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WEAK EDGE DOMINATION IN GRAPH

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Abstract

The paper is about weak edge domination in graph. The edge set of the graph is weak edge dominating set if the set of end vertices of edges of the edge set is a dominating set. It is proved that any weak edge dominating set of the graph gives rise to an edge dominating set in the corresponding dual hypergraph of the graph and vice versa. The bounds of weak edge domination number are obtained in terms of total domination number and edge domination number. The effect of a vertex (edge) removal operation on weak edge domination number of the graph is observed.

1. Introduction

Let $G = (V, E)$ be a finite, simple, undirected graph. The vertex set of G is denoted by $V(G)$ (or simply V) and edge set by $E(G)$ (or E). Each edge $e \in E$ is an unordered pair of distinct vertices of V. If an edge $e = uv$ then we say that a vertex u is adjacent to vertex v or the vertices u and v are neighbors and that e is incident to u and v. For an edge set $F \subseteq E$, $V(F)$ is the set of end vertices of edges of F.

A *subgraph* H of G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $H \subseteq V(G)$ then $\langle H \rangle$ denotes the subgraph induced by the vertices of H. For a vertex $v \in V(G)$, the graph $G - v$ is called a vertex-deleted subgraph of G. If $vu \in E(G)$ then $G - vu$ is called an edge-deleted subgraph of G.

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The degree of a vertex v (denoted by $\deg(v)$) is equal to the number of vertices that are adjacent to v. If there is a vertex $v \in V(G)$ such that $\deg(v) = 0$ then v is called an *isolated vertex*. If $\deg(v) = 1$ then v is called a *pendent vertex*. An edge $e = uv$ is an *isolated edge* of G if $\deg(u) = 1$ and $\deg(v) = 1$, also $e = uv$ is a *pendent edge* of G if the degree of exactly one of u and v is 1. The *open neighborhood* of a vertex v (denoted by $N(v)$) is the set of vertices of G that are adjacent to v, that is $N(v) = \{u \in V \mid uv \in E\}$. The *closed neighborhood* of a vertex v is $N[v] = N(v) \cup \{v\}$.

A *dominating set* $S \subseteq V$ of G is a set of vertices such that each vertex $v \in V$ is either in S or adjacent to a vertex of S. The *domination number* (denoted by $\gamma(G)$) of G is the minimum cardinality of a dominating set of G [3]. A subset $S \subseteq V$ is a *total dominating s*et if each vertex $v \in V$ is adjacent to a vertex of S. The *total domination number* (denoted by $\gamma_t(G)$) of G is the minimum cardinality of a total dominating set of G [3].

The concept of edge domination is well-known. An edge set $F \subseteq E$ is said to be an *edge dominating set* of G if for every edge e not in F is adjacent to some edge in F. An edge dominating set F of G is a *minimal* edge dominating set if F does not have a proper subset which is an edge dominating set. An edge dominating set with minimum cardinality is a *minimum* edge dominating set. The cardinality of a minimum edge dominating set is the *edge domination number* (denoted by $\gamma'(G)$) of the graph G [3]. Several papers have appeared related to edge domination and edge domination number [5, 6, 7].

Here, a concept called weak edge domination in graph is considered [4]. Every edge dominating set is a weak edge dominating set but the converse need not be true. The dual hypergraph of the graph is considered. It is proved that the weak edge dominating set of the graph is an edge dominating set of the corresponding dual hypergraph of the graph. The bounds of weak edge domination number are obtained in terms of total domination number and edge domination number. It is interesting to consider the vertex (edge) removal operation on graph, particularly, the effect of this operation on the weak edge domination number of the graph.

Definition 1.1[4]**.** An edge set $F \subseteq E(G)$ is said to be a *weak edge dominating set* if $V(F)$ is a dominating set.

Example 1.1. (1) Consider the following graphs.

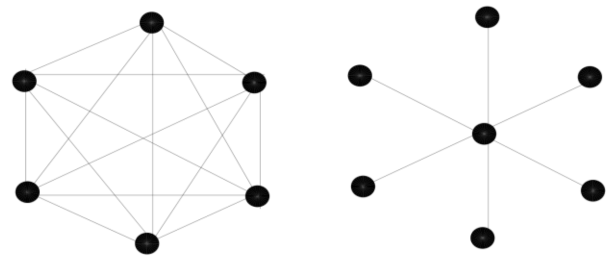

Figure 1. Complete graph and star graph.

An edge set containing any single edge of the graph (complete graph or star graph) is a weak edge dominating set. For each graph given in figure 1, every weak edge dominating set of the graph is an edge dominating set.

(2) Consider the cycle graph C_6 with vertices 1, 2, 3, 4, 5, 6 and edges 12, 23, 34, 45, 56, 16.

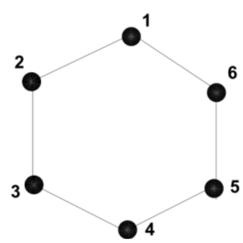

Figure 2. Cycle graph C_6.

Consider the edge set $F = \{12, 34\}$ of the graph C_6. For this set F, $V(F)$ is a dominating set but F is not an edge dominating set. In fact, $V(F)$ is a total dominating set.

Remark 1.1. Let G be a graph and $F \subseteq E$. If F is an edge dominating set then $V(F)$ is a dominating set. Therefore every edge dominating set of the graph is a weak edge dominating set but the converse need not be true.

Let G be a graph and $e = uv$ be an edge of G. We have defined that an edge set F is a weak edge dominating set if $V(F)$ is a dominating set. Therefore, in this section, we consider only those graphs G which do not have an isolated vertex.

Proposition 1.1. *Let G be a graph and $e = uv$ be an edge of G such that $e \in F \subseteq E(G)$.*

(a) *If $V(F - \{e\}) = V(F)$ then there are atleast two edges $uw_1 \in F$ and $vw_2 \in F$ (That is, uv is not an isolated or pendent edge in $\langle V(F) \rangle$).*

(b) *If $V(F - \{e\}) \subset V(F)$ then one of the following three holds.*

(1) *$u \notin V(F - \{e\})$ and $v \in V(F - \{e\})$ (That is, u is a pendent vertex in $\langle V(F) \rangle$).*

(2) *$u \in V(F - \{e\})$ and $v \notin V(F - \{e\})$ (That is, v is a pendent vertex in $\langle V(F) \rangle$).*

(3) *$u \notin V(F - \{e\})$ and $v \notin V(F - \{e\})$ (That is, uv is an isolated edge in $\langle V(F) \rangle$).*

Proof. Obvious. □

Definition 1.2[2]**.** Let G be a graph, $S \subset V(G)$ and $v \in S$. The *external private neighborhood* of v with respect to S is the set of vertices $\{w \in V(G) - S \mid N(w) \cap S = \{v\}\}$. It is denoted by $epn(v, S)$.

Definition 1.3. A weak edge dominating set $F \subseteq E(G)$ is *minimal* if for any edge $e \in F$, $V(F - \{e\})$ is not a dominating set.

Theorem 1.1. *Let G be a graph. An edge set $F \subseteq E(G)$. The edge set F is a minimal weak edge dominating set if and only if for every edge $e = uv \in F$, one of the following holds.*

(i) *e is an isolated edge of $\langle V(F) \rangle$.*

(ii) *$epn(u, V(F)) \neq \phi$ or $epn(v, V(F)) \neq \phi$.*

Proof. Suppose that F is a minimal weak edge dominating set with $e = uv \in F$. Since F is a minimal weak edge dominating set, $V(F - \{e\})$ is not a dominating set, for every edge $e = uv \in F$. Therefore $V(F - \{e\}) \subset V(F)$ for every edge $e = uv \in F$, by proposition 1.1(b), one of the following three holds.

(1) If $u \notin V(F - \{e\})$ and $v \in V(F - \{e\})$ then u is adjacent to v. Since F is a minimal weak edge dominating set, $V(F - \{e\})$ is not a dominating set. Therefore, there is a vertex w which is not in $V(F - \{e\})$ and also w is not adjacent with any vertex of $V(F - \{e\})$. This vertex w cannot be u because u is adjacent to v and $v \in V(F - \{e\})$. Therefore, $w \neq u$. Therefore w is a vertex outside $V(F)$ such that w is adjacent to some vertex of $V(F)$ but w is not adjacent to any vertex of $V(F - \{e\})$. So, this vertex to which w is adjacent must be u. Therefore, $w \in epn\,(u, V(F))$.

(2) If $u \in V(F - \{e\})$ and $v \notin V(F - \{e\})$ then by symmetric argument, it follows that, there is a vertex w outside $V(F)$ such that $w \in epn\,(u, V(F))$.

(3) If $u \notin V(F - \{e\})$ and $v \notin V(F - \{e\})$ then $e = uv$ is an isolated edge of $\langle V(F) \rangle$.

Conversely, Consider the weak edge dominating set $F \subseteq E(G)$ for which the conditions are satisfied and $e = uv \in F$. If e is an isolated edge of $\langle V(F) \rangle$ then the vertices u and v are not dominated by the vertex set $V(F - \{e\})$. If $epn\,(u, V(F)) \neq \phi$ then there exists at least one vertex outside the vertex set $V(F)$ which is not dominated by any vertex of $V(F - \{e\})$. Similarly, $epn\,(v, V(F)) \neq \phi$ also. □

Definition 1.4[1]**.** The *hypergraph* is an order pair $G = (V, E)$. The nonempty set V contains the elements $\{v_1, v_2, \ldots, v_n\}$ and $E = \{C_1, C_2, \ldots, C_m\}$ is a family of subsets of V such that $\bigcup_{i=1}^{m} C_i = V$ and each C_i is non empty. The elements of V and E are called the vertices and the edges of the hypergraph G respectively.

Definition 1.5. Let $G = (V, E)$ be a hypergraph. Two edges C_1 and C_2 of G are adjacent if they have at least one common vertex.

Definition 1.6. Let $G = (V, E)$ be a hypergraph. A subset F of $E(G)$ is said to be an *edge dominating set* of the hypergraph G if for every edge C not in F is adjacent to some edge in F.

If G is a graph, then G can also be regarded as a hypergraph. Therefore the dual hypergraph G^* of the graph G is defined as follows.

Definition 1.7[1]**.** Let $G = (V(G), E(G))$ be a graph without isolated vertices. The *dual hypergraph* G^* of the graph G is a hypergraph with vertex set $V(G^*) = E(G)$ and edge set $E(G^*) = \{\overline{v} \mid v \in V(G)\}$ where $\overline{v} = \{e \in E(G) \mid v$ is an end vertex of $e\}$.

Consider the dual hypergraph G^* of the graph G.

Theorem 1.2. *Let G be a graph without isolated vertices and $F \subseteq E(G)$. The edge set F is a weak edge dominating set of G if and only if the set $F^* = \{\overline{u} \mid u \in V(F)\}$ is an edge dominating set of G^*.*

Proof. First suppose that F is a weak edge dominating set of G and consider the subset F^* of $E(G^*)$. Let $\overline{x} \in E(G^*) - F^*$ then x is not an end vertex of any edge of F. This means that $x \notin V(F)$. Therefore, x is adjacent to some vertex $u \in V(F)$. Now consider the edge $e = xu$, then $e \in \overline{x}$ and $e \in \overline{u}$. That is, $\overline{x}$ is adjacent to $\overline{u}$ in hypergraph G^* and $\overline{u} \in F^*$. Thus F^* is an edge dominating set in G^*.

Conversely, suppose that $x \notin V(F)$. If $\overline{x} \in F^*$ then $\overline{x} = \overline{u}$ for some $u \in V(F)$. Therefore, x is adjacent to u for some $u \in V(F)$. Suppose that $\overline{x} \notin F^*$. Now F^* is an edge dominating set of G^* and therefore $\overline{x}$ is adjacent to $\overline{u}$ for some $u \in V(F)$. This means that there is an edge e of G such that $e \in \overline{x} \cap \overline{u}$. Therefore $e = xu$. Thus, x is adjacent to u, where $u \in V(F)$. Therefore F is a weak edge dominating set of G. □

Definition 1.8. The minimum cardinality of a weak edge dominating set is known as *weak edge domination number* of the graph. It is denoted by $\gamma'_w(G)$.

Remark 1.2. (1) If F is an edge dominating set, then $V(F)$ is a dominating set. It implies that

$$\gamma'_w(G) \leq \gamma'(G).$$

(2) Let $F \subseteq E(G)$ be an edge set. If $V(F)$ is a dominating set, then $V(F)$ is a total dominating set. It implies that

$$\frac{\gamma_t(G)}{2} \leq \gamma'_w(G), \qquad \text{if } \gamma_t(G) \text{ is even}$$

$$\frac{\gamma_t(G)+1}{2} \leq \gamma'_w(G), \text{ if } \gamma_t(G) \text{ is odd}$$

2. Vertex Removal from the Graph

We consider vertex removal operation on the graph and observe the effect of this operation on weak edge domination number of the graph.

Example 2.1. Consider the graphs G of row 2 in the following figure 3. The graphs after removal of a vertex from G are given in row 1.

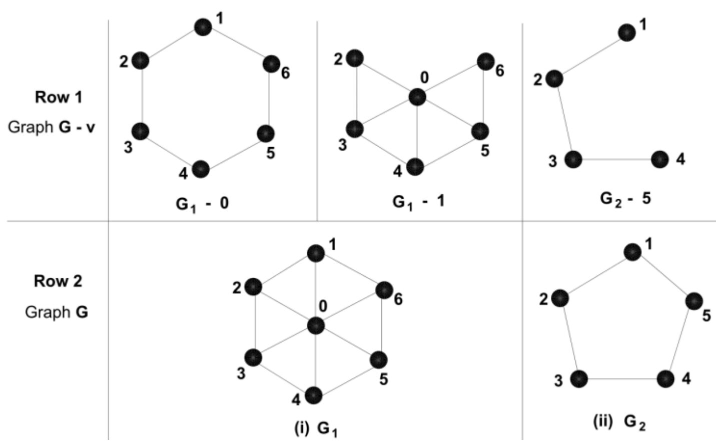

Figure 3. Graphs indicating vertex removal effect on weak edge domination.

The effect of vertex removal on the weak edge domination number of the graph is indicated with figure 3. The weak edge domination number of G_1 is 1. If a vertex 0 is removed from G_1 then the weak edge domination number of $G_1 - 0$ is 2. Thus, the weak edge domination number of G_1 increases when vertex 0 is removed from G_1. If any vertex v $(v \neq 0)$ is removed from G_1 then the weak edge domination number of $G_1 - v$ $(v \neq 0)$ is 1. Thus, the weak edge domination number of G_1 does not change when a vertex other than 0 is removed from G_1.

The weak edge domination number of G_2 is 2. If any vertex is removed from G_2 then the weak edge domination number of $G_2 - v$ (for any $v = 1, 2, 3, 4, 5$) is 1. Thus, the weak edge domination number of G_2 decreases when a vertex is removed from G_2.

Definition 2.1. Let G be a graph and $v \in V(G)$. The *edge neighborhood* of a vertex v is equal to $\{e \in E(G) \mid v \text{ is an end vertex of } e\}$. It is denoted by $N_E(v)$.

We prove a necessary and sufficient condition under which the weak edge domination number of the graph increases when a vertex is removed from the graph.

Theorem 2.1. *Let G be a graph without isolated vertex and a vertex $v \in V(G)$ such that v has no pendent neighbors then $\gamma'_w(G - v) > \gamma'_w(G)$ if and only if the following conditions are satisfied.*

(1) *v is an end vertex of some edge e in every minimum weak edge dominating set F of G.*

(2) *There is no subset F of $E(G) - N_E(v)$ such that $|F| \leq \gamma'_w(G)$ and F is a weak edge dominating set of $G - v$.*

Proof. Suppose that $\gamma'_w(G - v) > \gamma'_w(G)$.

(1) Suppose that there is a minimum weak edge dominating set F of G such that v is not an end vertex of any edge of F. Now consider the subgraph $G - v$. The given set F is a set of edges of $G - v$. It is obvious that F is a weak

edge dominating set of $G - v$. Therefore $\gamma'_w(G - v) \leq |F| = \gamma'_w(G)$. Which is a contradiction. Therefore, condition (1) is satisfied.

(2) Suppose that there is an edge set F of $E(G) - N_E(v)$ with $|F| \leq \gamma'_w(G)$ and F is a weak edge dominating set of $G - v$ then it follows that $\gamma'_w(G - v) \leq |F| = \gamma'_w(G)$. Which is a contradiction. Therefore, condition (2) is satisfied.

Conversely, suppose that conditions (1) and (2) are satisfied. Suppose that $\gamma'_w(G - v) = \gamma'_w(G)$. Let F be a minimum weak edge dominating set of $G - v$. Suppose that F is also a weak edge dominating set of G then v is adjacent to u for some $u \in V(F)$. Thus F is a minimum weak edge dominating set of G such that v is not an end vertex of any edge of F. This contradicts condition (1). Suppose that v is not adjacent with any vertex of $V(F)$ then $F \cap N_E(v) = \phi$. Therefore $F \subseteq E(G) - N_E(v)$ also $|F| = \gamma'_w(G)$ and F is a weak edge dominating set of $G - v$. This contradicts condition (2).

Suppose that $\gamma'_w(G - v) < \gamma'_w(G)$. Let F be a minimum weak edge dominating set of $G - v$ then F cannot be a weak edge dominating set of G. Therefore v is not adjacent to any vertex of $V(F)$. This implies that $F \subseteq E(G) - N_E(v)$ with $|F| = \gamma'_w(G)$ and F is a weak edge dominating set of $G - v$. Which is a contradiction.

Thus, the assumption $\gamma'_w(G - v) \leq \gamma'_w(G)$ leads to a contradiction. Therefore $\gamma'_w(G - v) > \gamma'_w(G)$. □

Theorem 2.2. *Let G be a graph without isolated vertex and a vertex* $v \in V(G)$ *such that v has no pendent neighbors then* $\gamma'_w(G - v) < \gamma'_w(G)$ *if and only if there is a minimum weak edge dominating set F of G such that the following conditions are satisfied.*

(1) *There is exactly one neighbor u of v such that* $uv \in F$.

(2) *Every neighbor w of v,* $N(w) \cap (V(F) - \{v\}) \neq \phi$.

Proof. Suppose that $\gamma'_w(G - v) < \gamma'_w(G)$.

Let F_1 be a minimum weak edge dominating set of $G - v$ then F_1 can not be a weak edge dominating set of G. Thus, $V(F_1)$ cannot be a dominating set of G. Therefore v is not adjacent to any vertex of $V(F_1)$. Select one neighbor u of v and consider the set $F = F_1 \cup \{uv\}$. Obviously, F is a weak edge dominating set of G. Note that if w is any other neighbor of v $(w \neq u)$ then $w \notin V(F_1)$. Since $V(F_1)$ is a dominating set of $G - v$, $N(w) \cap V(F_1) \neq \phi$. Therefore $N(w) \cap (V(F) - \{v\}) \neq \phi$.

Conversely, suppose that there is a minimum weak edge dominating set F of G such that conditions (1) and (2) hold.

Let $T = F - \{uv\}$. Let w be any vertex of $G - v$ which is not in $V(T)$. If w is not a neighbor of v then since $V(F)$ is a dominating set of G, w is adjacent to some vertex of $V(F)$ (different from v). Thus w is adjacent to some vertex of $V(T)$. If w is a neighbor of v then by the given condition, w is also adjacent to some other vertex of $V(F)$ different from v. Thus, w is adjacent to some vertex of $V(T)$. Hence $V(T)$ is a dominating set of $G - v$. Thus T is a weak edge dominating set of $G - v$. Therefore, $\gamma'_w(G - v) = |T| < \gamma'_w(G)$. □

Remark 2.1. In the theorem 2.2, the sentence 'there is a minimum weak edge dominating set F' cannot be replaced by 'every minimum weak edge dominating set F'. The following example provides an answer.

Example 2.2. Consider the graph C_5 with vertices 1, 2, 3, 4, 5.

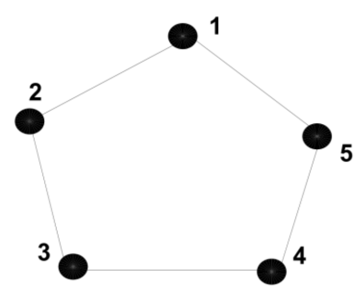

Figure 4. Cycle graph C_5.

The number $\gamma'_w(C_5) = 2$ and $\gamma'_w(C_5 - 5) = 1$.

(1) If we consider the set $F = \{15, 23\}$ then F is a minimum weak edge dominating set of G. Note that $u = 1$ satisfies the required condition.

(2) If we consider $F = \{45, 34\}$ then F is a minimum weak edge dominating set of G but condition 2 of the theorem 2.2 is not satisfied in this case.

3. Edge removal from the Graph

We consider the operation edge removal from the graph and observe the effect of this operation on weak edge domination number of the graph.

Example 3.1. Consider the graph G of row 2 in the following figure 5. The graphs after removal of an edge from G are given in row 1.

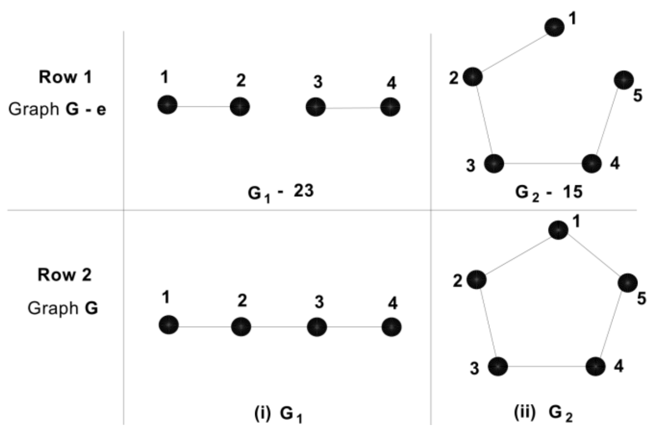

Figure 5. Graphs indicating edge removal effect on weak edge domination.

(i) The minimum weak edge domination number of G_1 is 1. If the edge 23 is removed from G_1 then the weak edge domination number of $G_1 - 23$ is 2. Thus, the weak edge domination number of G_1 increases when edge 23 is removed from G_1. Also, the weak domination number of $G_1 - e$ $(e \neq 23)$ is 1. Thus, the weak edge domination number of G_1 does not change when an edge other than 23 is removed from G_1.

(ii) The weak edge domination number of G_2 is 2. If any edge is removed from G_2 then the weak edge domination number of $G_2 - e$ is also 2. Thus, the weak edge domination number of G_2 does not change when any edge is removed from G_2.

We have defined that an edge set F is a weak edge dominating set if $V(F)$ is a dominating set. Therefore in the following theorems, we consider the graph G which does not have an isolated vertex.

Theorem 3.1. *Let G be a graph and $e = uv$ be an edge of G such that $G - e$ does not have an isolated vertex then $\gamma'_w(G - e) \geq \gamma'_w(G)$.*

Proof. Let F be a minimum weak edge dominating set of $G - e$. Therefore, $V(F)$ is a dominating set of $G - e$. Obviously, $V(F)$ is also a dominating set of G. Therefore, F is a weak edge dominating set of G. Thus, $\gamma'_w(G) \leq |F| = \gamma'_w(G - e)$. □

Theorem 3.2. *Let G be a graph and $e = uv$ be an edge of G such that $G - e$ does not have an isolated vertex then $\gamma'_w(G - e) > \gamma'_w(G)$ if and only if for every minimum weak edge dominating set F of G, $N(u) \cap (V(F) - \{v\}) = \phi$ or $N(v) \cap (V(F) - \{u\}) = \phi$.*

Proof. Suppose that $\gamma'_w(G - e) > \gamma'_w(G)$ and there is a minimum weak edge dominating set F of G such that $N(u) \cap (V(F) - \{v\}) \neq \phi$ and $N(v) \cap (V(F) - \{u\}) \neq \phi$.

Suppose that $e \in F$ and let $F_1 = F - \{e\}$. If x is any vertex of $G - e$ such that $x \notin V(F_1)$ then $x \notin V(F)$ also. Suppose that $u \notin V(F_1)$ and $x = u$. Now $N(u) \cap (V(F) - \{v\}) \neq \phi$ therefore there is a neighbor of $x = u$ in $V(F_1)$ (which is different from v). Similarly, if $v \notin V(F_1)$ and $x = v$ then x is adjacent to some vertex of $V(F_1)$ which is different from u. Suppose that $x \neq u$ and $x \neq v$. If $u \in V(F_1)$ and x is adjacent to u then we have done. Similarly, if $v \in V(F_1)$ and x is adjacent to v then we have done.

Suppose that $x \neq u$ and $x \neq v$ then $x \notin V(F_1)$. Therefore x is adjacent to some vertex z of $V(F)$ in G. If $z = u$ then x is adjacent to u in $G - e$ and $u \in V(F_1)$ and similarly if $z = v$ then $v \in V(F_1)$ and x is adjacent to v in

$G - e$. Suppose that x is adjacent to z with $z \neq u$ and $z \neq v$ in G then x is adjacent to z in $G - e$ also.

Thus, from all the cases, it follows that $V(F_1)$ is a dominating set of $G - e$ also $|F_1| < |F|$. This implies that $\gamma'_w(G - e) < \gamma'_w(G)$ which is a contradiction. Suppose that $e \notin F$ then by similar arguments, we can prove that F is a weak edge dominating set of $G - e$. This implies that $\gamma'_w(G - e) \leq \gamma'_w(G)$ which is a contradiction. Thus, the given condition is satisfied.

Conversely, suppose that the condition is satisfied and suppose that $\gamma'_w(G - e) = \gamma'_w(G)$. Suppose that there is a minimum weak edge dominating set F of $G - e$ such that $u \in V(F)$ and $v \in V(F)$. Suppose that $u \in V(F)$. Now there is an edge f of $G - e$ such that u is an end vertex of f and $f \in F$. Let the other end vertex of f is z.

Suppose that $v \notin V(F)$. Now $V(F)$ is a dominating set of $G - e$ so v is adjacent to some vertex x of $V(F)$. Now consider the set F in G. Obviously, $V(F)$ is a dominating set of G and therefore F is a weak edge dominating set of G also. Since $\gamma'_w(G - e) = \gamma'_w(G)$, F is a minimum weak edge dominating set of G. However, $N(u)$ contains the vertex z of $V(F)$ which is different from v and $N(v)$ contains the vertex x of $V(F)$ which is different from u. This contradicts the given condition.

Suppose that $u \in V(F)$ and $v \in V(F)$ then u is an end vertex of some edge uz which is in F and v is an end vertex of some edge vx which is in F. Obviously, $z \neq v$ and $x \neq u$. Thus, F is a minimum weak edge dominating set of $G - e$ such that $N(u) \cap (V(F) - \{v\}) \neq \phi$ and $N(v) \cap (V(F) - \{u\}) \neq \phi$. This is again a contradiction.

Suppose that $u \notin V(F)$ and $v \notin V(F)$. Since $V(F)$ is a dominating set of $G - e$, u is adjacent with some vertex z of $V(F)$ with $z \neq v$. Similarly, v is adjacent with some vertex x of $V(F)$ with $x \neq u$. Thus again the condition violates. Therefore $\gamma'_w(G - e) = \gamma'_w(G)$ is not possible. Hence, $\gamma'_w(G - e) < \gamma'_w(G)$. □

Corollary 3.1. *Let G be a graph and $e = uv$ be an edge of G such that $G - e$ does not have an isolated vertex then $\gamma'_w(G - e) = \gamma'_w(G)$ if and only if for every minimum weak edge dominating set F of G, $N(u) \cap (V(F) - \{v\}) \neq \phi$ and $N(v) \cap (V(F) - \{u\}) \neq \phi$.*

Example 3.2. Consider the following graph.

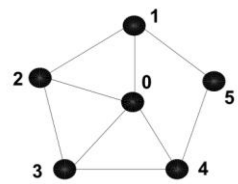

Figure 6. Graph G with 6 vertices.

The minimum weak edge dominating sets of the graph G are $F_1 = \{01\}$ and $F_2 = \{04\}$. Therefore the weak edge domination number of G is 1. If we consider any edge $e = uv \notin \{01, 04\}$ then $N(u) \cap (V(F) - \{v\}) \neq \phi$ and $N(v) \cap (V(F) - \{u\}) \neq \phi$. Thus, $\gamma'_w(G - e) = \gamma'_w(G)$ for any edge $e = uv \notin \{01, 04\}$. Since $F_1 = \{01\}$ and $F_2 = \{04\}$ are minimum weak edge dominating sets of G, the weak edge domination number of the graph $G - 01$ is 2. Also, the weak edge domination number of the graph $G - 04$ is 2.

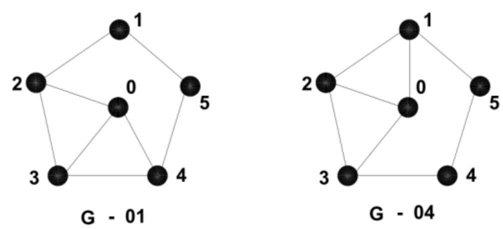

Figure 7. Graphs $G - 01$ and $G - 04$.

4. Concluding Remarks and Further Scope

The weak edge dominating set of the graph is an edge dominating set of the corresponding dual hypergraph of the graph. The further scope of research is to find the relation between the minimal weak edge dominating set of the graph and the minimal dominating set of the corresponding dual hypergraph of the graph also the relation between the number γ'_w of the graph and the number γ' of the corresponding dual hypergraph of the graph. The bounds of γ'_w are written in terms of total domination number and edge domination number. the vertex (edge) removal operation on graph is considered and the effect of this operation on γ'_w of the graph is discussed. How many units the number γ'_w change under these removal operations is a matter of further research.

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