Advances and Applications in Mathematical Sciences
Volume 22, Issue 2, December 2022, Pages 571-580

SELF-SIMILAR AND TRAVELLING-WAVE SOLUTION FOR UNSTEADY NAVIER-STOKES EQUATION

R. ILANGOVANE, K. KRISHNAKUMAR and R. SINUVASAN

[1]Department of Mathematics
Perunthalaivar Kamarajar Institute of
Engineering and Technology (PKIET)
Karaikal, 609 603, India
E-mail: ilangovane@gmail.com

[2]Department of Mathematics
Srinivasa Ramanujan Centre
SASTRA Deemed to be University
Kumbakonam, 612 001, India
E-mail: krishapril09@gmail.com

[3]Department of Mathematics
VIT-AP University
Amaravati, India 522237
E-mail: rsinuvasan@gmail.com

Abstract

We examine the unsteady Navier-Stokes equation for its Lie point symmetries. We find a six dimensional Lie algebra. When we use the self-similar Abelian sub algebra, we obtain the similarity solution for the Navier-Stokes equation. When we use a linear combination of three symmetries with specific values of the parametric functions contained therein, we obtain a travelling-wave solution for the Navier-Stokes equation.

1. Introduction

The mathematical modelling of physical phenomena frequently leads to differential equations and integral equations that have conserved quantities such as mass, momentum and energy. Most real-world problems are

2020 Mathematics Subject Classification: 76Nxx.
Keywords: Navier-Stokes equations, Symmetries, Integrability.
Received January 4, 2022; Accepted January 14, 2022

nonlinear also more complicated and complex problems. A large number of them can be modelled in the form of a system of nonlinear ordinary or partial differential equations. Computer simulations of such mathematical models are being used extensively to solve such problems appearing in fluid dynamics, as given by Navier-Stokes equations and also diffusion phenomena. The Navier-Stokes equations provide an accurate representation of viscous, laminar, turbulent, Isothermal, Newtonian, Non-Newtonian fluid flows and are used in numerous practical situations. Before presenting the Navier-Strokes equations, include the open problems of classifying nonlinear partial differential equations. In the fundamental Navier-Stokes equation for fluid dynamics, the solutions resent fundamental fluid flows. This inherently nonlinear set of partial differential equations has no general solution and only a small number of exact solutions [12]. Analytical solutions are limited to only a few idealised cases, and only a small number of closed-form solutions have been found. Fluid dynamics plays a major part in the design of all these components, modern engineering and analysis of aircraft, boats, biomedical devices, cooling systems for electronic components, jet engines, submarines, rockets, wind turbines, transportation systems moving water, crude oil and natural gas. Numerous natural phenomena such as the rain cycle, weather patterns, the rise of groundwater to the tops of trees, winds and ocean waves the Navier-Stokes equations were derived by following the principles of conservation of mass and momentum. The unknown variables are the velocity vector and the pressure. Through the solution of the formulation in terms of the primary variables, one can acquire the velocity vector and pressure of the fluid flow. Powerful techniques are available when the infinitesimal group is known and nontrivial, and ordinary differential equation reduces by one for each independent generator of the infinitesimal group [7]. The analytical solution of the Navier-Stokes nonlinear partial differential equations is a very challenging task. We propose to seek solutions to the incompressible flow equations for the Navier-Stokes system. As fluid dynamics abound in nonlinearity, it is not surprising that Lie symmetry analysis should find a way to construct analytical solutions to those equations of classical fluid flow.

2. The Navier-Stokes Equations

Today real world problem such as interesting system of nonlinear differential equation appearing in fluid dynamics is given by Navier-Stokes governing equations for unsteady two-dimensional incompressible flow, laminar, viscous and isothermal in tensor form in each of the x-and y-directions are shown as follows

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{3}$$

Where u and v are velocity components in the x-direction and y-direction respectively. ρ and ν are density and kinematic viscosity respectively. And also p is the pressure. Using the follows non-dimensional scales are employed:

$$x' = \frac{x}{L},\ y' = \frac{y}{L},\ t' = \frac{tW^2}{L},\ u' = \frac{uL}{W},\ v' = \frac{vL}{W},\ p' = \frac{p}{\rho W^2}$$

Where $(')$ designated a dimensionless quantity. Let L, W, p denoted a characteristic length, velocity and pressure respectively for a particular problem. The dimensionless from equation (1), (2) and (3) are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{4}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

with $(')$ being dropped for simplicity and Reynolds (Re) is defined as $\mathrm{Re} = \frac{VL}{\nu}$.

3. Symmetry Properties of the Unsteady Navier-Stokes Equation

The Lie point symmetries of (4) are 1

$$\Gamma_1 = \partial_t \quad (5)$$

$$\Gamma_2 = f(t)\partial_p \quad (6)$$

$$\Gamma_3 = -y\partial_x + x\partial_y - v\partial_u + u\partial_v \quad (7)$$

$$\Gamma_4 = g(t)\partial_y + g'(t)\partial_v - yg''(t)\partial_p \quad (8)$$

$$\Gamma_5 = h(t)\partial_x + h'(t)\partial_u - xh''(t)\partial_p \quad (9)$$

$$\Gamma_6 = 2t\partial_t + x\partial_x + y\partial_y - 2p\partial_p - u\partial_u - v\partial_v. \quad (10)$$

The algebra of these six symmetries conveys little meaning due to the presence of the arbitrary functions $f(t)$, $g(t)$ and $h(t)$. However, it does decompose into two three-dimensional sub algebras which individually do make sense. The sub algebra comprising Γ_1, Γ_3 and Γ_6 constitutes $A_1 \oplus A_2$ and that comprising Γ_2, Γ_4 and Γ_5 constitutes $3A_1$.

We use the Mathematica Addon, Sym [1-4], for the calculation of Lie symmetries throughout this paper. For the description of an algebra we make use of the Mubarakzyanov Classification Scheme [8-11].

4. Self-Similar Solution for Unsteady Navier-Stokes Equation

To determine a self-similar solution we make use of the obvious self-similar symmetry, Γ_6 which has the invariants

$$\{r_1 = \frac{x}{\sqrt{t}},\ r_2 = \frac{y}{\sqrt{t}},\ w_1 = u\sqrt{t},\ w_2 = v\sqrt{t},\ w_3 = pt\}.$$

Any invariant solution $w_1 = U(r_1, r_2)$, $w_2 = V(r_1, r_2)$ and $w_3 = P(r_1, r_2)$ respectively. The equivalent to

$$u = \frac{U\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right)}{\sqrt{t}} \quad (11)$$

$$v = \frac{V\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right)}{\sqrt{t}} \tag{12}$$

$$p = \frac{P\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right)}{t}. \tag{13}$$

Substitution of (11, 12) and (13) into (4) leads to the system,

$$(r_2 - 2V)U_{r_2} - 2P_{r_1} + r_1 U_{r_1} + U(1 - 2U_{r_1}) + \frac{2}{\mu}(U_{r_1 r_1} + U_{r_2 r_2}) = 0$$

$$(r_1 - 2U)V_{r_1} - 2P_{r_2} + r_2 V_{r_2} + V(1 - 2V_{r_2}) + \frac{2}{\mu}(V_{r_1 r_1} + V_{r_2 r_2}) = 0$$

$$U_{r_1} + V_{r_2} = 0 \tag{14}$$

in which the number of the independent variables has been reduced to two. The Lie point symmetries of (14) are

$$\Gamma_1 = \partial_P \tag{15}$$

$$\Gamma_2 = 4\partial_{r_1} + r_1\partial_P + 2\partial_U \tag{16}$$

$$\Gamma_3 = 4\partial_{r_2} + r_1\partial_P + 2\partial_V \tag{17}$$

$$\Gamma_4 = r_2\partial_{r_1} - r_1\partial_{r_2} + V\partial_U - U\partial_V \tag{18}$$

with the algebra $3A_1 \oplus A_1$. We look for some combination of the symmetries to obtain a suitable candidate for reduction. The possibility we choose is the linear combination of Γ_2 and Γ_3 we write

$$\Gamma = \Gamma_2 + a\Gamma_3$$

and note that there is no need to put a second constant in front of Γ_2 because the symmetry is written up to a constant multiplier. It follows that

$$\Gamma = 4\partial_{r_1} + 4a\partial_{r_2} + (r_1 + ar_2)\partial_P + 2\partial_U + 2a\partial_V.$$

Γ has the invariants

$$\{r = r_2 - ar_1, W_1 = U - \frac{r_1}{2}, W_2 = V - \frac{ar_1}{2}, W_3 = \frac{1}{8}(r_1((a^2 - 1)r_1 - 2ar_2) + 8P)\}.$$

Any invariant solution $W_1 = U1(r), W_2 = V1(r)$ and $W_3 = P1(r)$ respectively. The equivalent to

$$U = U1(r_2 - ar_1) + \frac{r_1}{2} \tag{19}$$

$$V = V1(r_2 - ar_1) + \frac{ar_1}{2} \tag{20}$$

$$P = \frac{1}{8}(r_1(a^2(-r_1) + 2ar_2 + r_1) + 8P_1(r_2 - ar_1)). \tag{21}$$

Substitution of (19, 20) and (21) into (14) leads to the system of ordinary differential equation namely,

$$ar - 4aP_1' - 2(r + 2aU_1 - 2V_1)U_1' - \frac{4}{\mu}(1 + a^2)U_1'' = 0$$

$$-aU_1 + V_1 - 2P_1' + (r + 2aU_1 - 2V_1)V_1' - \frac{2}{\mu}(1 + a^2)V_1'' = 0$$

$$\frac{1}{2} + V_1' - aU_1' = 0. \tag{22}$$

The general solution of the third equation is

$$U_1 = \frac{r}{2a} + k + \frac{V_1}{a}, \tag{23}$$

Where k is a constant of integration.

Substitution of (23) into (22) gives

$$-a^2 r\mu + 4a^2\mu P_1' + 2(ak + r)\mu(1 + 2V_1') + 4(1 + a^2)V_1'' = 0$$

$$-ak - \frac{r}{2} - 2P_1' + 2(ak + r)V_1' + \frac{2}{\mu}(1 + a^2)V_1'' = 0. \tag{24}$$

This is a coupled system of second-order linear ordinary differential equations. The general solutions of (24) are

$$P_1 = \frac{(a^2 - 3)r^2 - 8akr}{8(a^2+1)} + c_1 \tag{25}$$

$$V_1 = \sqrt{\frac{(a^2+1)}{2}} c_2 e^{\frac{a^2k^2\mu}{2a^2+2}} erf\left(\frac{\sqrt{\mu}(ak+r)}{\sqrt{2a^2+2}}\right) + \frac{(a^2-1)(ak+r)}{2(a^2+1)} + c_3. \tag{26}$$

From (26) we can find

$$U_1 = \frac{3a^2k + 2ar + k}{2a^2+2} + \frac{c_3}{a} c_2 \sqrt{\frac{\pi(a^2+1)}{2a^2\mu}} e^{\frac{a^2k^2\mu}{2a^2+2}} erf\left(\frac{\sqrt{\mu}(ak+r)}{\sqrt{2a^2+2}}\right). \tag{27}$$

Reversing the above procedure we can find the general solution of (4) which is

$$u = \frac{\sqrt{\mu}(2(a^2+1)c_3\sqrt{t} + a((3a^2+1)k\sqrt{t} + a^2(-x) + 2ay + x))}{2a(1+a^2)t\sqrt{\mu}}$$

$$\frac{\sqrt{2\pi}(a^2+1)^{3/2} c_2 \sqrt{t} e^{\frac{a^2k^2\mu}{2a^2+2}} erf\left(\frac{\sqrt{\mu}(ak\sqrt{t} - ax + y)}{\sqrt{2}\sqrt{a^2+1}\sqrt{t}}\right)}{2a(1+a^2)t\sqrt{\mu}} \tag{28}$$

$$v = c_2 e^{\frac{a^2k^2\mu}{2a^2+2}} \sqrt{\frac{\pi(a^2+1)}{2\mu t}} erf\left(\frac{\sqrt{\mu}(ak\sqrt{t} - ax + y)}{\sqrt{2}\sqrt{a^2+1}\sqrt{t}}\right)$$

$$+ \frac{a((a^2-1)k\sqrt{t} + ay + 2x)}{2(a^2+1)t} + \frac{c_3}{\sqrt{t}} \tag{29}$$

$$p = \frac{a^2(8c_1t + 8k\sqrt{t}x - 3x^2 + y^2) + 8ay(x - k\sqrt{t}) + 8c_1t + x^2 - 3y^2}{8(a^2+1)t^2}. \tag{30}$$

5. Travelling-Wave Solution

We assume $g(t) = b$ and $h(t) = a$ and combine Γ_1, Γ_4 and Γ_5 in the form $\Gamma_1 + a\Gamma_5 + b\Gamma_4$ and to obtain a traveling-wave solution with the independent variable $r_1 = x - at$ and $r_2 = y - bt$. The reduced system is

$$-bU_s + VU_s + P_r - aU_r + UU_r - \nu(U_{ss} + U_{rr}) = 0 \tag{31}$$
$$P_s - bV_s + VV_s - aV_r + UV_r + UV_r - \nu(V_{ss} + V_{rr}) = 0$$
$$V_s + U_r = 0.$$

The Lie point symmetries of (31) are

$$\Gamma_1 = \partial_P$$
$$\Gamma_2 = \partial_r$$
$$\Gamma_3 = \partial_s$$
$$\Gamma_4 = -s\partial_r + r\partial_s + (b - V)\partial_U - (a - U)\partial_V$$
$$\Gamma_5 = 2P\partial_P - r\partial_r - s\partial_s - (a - U)\partial_u - (b - V)\partial_v.$$

The algebra is $2A_1 \oplus_s 3A_1$.

The combination $\Gamma_2 + \alpha\Gamma_3$ gives $z = s - \alpha r$ is a new independent variable. The reduced system is

$$(-b + a\alpha + \Phi - \alpha\Phi)\Phi' - \alpha\Psi' - \nu\Phi''(1 + \alpha^2) = 0 \tag{32}$$
$$(-b + a\alpha + \phi - \alpha\phi)\phi' + \Psi' - \nu\phi''(1 + \alpha^2) = 0 \tag{33}$$
$$\phi' - \alpha\Phi' = 0, \tag{34}$$

where

$$P = \Psi(z),\ U = \Phi(z) \text{ and } V = \phi(z).$$

The solution of system (32) is

$$\Psi(z) = k \tag{35}$$
$$\Phi(z) = me^{-\frac{z(-a\alpha+b-l)}{(\alpha^2+1)\nu}} + n \tag{36}$$
$$\phi(z) = \alpha me^{-\frac{z(-a\alpha+b-l)}{(a^2+1)\nu}} + l + \alpha n, \tag{37}$$

Where k, l, m and n are constants of integration.

Reversing above proceeders we obtain the traveling-wave solutions

$$u = m - \frac{(a^2+1)\nu n\left(\exp\left(\frac{(a\alpha - b + l)(a\alpha t - bt - \alpha x + y)}{(a^2+1)}\right) - 1\right)}{-a\alpha + b - l} \tag{38}$$

$$v = \alpha\left(m - \frac{(a^2+1)\nu n\left(\exp\left(\frac{(a\alpha - b + l)(a\alpha t - bt - \alpha x + y)}{(a^2+1)}\right) - 1\right)}{-a\alpha + b - l}\right) + l \tag{39}$$

$$S = k. \tag{40}$$

6. Conclusion

The unsteady Navier-Stokes equation admits six Lie point symmetries. One of these symmetries of the self-similarity, this led to the set of solutions (28). The combination of the autonomy symmetry and symmetries for the variables x and enabled us to obtain travelling-wave solution if arbitrary functions of time in the latter symmetries are set as constants. Most interesting to note that reduction using the autonomy symmetry, ∂_t, leads to a five-dimensional algebra of the reduced system which is isomorphic to that given by the symmetries, (32), for the travelling-wave solution, i.e. $2A_1 \oplus_s 3A_1$. The coincidence is not surprising as it corresponds to setting $a = 0$ and $b = 0$ in (32) and this is equivalent to a constant translation in Φ and ϕ. Indeed the solution of the system is identical in form to that given in (35).

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