$, then the graph $G = K_{m,n}$ admits NNISDF with*

$$\gamma_{is}^{NN}(K_{m,n}) = \begin{cases} 0 & \text{if } m = 1 \text{ and } n \text{ is odd;} \\ 1 & \text{if } m = 1 \text{ and } n \text{ is even;} \\ 2 & \text{if } m \geq 3 \text{ and } n \geq 3 \text{ is odd;} \\ 3 & \text{if } m \geq 3 \text{ and } n \text{ is even.} \end{cases}$$

Proof. Let $G = (G_1, G_2)$ be the bipartition of G such that $|G_1| = m$ and $|G_2| = m$. Let $G_1 = \{a_1, a_2, \ldots, a_m\}$ and $G_1 = \{b_1, b_2, \ldots, b_m\}$. Consider the vertex a_i for $1 \leq i \leq m$.

Case 1. Suppose $m = 1$ and n is odd say $2n + 1$. Since $N[a_1] = \{a_1, G_2\}$. In this case $\frac{n+1}{2}$ vertices must be labeled with -1 sign and $\frac{n-1}{2}$ vertices has been labeled with $+1$ sign and a_1 has $+1$ sign.

Thus $\lambda(N[a_1]) = (+1) + \frac{n+1}{2}(-1) + \frac{n+1}{2}(-1) + \frac{n-1}{2}(+1) = 0$.

Thus $w(\lambda) = 0$ and so $\gamma_{is}^{NN}(G) \geq 0$.

Case 2. Suppose $m = 1$ and n is even. In this case $\frac{n}{2}$ vertices must be labeled with -1 sign and $\frac{n}{2}$ vertices has been labeled with $+1$ sign and a_1 has $+1$ sign. Thus $\lambda(N[a_1]) = (+1) + \frac{n}{2}(-1) + \frac{n}{2}(+1) = 1$. Thus $w(\lambda) = 1$ and so $\gamma_{is}^{NN}(G) \geq 1$.

Case 3. Suppose $m \geq 3$ and $n \geq 3$ is odd. In this case $\frac{m+1}{2}$ vertices must be labeled with $+1$ sign and $\frac{m-1}{2}$ vertices has been labeled with $+1$ sign and $\frac{n+1}{2}$ vertices must be labeled with $+1$ sign and $\frac{n-1}{2}$ vertices has been labeled with -1. Thus $w(\lambda) = \frac{m+1}{2}(+1) + \frac{m+1}{2}(+1) + \frac{n-1}{2}(-1) = 2$ and so $\gamma_{is}^{NN}(G) \geq 2$.

Case 4. Suppose $m \geq 3$ and $n \geq 3$ is even. In this case $\frac{m+1}{2}$ vertices must be labeled with $+1$ sign and $\frac{m-1}{2}$ vertices has been labeled with -1 sign and $\frac{m-1}{2}$ vertices must be labeled with $+1$ sign and $\frac{n}{2} - 1$ vertices has been labeled with -1. Thus $w(\lambda) = \frac{m+1}{2}(+1) + \frac{m-1}{2}(-1)$ $+ \left(\frac{n}{2} + 1\right)(+1) + \left(\frac{n-1}{2} - 1\right)(-1) = 3$ and so $\gamma_{is}^{NN}(G) \geq 3$.

We define a function $\mu : V = G_1 \cup G_2 \to \{-1, +1\}$ by

$$\mu(a_i) = \begin{cases} +1 & \text{when } i \text{ is odd} \\ -1 & \text{when } i \text{ is even.} \end{cases}$$

$$\mu(b_i) = \begin{cases} -1 & \text{when } i \text{ is odd} \\ +1 & \text{when } i \text{ is even.} \end{cases}$$

It is easy to verify that G is a NNISDF with

$$\gamma_{is}^{NN}(K_{m,n}) = \begin{cases} 0 & \text{if } m = 1 \text{ and } n \text{ is odd;} \\ 1 & \text{if } m = 1 \text{ and } n \text{ is even;} \\ 2 & \text{if } m \geq 3 \text{ and } n \geq 3 \text{ is odd.} \end{cases}$$

If $m \geq 3$ and $n \geq 3$, then we define a function $\mu : V = G_1 \cup G_2 \to \{-1, +1\}$ by

$$\mu(a_i) = \begin{cases} +1 & \text{when } i \text{ is odd} \\ -1 & \text{when } i \text{ is even.} \end{cases}$$

$$\mu(b_i) = \begin{cases} -1 & \text{when } i \text{ if } 3 \leq \text{i} \leq \text{ is odd} \\ -1 & \text{otherwise.} \end{cases}$$

From the above labeling, $\gamma_{is}^{NN}(K_{m,n}) = 3$.

Theorem 14. *Let $n \geq 2$ be an integer. Then the graph $G = K_{1,n}(n \geq 2)$ admits NNISDF with*

(i) $\gamma_{is}^{NN}(G) = 0$ *when n is odd*

(ii) $\gamma_{is}^{NN}(G) = 1$ *when n is even.*

Proof. Let $V(G) = \{a_0, a_1, \ldots, a_n\}$ and $E(G) = \{a_0 a_i : 1 \leq i \leq n\}$. Here, $a_1, a_2, \ldots, a_n$ an are pendent vertices. Let λ ba a minimum NNISDF of G. By the definition of NNISDF at least one vertex has $+1$. for a $a \in V(G)$.

Case 1. Suppose n is odd. Then $N[a_0] = |V(G)|$ is even. In this case a_0 must be labeled with $+1$, sign, otherwise a contra-diction to λ. Therefore remaining $\frac{n-1}{2}$ vertices has -1 sign and $\frac{n-1}{2}$ vertices has $+1$. In this case $\lambda(N[a_0]) = 0$. Thus $w(\lambda) = (+1) + \frac{n+1}{2}(-1) + \frac{n-1}{2}(+1) = 0$ and $\gamma_{is}^{NN}(G) \geq 0$.

Case 2. Suppose n is even. Then $N[a_0] = |V(G)|$ is odd. In this case a_0 must be labeled with $+1$ sign, otherwise a contradiction to λ. Therefore remaining $\frac{n}{2}$ vertices has -1, sign and $\frac{n}{2}$ vertices has $+1$. In this case $\lambda(N[a_0]) = 1$. Thus $w(\lambda) = (+1) + \frac{n}{2}(-1) + \frac{n}{2}(+1) = 1$ and so $\gamma_{is}^{NN}(G) \geq 1$.

We define a function $\mu : G \to \{-1, +1\}$ by $f(a_0) = +1$ and

$$\mu(a_i) = \begin{cases} +1 & \text{when } i \text{ is even} \\ -1 & \text{when } i \text{ is odd.} \end{cases}$$

Suppose n is odd. From the above labeling, we get $\mu(N[a_0]) = 0$ and $\mu(N[a_i]) \geq 1$ for $1 \leq i \leq n$. Thus μ is NNISDF with $w(\mu) = 0$ and so $\gamma_{is}^{NN}(G) \leq 0$.

Suppose n is even. From the above labeling, we get $\mu(N[a_0]) = 1$ and $\mu(N[a_i]) = 0$ for $i = 1, 3, 5, \ldots, n-1$ and $\mu(N[a_i]) = 2$ for $i = 2, 4, 6, \ldots, n$. Thus μ is NNISDF with $w(\mu) = 1$ and so $\gamma_{is}^{NN}(G) \leq 1$.

Theorem 15. *For* $n \geq 3$ *be an integer. Then the wheel graph* $G = W_n$ *admits NNISDF with*

(i) $\gamma_{is}^{NN}(G) = 0$ *when n is odd*

(ii) $\gamma_{is}^{NN}(G) = 1$ *when n is even.*

Proof. Let $V(G) = \{a_0, a_1, \ldots, a_n\}$ and $E(G) = \{a_0 a_i : 1 \leq i \leq n\} \cup \{a_i a_{i+1} : 1 \leq i \leq n-1\} \cup \{a_{n-1} a_n\}$. Since $|N[a_i]]|$ is even for all $1 \leq i \leq n$. Let λ be a minimum NNISDF of G. By the definition of NNISDF at least one vertex has $\lambda(N[a]) = 0$ for $a \in V(G)$.

Case 1. Suppose n is odd. Then $N[a_0] = |V(G)|$ is even. Since λ be a minimum NNISDF of G. In this case $\frac{n+1}{2}$ vertices has -1 sign and $\frac{n-1}{2}$ vertices has $+1$ and $\lambda(a_0) = +1$. Thus λ is NNISDF of G with $w(\lambda) = 0$ and so $\gamma_{is}^{NN}(G) \geq 0$.

Case 2. Suppose n is even. Then $N[a_0] = |V(G)|$ is odd. In this case $\frac{n}{2}$ vertices has -1 sign and $\frac{n}{2}$ vertices has $+1$ and $\gamma_{is}^{NN}(G) \geq 1$. Thus λ is NNISDF of G with $w(\lambda) = 1$ and so $\gamma_{is}^{NN}(G) \geq 1$.

We define a function $\mu : G \to \{-1, +1\}$ by $\mu(a_0) = +1$ and

$$\mu(a_i) = \begin{cases} +1 & \text{when } i \text{ is even} \\ -1 & \text{when } i \text{ is odd.} \end{cases}$$

Suppose n is odd. According to the above labeling, we get $\mu(N[a_0]) = 0$ and $\mu(N[a_i]) \geq 1$ for $1 \leq i \leq n$. Thus g is NNISDF with $w(\mu) = 0$ and so $\gamma_{is}^{NN}(G) \leq 0$.

Suppose n is even. From the above labeling, we get $\mu(N[a_0]) \geq 1$ and $\mu(N[a_i]) = 0$ for $i = 1, 3, 5, \ldots, n-1$ and $\mu(N[a_i]) = 2$ for $i = 2, 4, 6, \ldots, n$. Thus μ is NNISDF with $w(\mu) = 1$ and so $\gamma_{is}^{NN}(G) \leq 1$.

Acknowledgement

The authors thank the anonymous referees for their useful comments and suggestions which improved the quality and the read-ability of the paper.

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