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ON FUZZY SUBMAXIMAL AND QUASI-SUBMAXIMAL SPACES

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Abstract

In this paper, the notion of fuzzy quasi-submaximal spaces is introduced by means of fuzzy boundary. Several characterizations of fuzzy submaximal spaces are obtained. The conditions under which fuzzy quasi-submaximal spaces become fuzzy submaximal spaces are obtained. The conditions for fuzzy quasi-submaximal spaces to become fuzzy almost P-spaces and fuzzy Baire spaces are established. The condition under which fuzzy quasi-submaximal spaces, become fuzzy second category spaces is also obtained and it is shown that fuzzy σ-resolvable spaces are not fuzzy quasi-submaximal spaces and fuzzy quasi-submaximal, fuzzy nodef and fuzzy P-spaces are fuzzy submaximal spaces.

1. Introduction

The concept of fuzzy sets as a new approach for modelling uncertainties was introduced by L. A. Zadeh [32] in 1965 in his classic paper. Any application of mathematical concepts depends firmly and closely how one introduces basic ideas that may yield various theories in various directions. If the basic idea is suitably introduced, then not only the existing theories stand but also the possibility of emerging new theories increases and on these lines C. L. Chang [9] introduced the notion of fuzzy topological spaces in 1968. The

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paper of Chang paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts.

The term "submaximal space" was first used by Bourbaki in [8]. Since its birth, the class of submaximal spaces has attracted the attention of general topologists. The first systematic study of submaximal spaces was undertaken in the paper of A. V. Arhangel' Skiand P. J. Collins [2]. In 2001, B. Al-Nashef [1] introduced the concept of quasi-submaximal spaces which is weaker than submaximality. The notion of submaximal spaces in fuzzy topology was introduced by G. Balasubramanian in [6]. In this paper several characterizations of fuzzy submaximal spaces are obtained. It is obtained that fuzzy almost resolvable spaces with fuzzy submaximality are not fuzzy second category spaces. Also it is established that fuzzy submaximal spaces are fuzzy irresolvable and fuzzy quasi-maximal spaces and fuzzy hyperconnected spaces with fuzzy irresolvablity are fuzzy quasi-maximal spaces. The notion of fuzzy quasi-submaximal spaces is introduced and studied in this paper. Several characterizations of fuzzy quasi-submaximal spaces are obtained. It is obtained that fuzzy quasi-submaximal, fuzzy nodec space, and fuzzy quasi-submaximal, fuzzy globally disconnected spaces are fuzzy submaximal spaces. The conditions for fuzzy quasi-submaximal spaces to become fuzzy almost P-spaces and fuzzy Baire spaces are established. It is established that fuzzy quasi-submaximal spaces are fuzzy irresolvable, fuzzy strongly irresolvable, fuzzy weakly Volterra and fuzzy GID spaces.

2. Preliminaries

In order to make the exposition self-contained, some basic notions and results used in the sequel are given. In this work by (X, T) or simply by X, we will denote a fuzzy topological space due to Chang (1968). Let X be a non-empty set and I the unit interval $[0, 1]$. A fuzzy set λ in X is a mapping from X into I. The fuzzy set 0_X is defined as $0_X(x) = 0$, for all $x \in X$ and the fuzzy set 1_X is defined as $1_X(x) = 1$, for all $x \in X$.

Definition 2.1 [9]. Let λ be any fuzzy set in the fuzzy topological space (X, T). The fuzzy interior, the fuzzy closure and the fuzzy complement of λ are defined respectively as follows:

(i) $Int(\lambda) = \vee\{\mu/\mu \leq \lambda, \mu \in T\}$,

(ii) $Cl(\lambda) = \wedge\{\mu/\lambda \leq \mu, 1 - \mu \in T\}$,

(iii) $\lambda'(x) = 1 - \lambda(x)$, for all $x \in X$.

The notions union $\psi = \vee_i(\lambda_i)$ and intersection $\delta = \wedge_i(\lambda_i)$, are defined respectively, for the family $\{\lambda_i / i \in I\}$ of fuzzy sets in (X, T) as follows:

(iv) $\psi(x) = \sup_i \{\lambda_i(x)/x \in X\}$

(v) $\delta(x) = \inf_i \{\lambda_i(x)/x \in X\}$.

Lemma 2.1 [5]. *For a fuzzy set λ of a fuzzy topological space X,*

(i) $1 - Int(\lambda) = Cl(1 - \lambda)$ *and* (ii) $1 - Cl(\lambda) = Int(1 - \lambda)$.

Definition 2.2. A fuzzy set λ in a fuzzy topological space (X, T) is called an

(1) fuzzy pre-open if $\lambda \leq \operatorname{int} cl(\lambda)$ and fuzzy pre-closed if $cl \operatorname{int}(\lambda) \leq \lambda$ [7].

(2) fuzzy semi-open if $\lambda \leq cl \operatorname{int}(\lambda)$ and fuzzy semi-closed if $\operatorname{int} cl(\lambda) \leq \lambda$ [5].

(3) fuzzy G_δ-set if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, where $\lambda_i \in T$ for $i \in I$ [6].

(4) fuzzy F_σ-set if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $1 - \lambda_i \in T$ for $i \in I$ [6].

(5) fuzzy regular-open if $\lambda = \operatorname{int} cl(\lambda)$ and fuzzy regular-closed if $\lambda = cl \operatorname{int}(\lambda)$ [5].

Definition 2.3 [12]. Let λ be a fuzzy set in the fuzzy topological space (X, T). The fuzzy boundary of λ is defined as $Bd(\lambda) = cl(\lambda) \wedge cl(1 - \lambda)$.

Definition 2.4. A fuzzy set λ in a fuzzy topological space (X, T), is called a

(i) *fuzzy dense set* if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$. That is, $cl(\lambda) = 1$, in (X, T) [13].

(ii) *fuzzy nowhere dense set* if there exists no non-zero fuzzy open set μ in

(X, T) such that $\mu < cl(\lambda)$. That is, $\operatorname{int} cl(\lambda) = 0$, in (X, T) [13].

(iii) *fuzzy first category set* if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T). Any other fuzzy set in (X, T) is said to be of fuzzy second category [13].

(iv) *fuzzy simply open set* if $Bd(\lambda)$ is a fuzzy nowhere dense set in (X, T). That is, λ is a fuzzy simply open set in (X, T) if $[cl(\lambda) \wedge cl(1 - \lambda)]$, is a fuzzy nowhere dense set in (X, T) [22].

(v) *fuzzy* σ*-boundary set* if $\lambda = \vee_{i=1}^{\infty}(\mu_i)$, where $\mu_i = cl(\lambda_i) \wedge (1 - \lambda_i)$ and (μ_i)'s are fuzzy regular open sets in (X, T) [23].

(vi) *fuzzy* σ*-nowhere dense set* if λ is a fuzzy F_σ-set with $\operatorname{int}(\lambda) = 0$, in (X, T) [17].

Definition 2.5 [15]**.** Let λ be a fuzzy first category set in the fuzzy topological space (X, T). Then, $1 - \lambda$ is called a fuzzy residual set in (X, T).

Definition 2.6. Let (X, T) be the fuzzy topological space and (X, T) is called a

(i) *fuzzy Baire space* if $\operatorname{int}(\vee_{i=1}^{\infty} (\lambda_i)) = 0$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T) [15].

(ii) *fuzzy hyper connected space* if every non-null fuzzy open subset of (X, T) is fuzzy dense in (X, T) [11].

(iii) *fuzzy almost resolvable space* if $\vee_{i=1}^{\infty} (\lambda_i) = 1_X$ where the fuzzy sets (λ_i)'s in (X, T) are such that $\operatorname{int}(\lambda_i) = 0$. Otherwise (X, T) is called a *fuzzy almost irresolvable space* [30].

(iv) *fuzzy first category space* if $1_X = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T). A fuzzy topological space which is not of fuzzy first category is said to be of *fuzzy second category* [13].

(v) *fuzzy strongly irresolvable space* if for every fuzzy dense set λ in (X, T), $cl\operatorname{int}(\lambda) = 1$, in (X, T) [28].

(vi) *fuzzy nodec space* if each non-zero fuzzy nowhere dense set is fuzzy closed in (X, T) [16].

(vii) *fuzzy door space* if every fuzzy sub set of X is either fuzzy open or fuzzy closed [4].

(viii) *fuzzy globally disconnected space* if each fuzzy semi-open set in (X, T) is a fuzzy open set in (X, T) [24].

(ix) *fuzzy almost GP-space* if $\operatorname{int}(\lambda) \neq 0$, for each non-zero fuzzy dense and fuzzy G_δ-set λ in (X, T) [20].

(x) *fuzzy quasi-maximal space* if for every fuzzy dense set λ in (X, T) with $\operatorname{int}(\lambda) \neq 0$, $\operatorname{int}(\lambda)$ is fuzzy dense in (X, T) [3].

(xi) *fuzzy GID-space* if for each fuzzy dense and fuzzy G_δ-set λ in (X, T), $cl\operatorname{int}(\lambda) = 1$, in (X, T) [19].

(xii) *fuzzy weakly Volterra space* if $cl(\wedge_{i=1}^{N} (\lambda_i)) \neq 0$ where the fuzzy sets (λ_i)'s are fuzzy dense and fuzzy G_δ-sets in (X, T) [29].

(xiii) *fuzzy* σ*-resolvable space* if $\vee_{i=1}^{\infty} (\lambda_i) = 1$, where (λ_i)'s are fuzzy dense sets in (X, T) such that $\lambda_i \leq 1 - \lambda_j$ for $i \neq j$ [31].

(xiv) *fuzzy first* σ*-category space* if $1_X = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy σ-nowhere dense sets in (X, T) [17].

(xv) *fuzzy weakly Baire space* if $\operatorname{int}(\vee_{i=1}^{\infty} (\mu_i)) = 0$, where $\mu_i = cl(\lambda_i) \wedge (1 - \lambda_i)$ and (λ_i)'s are fuzzy regular open sets in (X, T) [23].

(xvi) *fuzzy* σ*-Baire space* if $\operatorname{int}(\vee_{i=1}^{\infty} (\lambda_i)) = 0$, where (λ_i)'s are fuzzy σ-nowhere dense sets in (X, T) [17].

(xvii) *fuzzy perfectly disconnected space* if for any two non-zero fuzzy sets λ and μ defined on X with $\lambda \leq 1 - \mu$, $cl(\lambda) \leq 1 - cl(\mu)$, in (X, T) [25].

(xviii) *fuzzy* DG_δ*-space* if each fuzzy dense (but not fuzzy open) set in (X, T) is a fuzzy G_δ-set in (X, T) [27].

(xix) *fuzzy ∂-space* if each fuzzy G_δ-set is a fuzzy simply-open set in (X, T) [26].

(xx) *fuzzy nodef space* if each fuzzy nowhere dense set is a fuzzy F_σ-set in (X, T) [27].

(xxi) *fuzzy P-space* if every non-zero fuzzy G_δ-set λ in (X, T), is a fuzzy open set in (X, T) [18].

(xxi) *fuzzy almost P-space* if for every non-zero fuzzy G_δ-set λ in (X, T), $\text{int}(\lambda) \neq 0$, in (X, T) [21].

Theorem 2.1 [30]. *If $cl\,\text{int}(\lambda) = 1$, for each fuzzy dense set λ in the fuzzy almost resolvable space (X, T), then (X, T) is the fuzzy first category space.*

Theorem 2.2 [16]. *If the fuzzy topological space (X, T) is a fuzzy Baire space, then (X, T) is the fuzzy second category space.*

Theorem 2.3 [20]. *If the fuzzy topological space (X, T) is the fuzzy irresolvable space, then (X, T) is the fuzzy almost GP-space.*

Theorem 2.4 [21]. *If the fuzzy topological space (X, T) is the fuzzy almost P-space, then (X, T) is the fuzzy second category space.*

Theorem 2.5 [20]. *If the fuzzy topological space (X, T) is the fuzzy almost GP-space, then (X, T) is the fuzzy weakly Volterra space.*

Theorem 2.6 [10]. *Let (X, T) be the fuzzy topological space. Then the following conditions are equivalent*:

(i) *(X, T) is fuzzy hyper connected.*

(ii) *Every fuzzy pre-open set is fuzzy dense set.*

Theorem 2.7 [23]. *If λ is the fuzzy σ-boundary set in the fuzzy weakly Baire space (X, T), then λ is the fuzzy σ-nowhere dense set in (X, T).*

Theorem 2.8 [25]. *If λ is the fuzzy σ-boundary set in the fuzzy perfectly disconnected space (X, T), then there exists a fuzzy F_σ-set η in (X, T) such* *that* $\eta \leq \lambda$.

Theorem 2.9 [26]. *If the fuzzy topological space* (X, T) *is the fuzzy* ∂-*space, then* $\operatorname{int}[\lambda \wedge (1-\lambda)] = 0$, *for the fuzzy* G_δ-*set in* (X, T).

3. Fuzzy Submaximal Spaces

Definition 3.1 [6]. A fuzzy topological space (X, T) is called the fuzzy submaximal space if for each fuzzy set λ in (X, T) such that $cl(\lambda) = 1$, $\lambda \in T$. That is, (X, T) is a fuzzy submaximal space if each fuzzy dense set in (X, T) is a fuzzy open set in (X, T).

Proposition 3.1. *If* $\operatorname{int}(\lambda) = 0$ *for a fuzzy set* λ *defined on X in the fuzzy submaximal space* (X, T), *then* λ *is the fuzzy nowhere dense set in* (X, T).

Proof. Suppose that $\operatorname{int}(\lambda) = 0$, for a fuzzy set λ defined on X in (X, T). Now $1 - \operatorname{int}(\lambda) = 1 - 0 = 1$ and by the lemma 2.1, $cl(1-\lambda) = 1$. Since (X, T) is the fuzzy submaximal space, $1-\lambda$ is a fuzzy open set in (X, T) and then λ is the fuzzy closed set in (X, T) and thus $cl(\lambda) = \lambda$. This implies that $\operatorname{int} cl(\lambda) = \operatorname{int}(\lambda) = 0$, in (X, T) and hence λ is the fuzzy nowhere dense set in (X, T).

Proposition 3.2. *If* λ *is the fuzzy nowhere dense set in the fuzzy submaximal space* (X, T), *then* $1-\lambda$ *is the fuzzy open set in* (X, T).

Proof. Let λ be the fuzzy nowhere dense set in (X, T). Then, $\operatorname{int} cl(\lambda) = 0$, in (X, T). But $\operatorname{int}(\lambda) \leq \operatorname{int} cl(\lambda)$, implies that $\operatorname{int}(\lambda) = 0$. Then, $1 - \operatorname{int}(\lambda) = 1$ and by the lemma 2.1, $cl(1-\lambda) = 1$. Since (X, T) is the fuzzy submaximal space, $1-\lambda$ is a fuzzy open set in (X, T).

Proposition 3.3. *If* (X, T) *is the fuzzy submaximal space then* $\operatorname{int}(\lambda) = 0$ *if and only if* $\operatorname{int} cl(\lambda) = 0$, *for the fuzzy set* λ *defined on X in* (X, T).

Proof. Suppose that $\operatorname{int}(\lambda) = 0$, for a fuzzy set λ defined on X in (X, T). Then, by the proposition 3.1, $\operatorname{int} cl(\lambda) = 0$, in (X, T). Conversely, suppose that $\operatorname{int} cl(\lambda) = 0$, in (X, T). Now $\operatorname{int}(\lambda) \leq \operatorname{int} cl(\lambda)$, implies that $\operatorname{int}(\lambda) = 0$, in (X, T).

Proposition 3.4. *If* (X, T) *is the fuzzy submaximal space, then* (X, T) *is a fuzzy strongly irresolvable space.*

Proof. Let λ be a fuzzy dense set in (X, T). Since (X, T) is a fuzzy submaximal space, the fuzzy dense set λ in (X, T) is a fuzzy open set and then $\operatorname{int}(\lambda) = \lambda$, in (X, T). Now $cl \operatorname{int}(\lambda) = cl(\lambda) = 1$, in (X, T). Hence (X, T) is a fuzzy strongly irresolvable space.

Proposition 3.5. *If* (X, T) *is the fuzzy door space, then each fuzzy pre-open set is a fuzzy open set in* (X, T).

Proof. Let λ be a fuzzy pre-open set in (X, T). Then, $\lambda \leq \operatorname{int} cl(\lambda)$ in (X, T). Suppose that λ is not a fuzzy open set in (X, T). Since (X, T) is the fuzzy door space, λ is a fuzzy closed set in (X, T) and then $cl(\lambda) = \lambda$ and $\operatorname{int} cl(\lambda) = \operatorname{int}(\lambda) \leq \lambda$, a contradiction. Thus λ is a fuzzy open set in (X, T). Hence the fuzzy pre-open set λ is a fuzzy open set in (X, T).

Proposition 3.6. *If* (X, T) *is the fuzzy door space, then each fuzzy dense set is a fuzzy open set in* (X, T).

Proof. Let λ be a fuzzy dense set in (X, T). Then, $cl(\lambda) = 1$, in (X, T) and $\operatorname{int} cl(\lambda) = 1$ implies that $\lambda \leq \operatorname{int} cl(\lambda)$ in (X, T). Thus λ is a fuzzy pre-open set in (X, T). Since (X, T) is the fuzzy door space, by the proposition 3.5, λ is a fuzzy open set in (X, T).

Proposition 3.7. *If* (X, T) *is the fuzzy door space, then* (X, T) *is the fuzzy submaximal space.*

Proof. Let λ be a fuzzy dense set in (X, T). Since (X, T) is the fuzzy door space, by the proposition 3.6, λ is a fuzzy open set in (X, T). Hence (X, T) is the fuzzy submaximal space.

Proposition 3.8. *If each fuzzy pre-open set is a fuzzy open set in* (X, T), *then* (X, T) *is a fuzzy submaximal space.*

Proof. Let λ be a fuzzy dense set in (X, T). Then, $cl(\lambda) = 1$, in (X, T) and $\operatorname{int} cl(\lambda) = \operatorname{int}(1) = 1$ and this implies that $\lambda \leq \operatorname{int} cl(\lambda)$ in (X, T). Thus λ is a fuzzy pre-open set in (X, T). By the hypothesis, each fuzzy pre-open set is a fuzzy open set in (X, T) and hence λ is a fuzzy open set in (X, T). Hence (X, T) is a fuzzy submaximal space.

Proposition 3.9. *If λ is a fuzzy pre-open set in the fuzzy submaximal space (X, T), then*

(i) $[(1-\mu) \vee \lambda] \wedge \mu$ *is a fuzzy open set in* (X, T), *where* $\mu = \operatorname{int} cl(\lambda)$.

(ii) *There exists a fuzzy open set λ in (X, T) such that $\lambda \leq \eta \leq cl(\lambda)$.*

Proof. (i) Let (X, T) be a fuzzy submaximal space and λ be a fuzzy pre-open set in (X, T). Then, $\lambda \leq \operatorname{int} cl(\lambda)$ in (X, T). Let $\mu = \operatorname{int} cl(\lambda)$. Then, μ is a fuzzy open set in (X, T). Now $\lambda \leq \mu \leq cl(\lambda)$ implies that $cl(\lambda) \leq cl(\mu) \leq cl\ cl(\lambda)$ and thus $cl(\lambda) \leq cl(\mu) \leq cl(\lambda)$. Thus, $cl(\mu) = cl(\lambda)$, in (X, T).

It is claimed that $(1-\mu) \vee \lambda$ is a fuzzy dense set in (X, T). Suppose that $cl[(1-\mu) \vee \lambda] \neq 1$ and then, $1 - cl[(1-\mu) \vee \lambda] \neq 0$. This will imply that $1 - [cl(1-\mu) \vee cl(\lambda)] \neq 0$ and $(1 - [cl(1-\mu)) \wedge (1 - cl(\lambda)) \neq 0$. Then, $(1 - cl(1-\mu)) \wedge (1 - cl(\lambda)) \neq 0$ and $(1 - (1 - \operatorname{int}(\mu))) \wedge (1 - cl(\mu)) \neq 0$. [since $cl(\mu) = cl(\lambda)$, in (X, T)]. This will imply that $\operatorname{int}(\mu) \wedge (1 - cl(\mu)) \neq 0$ and then $\operatorname{int}(\mu) \not\leq 1 - [1 - cl(\mu)]$ and $\operatorname{int}(\mu) \not\leq cl(\mu)$, a contradiction. Hence $cl[(1-\mu) \vee \lambda] = 1$, in (X, T). Hence $(1-\mu) \vee \lambda$ is a fuzzy dense set in (X, T). Since (X, T) is the fuzzy submaximal space, $(1-\mu) \vee \lambda$ is a fuzzy open set in (X, T). Then, since μ is a fuzzy open set in (X, T), $[(1-\mu) \vee \lambda] \wedge \mu$ is a fuzzy open set in (X, T).

(ii). Now $\lambda \leq [(1-\mu) \vee \lambda]$ implies that $\lambda \wedge \mu \leq [(1-\mu) \vee \lambda] \wedge \mu$ and then $\lambda \leq [(1-\mu) \vee \lambda] \wedge \mu$ (since $\lambda \leq \mu, \lambda \wedge \mu = \lambda$). Also $[(1-\mu) \vee \lambda] \wedge \mu = [(1-\mu) \wedge \mu] \vee [\lambda \wedge \mu] = [(1-\mu) \wedge \mu] \vee \lambda$. Now $(1-\mu) \wedge \mu = (1 - \operatorname{int} cl(\lambda) \wedge \operatorname{int} cl(\lambda) = cl \operatorname{int}(1-\lambda) \wedge \operatorname{int} cl(\lambda) \leq cl(1-\lambda) \wedge cl(\lambda) \leq cl(\lambda)$ and thus

$[(1-\mu)\vee\lambda]\wedge\mu \leq cl(\lambda)$. Let $\eta = [(1-\mu)\vee\lambda]\wedge\mu$. Hence $\lambda \leq \eta \leq cl(\lambda)$, in (X, T).

Proposition 3.10. *If (X, T) is a fuzzy almost resolvable and fuzzy submaximal space, then (X, T) is not a fuzzy second category space.*

Proof. Let (X, T) be a fuzzy almost resolvable space. Then, $\vee_{i=1}^{\infty}(\lambda_i) = 1_X$, where the fuzzy sets (λ_i)'s in (X, T) are such that $\operatorname{int}(\lambda_i) = 0$. Since (X, T) is the fuzzy submaximal space, by the proposition 3.3, $\operatorname{int}(\lambda_i) = 0$ in (X, T), implies that $\operatorname{int} cl(\lambda_i) = 0$, in (X, T) and hence (λ_i)'s are fuzzy nowhere dense sets in (X, T). Thus, $\vee_{i=1}^{\infty}(\lambda_i) = 1_X$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T), implies that (X, T) is a fuzzy first category space and hence (X, T) is not a fuzzy second category space.

Proposition 3.11. *If λ is the fuzzy simply open set in the fuzzy submaximal space (X, T), then $bd(\lambda)$ is the fuzzy closed set in (X, T).*

Proof. Let λ be the fuzzy simply open set in (X, T). Then, $Bd(\lambda)$ is the fuzzy nowhere dense set in (X, T). Since (X, T) is the fuzzy submaximal space, by the proposition 3.2, $1 - bd(\lambda)$ is the fuzzy open set in (X, T) and hence $bd(\lambda)$ is the fuzzy closed set in (X, T).

Proposition 3.12. *If (X, T) is a fuzzy hyper connected and fuzzy submaximal space, then each fuzzy pre-open set is a fuzzy open set in (X, T).*

Proof. Let λ be the fuzzy pre-open set in (X, T). Since (X, T) is a fuzzy hyper connected space, by the theorem 2.6, λ is a fuzzy dense set in (X, T). Since (X, T) is a fuzzy submaximal space, the fuzzy dense set λ is a fuzzy open set in (X, T).

Proposition 3.13. *If (X, T) is a fuzzy submaximal space, then for any fuzzy set λ defined on X, $cl(\lambda)\wedge(1-\lambda)$ is the fuzzy closed set in (X, T).*

Proof. Let (X, T) be a fuzzy submaximal space and λ be a fuzzy set defined on X. It is claimed that $[(1 - cl(\lambda))\vee\lambda]$ is a fuzzy dense set in (X, T).

Assume the contrary. Suppose that $cl([1 - cl(\lambda)] \vee \lambda) \neq 1$. Then, $1 - cl([1 - cl(\lambda)] \vee \lambda) \neq 0$ and this implies that $1 - (cl[1 - cl(\lambda)] \vee cl(\lambda)) \neq 0$ and $\{1 - cl[1 - cl(\lambda)]\} \wedge \{1 - cl(\lambda)\} \neq 0$. Then, $\operatorname{int} cl(\lambda) \wedge \{1 - cl(\lambda)\} \neq 0$. This will imply that $\operatorname{int} cl(\lambda) \not\leq 1 - \{1 - cl(\lambda)\}$. Then, $\operatorname{int} cl(\lambda) \not\leq cl(\lambda)$, a contradiction. Hence it must be that $cl([1 - cl(\lambda)] \vee \lambda) = 1$, in (X, T). Since (X, T) is the fuzzy submaximal space, $[1 - cl(\lambda)] \vee \lambda$ is the fuzzy open set in (X, T). Then, $1 - ([1 - cl(\lambda)] \vee \lambda)$ is the fuzzy closed set in (X, T) and hence $cl(\lambda) \wedge (1 - \lambda)$ is the fuzzy closed set in (X, T).

Proposition 3.14. *If (X, T) is a fuzzy submaximal space, then (X, T) is a fuzzy irresolvable and fuzzy quasi-maximal space.*

Proof. Let λ be a fuzzy dense set in (X, T). Then, $cl(\lambda) = 1$, in (X, T).

Since (X, T) is the fuzzy submaximal space, λ is a fuzzy open set in (X, T) and $\operatorname{int}(\lambda) = \lambda \neq 0$. Now $cl(1 - \lambda) = 1 - \operatorname{int}(\lambda) \neq 1$ and hence (X, T) is a fuzzy irresolvable space. Now, for the fuzzy dense set λ with $\operatorname{int}(\lambda) \neq 0$, $cl \operatorname{int}(\lambda) = cl(\lambda) = 1$ and hence (X, T) is a fuzzy quasi-maximal space.

Proposition 3.15. *If (X, T) is a fuzzy hyper connected and fuzzy irresolvable space, then (X, T) is a fuzzy quasi-maximal space.*

Proof. Let λ be a fuzzy dense set in (X, T). Then, $cl(\lambda) = 1$, in (X, T). Since (X, T) is a fuzzy irresolvable space, $cl(1 - \lambda) \neq 1$ and then $1 - \operatorname{int}(\lambda) \neq 1$. Thus $\operatorname{int}(\lambda) \neq 0$. Now $\operatorname{int}(\lambda)$ is a fuzzy open set in the fuzzy hyper connected and hence $\operatorname{int}(\lambda)$ is a fuzzy dense set in (X, T). Thus, for the fuzzy dense set λ with $\operatorname{int}(\lambda) \neq 0$, $cl \operatorname{int}(\lambda) = 1$, implies that (X, T) is the fuzzy quasi-maximal space.

Proposition 3.16. *If λ is a fuzzy σ-boundary set in the fuzzy submaximal space (X, T), then λ is a fuzzy F_σ-set in (X, T).*

Proof. Let λ be a fuzzy σ-boundary set in (X, T). Then, $\lambda = \vee_{i=1}^{\infty}(\mu_i)$, where $\mu_i = cl(\lambda_i) \wedge (1 - \lambda_i)$ and (λ_i)'s are fuzzy regular open sets in (X, T). Since (X, T) is the fuzzy submaximal space, by the proposition 3.13, for the fuzzy regular open sets (λ_i)'s, $[cl(\lambda_i) \wedge (1 - \lambda_i)]$'s are fuzzy closed sets in

(X, T). Then, $\lambda = \vee_{i=1}^{\infty}(\mu_i)$, where (μ_i)'s are fuzzy closed sets in (X, T), implies that λ is a fuzzy F_σ-set in (X, T).

4. Fuzzy Quasi-Submaximal Spaces

Definition 4.1. A fuzzy topological space (X, T) is called the fuzzy quasi-submaximal space if $cl(\lambda) \wedge (1 - \text{int}(\lambda),$ is a fuzzy nowhere dense set in (X, T), for each fuzzy dense set λ in (X, T). That is, (X, T) is a fuzzy quasi-submaximal space if for each fuzzy dense set λ in (X, T), $\text{int}\, cl[bd(\lambda)] = 0$, in (X, T).

Example 4.1. Let $X = \{a, b, c\}$. Consider the fuzzy sets λ, μ, γ and α defined on X as follows:

$\lambda : X \to [0, 1]$ is defined as $\lambda(a) = 0.6, \lambda(b) = 0.7, \lambda(c) = 0.4,$

$\mu : X \to [0, 1]$ is defined as $\mu(a) = 0.7, \mu(b) = 0.4, \mu(c) = 0.6,$

$\gamma : X \to [0, 1]$ is defined as $\gamma(a) = 0.4, \gamma(b) = 0.6, \gamma(c) = 0.7,$

$\alpha : X \to [0, 1]$ is defined as $\alpha(a) = 0.7, \alpha(b) = 0.7, \alpha(c) = 0.5.$

Then, $T = \{0, \lambda, \mu, \gamma, \lambda \vee \mu, \lambda \vee \gamma, \mu \vee \gamma, \lambda \vee \gamma, \mu \wedge \gamma, [\gamma \wedge (\lambda \vee \mu)], [\mu \wedge (\lambda \vee \gamma)], [\lambda \wedge (\mu \vee \gamma)], [\gamma \vee (\lambda \wedge \mu)], [\mu \vee (\lambda \wedge \gamma)], [\lambda \vee (\mu \wedge \lambda)], [\lambda \wedge \mu \wedge \gamma], [\lambda \vee \mu \vee \gamma], 1\}$ is the fuzzy topology on X.

On computation $\lambda, \mu, \gamma, \lambda \vee \mu, \lambda \vee \gamma, \mu \vee \gamma, \lambda \vee (\mu \wedge \gamma), \mu \vee (\gamma \wedge \gamma), \gamma \vee (\lambda \wedge \mu), \lambda \vee \mu \vee \gamma$ and α are fuzzy dense sets in (X, T). For the fuzzy dense set λ, $[\eta = \lambda, \mu, \gamma, \lambda \vee \mu, \lambda \vee \gamma, \mu \vee \gamma, \lambda \vee (\mu \vee \gamma), \mu \vee (\lambda \wedge \gamma), \gamma \vee (\lambda \wedge \mu), \lambda \vee \mu \vee \gamma]$, on computation $\text{int}\, cl[bd(\lambda)] = \text{int}\, cl(1 - \eta) = 1 - cl\, \text{int}(\eta) = 1 - cl(\eta) = 1 - 1 = 0$, in (X, T).

Also $\text{int}\, cl[bd(\alpha)] = \text{int}\, cl[cl(\alpha) \wedge cl(1 - \alpha)] = \text{int}\, cl[1 \wedge cl(1 - \alpha)] = \text{int}\, cl[cl(1 - \alpha)] = \text{int}\, cl(1 - \alpha) = 1 - cl\, \text{int}(\alpha) = 1 - cl(\lambda) = 1 - 1 = 0,$ in (X, T). Hence (X, T) is a fuzzy quasi-submaximal space but not a fuzzy submaximal space, since $\alpha \notin T$.

Example 4.2. Let $X = \{a, b, c\}$. Consider the fuzzy sets α, β, γ and δ defined on X as follows:

$\alpha : X \to [0, 1]$ is defined as $\alpha(a) = 0.4,\ \alpha(b) = 0.5,\ \alpha(c) = 0.7,$

$\beta : X \to [0, 1]$ is defined as $\beta(a) = 0.6,\ \beta(b) = 0.7,\ \beta(c) = 0.5,$

$\gamma : X \to [0, 1]$ is defined as $\gamma(a) = 0.6,\ \gamma(b) = 0.5,\ \gamma(c) = 0.5,$

$\delta : X \to [0, 1]$ is defined as $\delta(a) = 0.6,\ \delta(b) = 0.6,\ \delta(c) = 0.5.$

Then, $T = \{0, \alpha, \beta, \gamma, \alpha\beta, \alpha \vee \gamma, \alpha \wedge \beta, 1\}$ is the fuzzy topology on X. On computation, $cl(\delta) = 1$ and thus δ is the fuzzy dense set in (X, T) and $\operatorname{int}(\delta) = \gamma$ and $\operatorname{int} cl[bd(\delta)] = \operatorname{int} cl[cl(\delta) \wedge cl(1 - \delta)] = \operatorname{int} cl[1 \wedge cl(1 - \delta)]$ $= \operatorname{int} cl[cl(1 - \delta)] = \operatorname{int} cl(1 - \delta) = 1 - cl \operatorname{int}(\delta) = 1 - cl(\gamma) = 1 - (1 - [\alpha \wedge \beta])$ $= [\alpha \wedge \beta] \neq 0$. Hence (X, T) is not a fuzzy quasi-submaximal space.

Proposition 4.1. *If* (X, T) *is a fuzzy quasi-submaximal space, then for the fuzzy dense set* λ *in* (X, T), $cl \operatorname{int}(\lambda) = 1$, *in* (X, T).

Proof. Let λ be a fuzzy dense set in (X, T). Since (X, T) is a fuzzy quasi-submaximal space, for the fuzzy dense set λ in (X, T), $[cl(\lambda) \wedge (1 - \operatorname{int}(\lambda))]$, is a fuzzy nowhere dense set in (X, T) and $\operatorname{int} cl[cl(\lambda) \wedge (1 - \operatorname{int}(\lambda))] = 0$ in (X, T). Now $\operatorname{int}[cl(\lambda) \wedge (1 - \operatorname{int}(\lambda))] \leq \operatorname{int} cl[cl(\lambda) \wedge (1 - \operatorname{int}(\lambda))]$ implies that $\operatorname{int}[cl(\lambda) \wedge (1 - \operatorname{int}(\lambda))] = 0$ and then $\operatorname{int} cl(\lambda) \wedge \operatorname{int}(1 - \operatorname{int}(\lambda)) = 0$. This implies that $\operatorname{int}(1) \wedge \operatorname{int}(1 - \operatorname{int}(\lambda)) = 0$ and $1 \wedge \operatorname{int} cl(1 - \lambda) = 0$. Thus $\operatorname{int} cl(1 - \lambda) = 0$ and thus $1 - cl \operatorname{int}(\lambda) = 0$. Hence $cl \operatorname{int}(\lambda) = 1$, in (X, T).

Proposition 4.2. *If* (X, T) *is a fuzzy quasi-submaximal space, then for any two fuzzy dense sets* λ *and* μ *in* (X, T), $\lambda \wedge \mu \neq 0$, *in* (X, T).

Proof. Suppose that $\lambda \wedge \mu = 0$, for the fuzzy dense sets λ and μ in (X, T). Then, $\mu \leq 1 - \lambda$. This will imply that $cl(\mu) \leq cl(1 - \lambda)$, in (X, T). Now $\operatorname{int} cl[bd(\lambda)] = \operatorname{int} cl[cl(\lambda) \wedge cl(1 - \lambda)] = \operatorname{int} cl[1 \wedge cl(1 - \lambda)] = \operatorname{int} cl[cl(1 - \lambda)]$ $\geq \operatorname{int} cl[cl(\mu)] \geq \operatorname{int} cl[1] = 1$. Then, $\operatorname{int} cl[bd(\lambda)] \neq 0$, a contradiction, since

(X, T) being a fuzzy quasi-submaximal space in which for the fuzzy dense set λ in (X, T), $\text{int}\, cl[bd(\lambda)] = 0$, in (X, T). Hence the supposition $\lambda \wedge \mu = 0$ does not hold in (X, T). Hence $\lambda \wedge \mu \neq 0$, for the fuzzy dense sets λ and μ in (X, T).

Proposition 4.3. *If* (X, T) *is a fuzzy submaximal space, then* (X, T) *is a fuzzy quasi-submaximal space.*

Proof. Let (X, T) be a fuzzy submaximal space. Then, the fuzzy dense set λ is a fuzzy open set in (X, T). Now $\text{int}\, cl[bd(\lambda)] = \text{int}\, cl[cl(\lambda) \wedge cl(1-\lambda)]$ $= \text{int}\, cl[1 \wedge cl(1-\lambda)] = \text{int}\, cl[cl(1-\lambda)] = \text{int}\, cl(1-\lambda) = 1 - cl\, \text{int}(\lambda) = 1 - cl(\lambda)$ $= 1 - 1 = 0$. Hence, for the fuzzy dense set λ in (X, T), $\text{int}\, cl[bd(\lambda)] = 0$, implies that (X, T) is a fuzzy quasi-submaximal space.

Remark 4.1. The converse of the above proposition need not be true. That is, fuzzy quasi-submaximal spaces need not be fuzzy submaximal spaces. For, in example 3.1, $cl(\alpha) = 1$ and $\alpha \notin T$ and hence (X, T) is not a fuzzy submaximal even though (X, T) is a fuzzy quasi-submaximal space.

Under what conditions does the converse of the proposition 4.3, hold?. In the class of fuzzy nodec spaces and fuzzy globally disconnected spaces, the answer is "yes".

Proposition 4.4. *If* (X, T) *is a fuzzy quasi-submaximal and fuzzy nodec space, then* (X, T) *is a fuzzy submaximal space.*

Proof. Let λ be a fuzzy dense set in (X, T). Since (X, T) is the fuzzy quasi-submaximal space, for the fuzzy dense set λ in (X, T), $\text{int}\, cl[bd(\lambda)] = 0$, in (X, T). Then, $\text{int}\, cl[cl(\lambda) \wedge cl(1-\lambda)] = 0$. Then, $\text{int}\, cl[1 \wedge cl(1-\lambda)] = 0$ and $\text{int}\, cl[cl(1-\lambda)] = 0$. Thus $\text{int}\, cl(1-\lambda) = 0$ and this implies that $1-\lambda$ is the fuzzy nowhere dense set in (X, T). Since (X, T) is the fuzzy nodec space, $1-\lambda$ is the fuzzy closed set and thus λ is the fuzzy open set in (X, T). Hence (X, T) is a fuzzy submaximal space.

Proposition 4.5. *If* λ *is a fuzzy dense set in the fuzzy quasi-submaximal space* (X, T), *then* λ *is a fuzzy semi-open set in* (X, T).

Proof. Let λ be a fuzzy dense set in (X, T). Since (X, T) is a fuzzy quasi-submaximal space, by the proposition 4.1, for the fuzzy dense set λ in (X, T), $cl\,\text{int}(\lambda) = 1$, in (X, T). This implies that $\lambda \leq cl\,\text{int}(\lambda)$ and hence λ is a fuzzy semi-open set in (X, T).

Proposition 4.6. *If (X, T) is the fuzzy quasi-submaximal and fuzzy globally disconnected space, then (X, T) is a fuzzy submaximal space.*

Proof. Let λ be a fuzzy dense set in (X, T). Since (X, T) is a fuzzy quasi-submaximal space, by the proposition 4.5, λ is a fuzzy semi-open set in (X, T). Since (X, T) is the fuzzy globally disconnected space, the fuzzy semi-open set λ is a fuzzy open in (X, T) and hence (X, T) is a fuzzy submaximal space.

Proposition 4.7. *If $\text{int}(\lambda) = 0$ for the fuzzy set defined on X in the fuzzy quasi-submaximal space (X, T), then λ is a fuzzy semi-closed set in (X, T).*

Proof. Suppose that $\text{int}(\lambda) = 0$, in (X, T). Then, $cl(1 - \lambda) = 1 - \text{int}(\lambda) = 1$ and then $1 - \lambda$ is a fuzzy dense set in (X, T). Since (X, T) is a fuzzy quasi-submaximal space, by the proposition 4.1, for the fuzzy dense set $1 - \lambda$ in (X, T), $cl\,\text{int}(1 - \lambda) = 1$, in (X, T). This implies that $1 - \text{int}\,cl(\lambda) = 1$ and then $\text{int}\,cl(\lambda) = 0$. This implies that $\text{int}\,cl(\lambda) \leq \lambda$, in (X, T) and hence λ is a fuzzy semi-closed set in (X, T).

Proposition 4.8. *If λ is a fuzzy dense set in the fuzzy quasi-submaximal space (X, T), then $1 - \lambda$ is a fuzzy nowhere dense set in (X, T).*

Proof. Let λ be a fuzzy dense set in (X, T). Since (X, T) is a fuzzy quasi-submaximal space, by the proposition 4.1, for the fuzzy dense set λ in (X, T), $cl\,\text{int}(\lambda) = 1$, in (X, T). This implies that $1 - cl\,\text{int}(\lambda) = 0$, in (X, T). By the lemma 2.1, $\text{int}\,cl(1 - \lambda) = 0$. Hence $1 - \lambda$ is a fuzzy nowhere dense set in (X, T).

Proposition 4.9. *If (X, T) is a fuzzy quasi-submaximal space, then (X, T) is a fuzzy strongly irresolvable space.*

Proof. Let λ be a fuzzy dense set in (X, T). Since (X, T) is a fuzzy quasi-submaximal space, by the proposition 4.1, for the fuzzy dense set λ in (X, T), $cl\,\text{int}(\lambda) = 1$, in (X, T). This implies that (X, T) is a fuzzy strongly irresolvable space.

Proposition 4.10. *If (X, T) is a fuzzy quasi-submaximal space, then for the fuzzy dense set λ in (X, T), $cl(1 - \lambda) \neq 1$, in (X, T).*

Proof. Suppose that $cl(1 - \lambda) = 1$, for the fuzzy dense set λ in (X, T). Since (X, T) is a fuzzy quasi-submaximal space, by the proposition 4.1, for the fuzzy dense set $1 - \lambda$ in (X, T), $cl\,\text{int}(1 - \lambda) = 1$, in (X, T). This will imply that $1 - \text{int}\,cl(\lambda) = 1$ and then $\text{int}\,cl(\lambda) = 0$. Now $cl(\lambda) = 1$ will imply that $\text{int}[1] = 0$, a contradiction. Hence, for the fuzzy dense set λ in (X, T), $cl(1 - \lambda) \neq 1$, in (X, T).

Proposition 4.11. *If (X, T) is a fuzzy quasi-submaximal space, then (X, T) is a fuzzy irresolvable space.*

Proof. Let λ be a fuzzy dense set in (X, T). Since (X, T) is a fuzzy quasi-submaximal space, by the proposition 4.10, for the fuzzy dense set λ in (X, T), $cl(1 - \lambda) \neq 1$, in (X, T). Hence (X, T) is a fuzzy irresolvable space.

Remark 4.2. From the proposition 4.11, it is to be observed that if (X, T) is a fuzzy resolvable space, then (X, T) is not a fuzzy quasi-submaximal space.

For, if (X, T) is a fuzzy resolvable space, then there exists a fuzzy dense set λ in (X, T) such that $cl(1 - \lambda) = 1$, in (X, T) and $\text{int}\,cl[bd(\lambda)] = \text{int}\,cl\,[cl(\lambda) \wedge cl(1 - \lambda)] = \text{int}\,cl[1 \wedge 1] = \text{int}\,cl[1] = 1 \neq 0$, in (X, T).

Proposition 4.12. *If $\lambda \leq 1 - \mu$, where λ is a fuzzy dense set and μ is the fuzzy set in the fuzzy quasi-submaximal space (X, T), then μ is a fuzzy nowhere dense set in (X, T).*

Proof. Suppose that $\lambda \leq 1 - \mu$, where λ is a fuzzy dense set in (X, T). Since (X, T) is a fuzzy quasi-submaximal space, by the proposition 4.1, for the fuzzy dense set λ in (X, T), $cl\operatorname{int}(\lambda) = 1$, in (X, T). Now $\lambda \leq 1 - \mu$, implies that $cl\operatorname{int}(\lambda) \leq cl\operatorname{int}(1 - \mu)$ and then $1 \leq cl\operatorname{int}(1 - \mu)$. That is, $cl\operatorname{int}(1 - \mu) = 1$. By the lemma 2.1, $1 - \operatorname{int} cl(\mu) = 1$ and thus $\operatorname{int} cl(\mu) = 0$. Hence μ is a fuzzy nowhere dense set in (X, T).

Proposition 4.13. *If* (X, T) *is the fuzzy quasi-submaximal and fuzzy almost resolvable space, then* (X, T) *is a fuzzy first category space.*

Proof. Let λ be a fuzzy dense set in (X, T). Since (X, T) is a fuzzy quasi-submaximal space, by the proposition 4.1, for the fuzzy dense set λ in (X, T), $cl\operatorname{int}(\lambda) = 1$, in (X, T). By the hypothesis, (X, T) is the fuzzy almost resolvable space and then, by the theorem 2.1, (X, T) is the fuzzy first category space.

Remark 4.3. From the proposition 4.13, it is to be observed that if (X, T) is the fuzzy quasi-submaximal and fuzzy almost irresolvable space, then (X, T) is a fuzzy second category space.

Proposition 4.14. *If* (X, T) *is the fuzzy quasi-submaximal and fuzzy almost resolvable space, then* (X, T) *is not a fuzzy Baire space.*

Proof. Suppose that (X, T) is a fuzzy Baire space. Then, by the theorem 2.2, (X, T) will be a fuzzy second category space, a contradiction to (X, T) being the fuzzy first category space, [by the proposition 4.13]. Hence (X, T) is not a fuzzy Baire space.

Proposition 4.15. *If* (X, T) *is a fuzzy quasi-submaximal space, then* (X, T) *is the fuzzy almost GP-space.*

Proof. Let λ be a fuzzy dense and fuzzy G_δ-set in (X, T). Since (X, T) is a fuzzy quasi-submaximal space, by the proposition 4.10, for the fuzzy dense set λ in (X, T), $cl(1 - \lambda) \neq 1$, in (X, T). This implies that $1 - \operatorname{int}(\lambda) \neq 1$ and then $\operatorname{int}(\lambda) \neq 0$, in (X, T). Thus, for the fuzzy dense and fuzzy G_δ-set λ in (X, T), $\operatorname{int}(\lambda) \neq 0$, implies that (X, T) is the fuzzy almost GP-space.

Proposition 4.16. *If* (X, T) *is a fuzzy quasi-submaximal space, then* (X, T) *is the fuzzy quasi-maximal space.*

Proof. Let λ be a fuzzy dense set in (X, T) with $\operatorname{int}(\lambda) \neq 0$. Since (X, T) is the fuzzy quasi-submaximal space, by the proposition 4.1, $cl\operatorname{int}(\lambda) = 1$, in (X, T). Hence (X, T) is the fuzzy quasi-maximal space.

Proposition 4.17. *If* (X, T) *is a fuzzy quasi-submaximal space, then* (X, T) *is the fuzzy GID-space.*

Proof. Let λ be a fuzzy dense and fuzzy G_δ-set λ in (X, T). Since (X, T) is the fuzzy quasi-submaximal space, by the proposition 4.1, $cl\operatorname{int}(\lambda) = 1$, in (X, T). Hence (X, T) is the fuzzy GID-space.

Proposition 4.18. *If* λ *is a fuzzy* σ*-nowhere dense set in the fuzzy quasi-submaximal space* (X, T), *then* λ *is a fuzzy nowhere dense set in* (X, T).

Proof. Let λ be a fuzzy σ-nowhere dense set in (X, T). Then, λ is a fuzzy F_σ-set with $\operatorname{int}(\lambda) = 0$, in (X, T). Now $cl(1 - \lambda) = 1 - \operatorname{int}(\lambda) = 1 - 0 = 1$. Thus, $1 - \lambda$ is a fuzzy dense set λ in (X, T). Since (X, T) is the fuzzy quasi-submaximal space, by the proposition 4.1, $cl\operatorname{int}(1 - \lambda) = 1$, in (X, T) and $1 - \operatorname{int}cl(\lambda) = 1$. Then $\operatorname{int}cl(\lambda) = 0$. Hence λ is a fuzzy nowhere dense set in (X, T).

Proposition 4.19. *If* λ *is a fuzzy* σ*-first category set in the fuzzy quasi-submaximal space* (X, T), *then* λ *is a fuzzy first category set in* (X, T).

Proof. Let λ be a fuzzy σ-first category set in (X, T). Then, $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy σ-nowhere dense sets in (X, T). Since (X, T) is the fuzzy quasi-submaximal space, by the proposition 4.18, the fuzzy σ-nowhere dense sets (λ_i)'s are fuzzy nowhere dense sets in (X, T). Thus, $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T), implies that λ is a fuzzy first category set in (X, T).

Proposition 4.20. *If* (X, T) *is a fuzzy* σ*-first category and fuzzy quasi-submaximal space, then* (X, T) *is the fuzzy first category space.*

Proof. Let (X, T) be a fuzzy σ-first category space. Then, $\vee_{i=1}^{\infty} (\lambda_i) = 1_X$, where (λ_i)'s are fuzzy σ-nowhere dense sets in (X, T). Since (X, T) is the fuzzy quasi-submaximal space, by the proposition 4.18, the fuzzy σ-nowhere dense sets (λ_i)'s are fuzzy nowhere dense sets in (X, T). Thus, $\vee_{i=1}^{\infty} (\lambda_i) = 1_X$ where (λ_i)'s are fuzzy nowhere dense sets in (X, T), implies that (X, T). is the fuzzy first category space.

The following proposition gives the condition for the fuzzy quasi-submaximal spaces to become fuzzy almost P-spaces.

Proposition 4.21. *If each fuzzy G_δ-set is a fuzzy pre-open set in the fuzzy quasi-submaximal space, then (X, T) is the fuzzy almost P-space.*

Proof. Let λ be a fuzzy G_δ-set in (X, T). By the hypothesis λ is a fuzzy pre-open set in (X, T) and $\lambda \leq \operatorname{int} cl(\lambda)$. It is claimed that $\operatorname{int}(\lambda) \neq 0$, in (X, T). Suppose that $\operatorname{int}(\lambda) = 0$, in (X, T). Then, $cl(1 - \lambda) = 1 - \operatorname{int}(\lambda) = 1$ and then $1 - \lambda$ is a fuzzy dense set in (X, T). Since (X, T) is the fuzzy quasi-submaximal space, by the proposition 4.1, $cl \operatorname{int}(1 - \lambda) = 1$, in (X, T) and this implies that $\operatorname{int} cl(\lambda) = 0$ and then $\operatorname{int} cl(\lambda) \leq \lambda$ which results in the fuzzy semi-closedness of λ, a contradiction. Thus, $\operatorname{int}(\lambda) \neq 0$, for the fuzzy G_δ-set in (X, T). Hence (X, T) is the fuzzy almost P-space.

Proposition 4.22. *If each fuzzy G_δ-set is a fuzzy pre-open set in the fuzzy quasi-submaximal space, then (X, T) is the fuzzy second category space.*

Proof. Let λ be a fuzzy G_δ-set in (X, T) such that $\lambda \leq \operatorname{int} cl(\lambda)$ in (X, T). Since (X, T) is the fuzzy quasi-submaximal space, by the proposition 4.21, (X, T) is the fuzzy almost P-space. Then, by the theorem 2.4, (X, T) is the fuzzy second category space.

Proposition 4.23. *If (X, T) is a fuzzy quasi-submaximal space, then (X, T) is the fuzzy weakly Volterra space.*

Proof. Let (X, T) be a fuzzy quasi-submaximal space. Then, by the proposition 4.15, (X, T) is the fuzzy almost *GP*-space. By the theorem 2.5, (X, T) is the fuzzy weakly Volterra space.

Proposition 4.24. *If* (X, T) *is a fuzzy* σ*-resolvable space, then* (X, T) *is not a fuzzy quasi-submaximal space.*

Proof. Let (X, T) be a fuzzy σ-resolvable space. Then, $\vee_{i=1}^{\infty} (\lambda_i) = 1$, where (λ_i)'s are fuzzy dense sets in (X, T) such that $\lambda_i \leq 1 - \lambda_j$, for $1 \neq j$. Now $\lambda_i \leq 1 - \lambda_j$, implies that $cl(\lambda_i) \leq cl(1 - \lambda_j)$ and then $1 \leq cl(1 - \lambda_j)$ and $cl(1 - \lambda_j) = 1$, in (X, T). Now $\operatorname{int} cl[bd(\lambda_j)] = \operatorname{int} cl[cl(\lambda_j) \wedge cl(1 - \lambda_j)] = \operatorname{int} cl[1 \wedge 1] = \operatorname{int} cl[1] = 1 \neq 0$. Hence, for the fuzzy dense set λ_j in (X, T), $\operatorname{int} cl[bd(\lambda_j)] \neq 0$, implies that (X, T) is not a fuzzy quasi-submaximal space.

The following proposition gives the condition for the fuzzy quasi-submaximal spaces to become fuzzy Baire spaces.

Proposition 4.25. *If* $cl[\wedge_{i=1}^{\infty} (\lambda_i)] = 1$, *where* (λ_i)*'s are fuzzy sets in the fuzzy quasi-submaximal space* (X, T), *then* (X, T) *is the fuzzy Baire space.*

Proof. Suppose that $cl[\wedge_{i=1}^{\infty} (\lambda_i)] = 1$, where (λ_i)'s are fuzzy sets defined on *X*. Since $cl[\wedge_{i=1}^{\infty} (\lambda_i)] \leq \wedge_{i=1}^{\infty} cl(\lambda_i)$, $\vee_{i=1}^{\infty} cl(\lambda_i) = 1$ and then $cl(\lambda_i) = 1$, in (X, T). Thus, (λ_i)'s are fuzzy dense sets in (X, T). Since (X, T) is the fuzzy quasi-submaximal space, by the proposition 4.1, $cl \operatorname{int}(\lambda_i) = 1$, in (X, T). Then, $1 - cl \operatorname{int}(\lambda_i) = 0$ and $\operatorname{int} cl(1 - \lambda_i) = 0$. Now $cl[\wedge_{i=1}^{\infty} (\lambda_i)] = 1$, implies that $\operatorname{int}[\vee_{i=1}^{\infty} (1 - \lambda_i)] = 0$, where $(1 - \lambda_i)$'s are the fuzzy nowhere dense sets in (X, T), implies that (X, T) is the fuzzy Baire space.

The following proposition gives the condition under which fuzzy σ-Baire spaces become fuzzy Baire spaces.

Proposition 4.26. *If* (X, T) *is a fuzzy quasi-submaximal and fuzzy* σ-*Baire space, then* (X, T) *is the fuzzy Baire space.*

Proof. Let (X, T) be a fuzzy σ-Baire space. Then, $\operatorname{int}[\vee_{i=1}^{\infty} (\lambda_i)] = 0$, where (λ_i)'s are fuzzy σ-nowhere dense sets in (X, T). Since (X, T) is the fuzzy quasi-submaximal space, by the proposition 4.18, the fuzzy σ-nowhere dense sets (λ_i)'s are fuzzy nowhere dense sets in (X, T). Thus, $\operatorname{int}[\vee_{i=1}^{\infty} (\lambda_i)] = 0$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T), implies that (X, T) is the fuzzy Baire space.

Proposition 4.27. *If* λ *is the fuzzy* σ*-boundary set in the fuzzy weakly Baire and fuzzy quasi-submaximal space* (X, T), *then* λ *is the fuzzy nowhere dense set in* (X, T).

Proof. Let λ be the fuzzy σ-boundary set in (X, T). Since (X, T) is the fuzzy weakly Baire space, by the theorem 2.7, λ is the fuzzy σ-nowhere dense set in (X, T). Since (X, T) is the fuzzy quasi-submaximal space, by the proposition 4.18, the fuzzy σ-nowhere dense set λ is the fuzzy nowhere dense set in (X, T).

Proposition 4.28. *If* λ *is the fuzzy* σ*-boundary set in the fuzzy perfectly disconnected, fuzzy weakly Baire and fuzzy quasi-submaximal space* (X, T), *then there exists a fuzzy* σ*-nowhere dense* η *in* (X, T) *such that* $\eta \leq \lambda$.

Proof. Let λ be the fuzzy σ-boundary set in (X, T). Since (X, T) is the fuzzy perfectly disconnected space, by the theorem 2.8, there exists a fuzzy F_σ-set η in (X, T) such that $\eta \leq \lambda$. Since (X, T) is the fuzzy weakly Baire and fuzzy quasi-submaximal space, by the proposition 4.27, λ is the fuzzy nowhere dense set in (X, T). Then, $\operatorname{int} cl(\lambda) = 0$, in (X, T). Now $\operatorname{int}(\lambda) \leq \operatorname{int} cl(\lambda)$, implies that $\operatorname{int}(\lambda) = 0$, in (X, T). Also $\eta \leq \lambda$, implies that $\operatorname{int}(\eta) \leq \operatorname{int}(\lambda)$ and thus $\operatorname{int}(\eta) = 0$, in (X, T). Hence η is the fuzzy F_σ-set with $\operatorname{int}(\eta) = 0$, in (X, T). Hence η is the fuzzy σ-nowhere dense set in (X, T).

Proposition 4.29. *If* λ *is a fuzzy dense set in the fuzzy quasi-submaximal and fuzzy globally disconnected space* (X, T), *then* λ *is the fuzzy simply-open set in* (X, T).

Proof. Let λ be a fuzzy dense set in (X, T). Since (X, T) is the fuzzy quasi-submaximal space, by the proposition 4.5, λ is a fuzzy semi-open set in

(X, T). Since (X, T) is the fuzzy globally disconnected space, the fuzzy semi-open set λ is the fuzzy open set in (X, T). Now $\operatorname{int} cl[bd(\lambda)] = \operatorname{int} cl[cl(\lambda) \wedge cl(1-\lambda)] = \operatorname{int} cl[1 \wedge cl(1-\lambda)] = \operatorname{int} cl[cl(1-\lambda)] = \operatorname{int} cl(1-\lambda)1 - cl \operatorname{int}(\lambda) = 1 - cl(\lambda) = 1 - 1 = 0$. Hence λ is the fuzzy simply-open set in (X, T).

Proposition 4.30. *If each fuzzy G_δ-set is a fuzzy dense set in the fuzzy quasi-submaximal space (X, T), then (X, T) is the fuzzy ∂-space.*

Proof. Let λ be a fuzzy G_δ-set in (X, T). Then, by the hypothesis, λ is a fuzzy dense set in (X, T). Since (X, T) is the fuzzy quasi-submaximal space, by the proposition 4.1, $cl \operatorname{int}(\lambda) = 1$, in (X, T). Now $\operatorname{int} cl[bd(\lambda)] = \operatorname{int} cl[cl(\lambda) \wedge cl(1-\lambda)] = \operatorname{int} cl[1 \wedge cl(1-\lambda)] = \operatorname{int} cl[cl(1-\lambda)] = \operatorname{int} cl(1-\lambda) = 1 - cl \operatorname{int}(\lambda) = 1 - 1 = 0$. Hence λ is the fuzzy simply-open set in (X, T). Thus, the fuzzy G_δ-set λ is the fuzzy simply-open set in (X, T) and hence (X, T) is the fuzzy ∂-space.

Proposition 4.31. *If λ is a fuzzy G_δ-set in (X, T) such that $cl(\lambda) = 1$, in the fuzzy quasi-submaximal space (X, T), then $\operatorname{int}[\lambda \wedge (1-\lambda)] = 0$, in (X, T).*

Proof. Let λ be a fuzzy G_δ-set in (X, T). Then, by the hypothesis, λ is a fuzzy dense set in (X, T). Then, by the proposition 4.30, (X, T) is the fuzzy ∂-space and by the theorem 2.9, $\operatorname{int}[\lambda \wedge (1-\lambda)] = 0$, in (X, T).

Proposition 4.32. *If (X, T) is a fuzzy quasi-submaximal and fuzzy nodef space, then (X, T) is the fuzzy DG_δ-space.*

Proof. Let λ be a fuzzy dense (but not fuzzy open) set in (X, T). Since (X, T) is the fuzzy quasi-submaximal space, by the proposition 4.1, $cl \operatorname{int}(\lambda) = 1$, in (X, T). Then, $1 - cl \operatorname{int}(\lambda) = 0$ and $\operatorname{int} cl(1-\lambda) = 0$. Thus, $1 - \lambda$ is the fuzzy nowhere dense set in (X, T). Since (X, T) is the fuzzy nodef space, $1 - \lambda$ is the fuzzy F_σ-set in (X, T) and thus λ is the fuzzy G_δ-set in (X, T). Hence, the fuzzy dense (but not fuzzy open) set λ is the fuzzy G_δ-set in (X, T), implies that (X, T) is the fuzzy DG_δ-space.

Proposition 4.33. *If λ is a fuzzy dense set in the fuzzy quasi-submaximal and fuzzy nodef space, then λ is the fuzzy G_δ-set such that $\operatorname{int}(\lambda) \neq 0$, in (X, T).*

Proof. Let λ be a fuzzy dense set in (X, T). Since (X, T) is the fuzzy quasi-submaximal space, by the proposition 4.1, $cl\operatorname{int}(\lambda) = 1$, in (X, T). Then, $1 - cl\operatorname{int}(\lambda) = 0$ and $\operatorname{int}cl(1 - \lambda) = 0$. Thus, $1 - \lambda$ is the fuzzy nowhere dense set in (X, T). Since (X, T) is the fuzzy nodef space, $1 - \lambda$ is the fuzzy F_σ-set in (X, T) and thus λ is the fuzzy G_δ-set in (X, T). Hence, the fuzzy dense set λ is the fuzzy G_δ-set in (X, T). Since (X, T) is the fuzzy quasi-submaximal space, by the proposition 4.10, for the fuzzy dense set λ in (X, T), $cl(1 - \lambda) \neq 1$, in (X, T). Then, by the lemma 2.1, $1 - \operatorname{int}(\lambda) \neq 1$ and thus $\operatorname{int}(\lambda) \neq 0$, in (X, T). Hence the fuzzy dense set λ is the fuzzy G_δ-set such that $\operatorname{int}(\lambda) \neq 0$, in (X, T).

Remark 4.4. In view of the proposition 4.33, one will have the following result: "Fuzzy dense sets are fuzzy G_δ-sets with non-zero interiors in fuzzy quasi-submaximal and fuzzy nodef spaces".

Proposition 4.34. *If λ is a fuzzy nowhere dense set in the fuzzy quasi-submaximal and fuzzy nodef space, then λ is the fuzzy F_σ-set such that $cl(\lambda) \neq 1$, in (X, T).*

Proof. Let λ be a fuzzy nowhere dense set in (X, T). Since (X, T) is the fuzzy nodef space, λ is the fuzzy F_σ-set in (X, T). It is claimed that $cl(\lambda) \neq 1$, in (X, T). Assume the contrary. Suppose that $cl(\lambda) = 1$, in (X, T). Since (X, T) is the fuzzy quasi-submaximal space, by the proposition 4.1, $cl\operatorname{int}(\lambda) = 1$, in (X, T) and $1 - \lambda$ is the fuzzy nowhere dense set in (X, T). Since (X, T) is the fuzzy nodef space, $1 - \lambda$ is the fuzzy F_σ-set in (X, T) and then λ is the fuzzy G_δ-set in (X, T), a contradiction. Hence it must be that $cl(\lambda) \neq 1$, in (X, T). Hence λ is the fuzzy F_σ-set such that $cl(\lambda) \neq 1$, in (X, T).

Proposition 4.35. *If λ is a fuzzy dense set in the fuzzy quasi-submaximal, fuzzy nodef and fuzzy P-space, then λ is a fuzzy open set in (X, T).*

Proof. Let λ be a fuzzy dense set in (X, T). Since (X, T) is the fuzzy quasi-submaximal and fuzzy nodef space, by the proposition 4.33, λ is the fuzzy G_δ-set such that $\text{int}(\lambda) \neq 0$, in (X, T). Since (X, T) is the fuzzy P-space, the fuzzy G_δ-set λ is a fuzzy open set in (X, T).

Remark 4.5. In view of the proposition 4.35, one will have the following result: "Fuzzy quasi-submaximal, fuzzy nodef and fuzzy P-spaces are fuzzy submaximal spaces".

Proposition 4.36. *If λ is a fuzzy open set in the quasi-submaximal fuzzy hyper connected and fuzzy nodef space, then λ is a fuzzy dense and fuzzy G_δ-set in (X, T).*

Proof. Let λ be a fuzzy open set in (X, T). Since (X, T) is the fuzzy hyper connected space, λ is a fuzzy dense set in (X, T). Since (X, T) is the fuzzy quasi-submaximal and fuzzy nodef space, by the proposition 4.33, λ is the fuzzy G_δ-set such that $\text{int}(\lambda) \neq 0$, in (X, T). Thus then λ is an fuzzy dense and fuzzy G_δ-set in (X, T).

Conclusions

In this paper several characterizations of fuzzy submaximal spaces are obtained. It is obtained that fuzzy almost resolvable spaces with fuzzy submaximality are not fuzzy second category spaces. Also it is established that fuzzy submaximal spaces are fuzzy irresolvable and fuzzy quasi-maximal spaces and fuzzy hyper connected spaces with fuzzy irresolvability, are fuzzy quasi-maximal spaces.

The notion of fuzzy quasi-submaximal spaces is introduced by means of fuzzy boundary and studied in this paper. It is established that fuzzy submaximal spaces are fuzzy quasi-submaximal spaces. The conditions under which fuzzy quasi-submaximal spaces become fuzzy submaximal spaces are obtained. The conditions for fuzzy quasi-submaximal spaces to become fuzzy almost P-spaces and fuzzy Baire spaces are established. It is established that fuzzy quasi-submaximal spaces are fuzzy irresolvable, fuzzy strongly irresolvable, fuzzy weakly Volterra and fuzzy GID spaces. The condition under which fuzzy quasi-submaximal spaces, become fuzzy second category spaces is also obtained and it is shown that fuzzy σ-resolvable spaces are not fuzzy quasi-submaximal spaces and fuzzy quasi-submaximal, fuzzy nodef and fuzzy P-spaces are fuzzy submaximal spaces.

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