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COVER EDGE PEBBLING NUMBER FOR JAHANGIR GRAPHS $J_{1,m}$, $J_{2,m}$, $J_{3,m}$, $J_{4,m}$ AND $J_{5,m}$

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Abstract

Let G be a connected graph. An edge pebbling move on G is the process of removing two pebbles from one edge and placing one pebble on the adjacent edge. The cover edge pebbling number of G, denoted by $CP_E(G)$ is the minimum number of pebbles required to place a pebble on all the edges of G, however might be the initial configuration is. In this paper, we determine the cover edge pebbling number for Jahangir graphs $J_{1,m}$, $J_{2,m}$, $J_{3,m}$, $J_{4,m}$ and $J_{5,m}$.

1. Introduction

Lagarias and Saks first suggested the game of pebbling. Later by Chung [1], it was introduced into the literature. Removal of two pebbles from one vertex and placement of one pebble on the adjacent vertex is called pebbling move. Given a connected graph G. The pebbling number $\pi(G)$ of G is the

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least number of pebbles needed in a graph G so that we can move a pebble to any arbitrary target vertex by a sequence of pebbling move whatever might be the initial configuration is.

The concept of cover pebbling was first introduced by Crull [2]. The cover pebbling number $CP(G)$ is the least number of pebbles needed in a graph G so that we can move one pebble to all the vertices of the graph G.

In [6] a new concept namely edge pebbling number and cover edge pebbling number has been introduced and cover edge pebbling number for certain standard graphs namely path, complete graph, friendship graph and star graph have been determined.

In edge pebbling, pebbles will be distributed on the edges of the graph instead of the vertices. An edge pebbling move is the process of removing two pebbles from one edge and placing one pebble on the adjacent edge. Edge pebbling number $P_E(G)$ is the minimum number of pebbles needed in a graph G to reach any arbitrary target edge by a sequence of edge pebbling move regardless of initial configuration of pebbles. The cover edge pebbling number $CP_E(G)$ is the least number of pebbles needed in a graph G so that we can move one pebble to all the edges of the graph G. In this paper we establish the cover edge pebbling number for certain classes of Jahangir graph.

2. Cover Edge Pebbling Number

Definition 2.1[6]**.** A cover edge pebbling number $CP_E(G)$ of a graph G is defined as, however the pebbles are initially placed in the edges, the minimum number of pebbles required to place a pebble in all the edges.

Definition 2.2[6]**.** The distance between two edges x and y is defined as, $d(x, y) = d(v_i, v_j) - 2$ where $x = v_i v_{i+1}$, $y = v_{j-1} v_j$ and $d(v_i, v_j)$ is the length of the shortest path between v_i and v_j.

Definition 2.3[6]**.** The distance $d(x)$ of an edge x in a graph G is the sum of the distances from x to each other edge of $E(G)$, where $E(G)$ is the edge set of G.

(i.e.) $d(x) = \sum_{x \in E(G)} d(y, x) \ \forall \ y \in E(G), \ x \neq y.$

Definition 2.4[6]**.** Let $x \in E(G)$, then x is called a key edge if $d(x)$ is a maximum.

Result [6]**.** After finding the key edge of a graph find the minimum number of pebbles to be placed on that key edge such that a pebble is placed on each of the edges of the given graph. Then the minimum number of pebbles placed on the key edge is the cover edge pebbling number. Using this result the following theorems are proved.

Definition 2.5[5]**.** Jahangir graph $J_{n,m}$ for $m \geq 3$ is a graph on $nm+1$ vertices, that is, a graph consisting of a cycle C_{nm} with one additional vertex which is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm}.

Note. In this paper, one additional vertex which is at the center is mentioned as central vertex.

Theorem 2.6. *For* $J_{1,m}(m = 3, 4)$, $CP_E(J_{1,m}(m = 3, 4)) = 8m - 11$.

Proof. $J_{1,m}(m = 3, 4)$ has $m+1$ vertices and $2m$ edges. For $J_{1,3}$ all the edges are key edges. For $J_{1,4}$ the edges which lies on the cycle C_m are key edges. Choose any one of the key edges from $J_{1,m}(m = 3, 4)$. Let it be e_1. Each of the key edges are adjacent with 4 edges (i.e.) e_1 is adjacent with 4 edges. To place one pebble on each of these 4 edges, $4 * 2 = 8$ pebbles are needed in e_1 because of adjacency. There are remaining $2m - 5$ edges. On finding the shortest path these $2m - 5$ edges can be reached by crossing exactly one edge. Therefore $2^2(2m-5)$ pebbles are needed in e_1. All the edges except the chosen key edge e_1 have been dealt with. After covering all the edges with one pebble, one more pebble has to be placed on e_1.

Hence the cover edge pebbling number of $J_{1,m}(m = 3, 4)$ is

$$= 8 + 2^2(2m-5) + 1$$

$$= 8m - 11.$$

Theorem 2.7. *For* $J_{1,m}(m \geq 5)$, $CP_E(J_{1,m}(m \geq 5)) = 12m - 31$.

Proof. $J_{1,m}(m \geq 5)$ has $m+1$ vertices and $2m$ edges. For $J_{1,m}(m \geq 5)$ the edges which lies on the cycle C_m are key edges. Choose any one of the key edges. Let it be e_1. e_1 is adjacent with 4 edges. ((i.e.) 2 edges from the cycle C_m and 2 edges which are incident with the central vertex). To place one pebble on each of these 4 edges, $4 * 2 = 8$ pebbles are needed in e_1 because of adjacency. There are remaining $m-3$ edges from C_m and $m-2$ edges which are incident with the central vertex. Among these $m-3$ edges from C_m, 2 edges can be reached by crossing exactly one edge on finding the shortest path. Therefore $2(2^2)$ pebbles are needed in e_1. The remaining $m-5$ edges can be reached by crossing exactly 2 edges on finding the shortest path. Therefore $2^3(m-5)$ pebbles are needed in e_1. To reach the $m-2$ edges which are incident with the central vertex one edge has to be crossed. Here in this case $2^3(m-2)$ pebbles are needed in e_1. Altogether, $8 + 2(2^2) + 2^3(m-5) + 2^2(m-2)$ pebbles are needed in e_1. All the edges except the chosen key edge e_1 have been dealt with. After covering all the edges with one pebble, one more pebble has to be placed on e_1.

Hence the cover edge pebbling number of $J_{1,m}(m \geq 5)$

$$= 8 + 2(2^2) + 2^3(m-5) + 2^2(m-2) + 1$$

$$= 12m - 31.$$

Theorem 2.8. *For* $J_{2,m}(m \geq 3)$, $CP_E(J_{2,m}) = 20m - 29$.

Proof. $J_{2,m}(m \geq 3)$ has $2m+1$ vertices and $3m$ edges. For $J_{2,m}(m \geq 3)$ the edges which lies on the cycle C_{2m} are key edges. Choose any one of the key edges. Let it be e_1. Now, e_1 is exactly adjacent with 3 edges. (i.e.) 2 edges from the cycle C_{2m} and 1 edge from the edges which are incident with the central vertex. To place one pebble on each of these 3 edges, $3 * 2 = 6$ pebbles are needed in e_1. There are remaining $2m-3$ edges from the cycle C_{2m} and

$m-1$ edges which are incident with the central vertex. Among these $2m-3$ edges from the cycle C_{2m}, 2 edges can be reached by crossing exactly one edge on finding the shortest path. Therefore $2(2^2)$ pebbles are needed in e_1. Remaining $2m-5$ edges from the cycle C_{2m} can be reached by crossing exactly 2 edges on finding the shortest path. Therefore $2^3(2m-5)$ pebbles are needed in e_1. To reach the $m-1$ edges which are incident with the central vertex exactly one edge has to be crossed. Here in this case $2^2(m-1)$ pebbles are needed in e_1. Altogether, for all the cases considered above $6+2(2^2)+2^3(2m-5)+2^2(m-1)$ pebbles are needed in e_1. All the edges except the chosen key edge e_1 have been dealt with. After covering all the edges with one pebble, one more pebble has to be placed on e_1.

Hence the cover edge pebbling number of $J_{2,m}(m \geq 3)$

$$= 6+2(2^2)+2^3(2m-5)+2^2(m-1)+1$$

$$= 20m-29.$$

Theorem 2.9. *For* $J_{3,m}(m \geq 3)$, $CP_E(J_{3,m}) = 72m-139$.

Proof. $J_{3,m}(m \geq 3)$ has $3m+1$ vertices and $4m$ edges. For $J_{3,m}(m \geq 3)$ the edges of the cycle C_{3m} for which both the endpoints are not adjacent with the central vertex are the key edges. Choose any one of the key edges. Let it be e_1. Now e_1 is exactly adjacent with 2 edges which are from the cycle C_{3m}. To place one pebble on each of these 2 edges, $2*2=4$ pebbles are needed in e_1 because of adjacency. Another two edges from the cycle C_{3m} can be reached by crossing exactly one edge and again another two edges can be reached by crossing exactly two edges on finding the shortest path. As a whole to reach these four edges $2(2^2)+2(2^3)$ pebbles are needed in e_1.

We know that m edges are incident with the central vertex. Out of these m edges, 2 edges can be reached by crossing exactly one edge and the remaining $m-2$ edges can be reached by crossing exactly two edges. Also, with each of these $m-2$ edges, a pair of edges from the cycle C_{3m} are adjacent. In order to reach these $2(m-2)$ edges from cycle C_{3m}, three edges have to be crossed on finding the shortest path. Therefore $2(2^2)+2^3(m-1)$ $+2(m-2)(2^4)$ pebbles are needed in e_1 to reach the edges considered above. All the edges which are incident with the central vertex are dealt with. Only few more edges from the cycle C_{3m} are left. In C_{3m}, totally there are $3m$ edges. Among these $3m$ edges, $7+2(m-2)$ edges are already dealt with. Now, $[3m-7-2(m-2)]$ edges are there to deal with. On finding the shortest path four edges has to be crossed to reach these edges. Hence $[3m-7-2(m-2)]2^5$ pebbles are needed in e_1. All the edges except the chosen key edge e_1 have been dealt with. After covering all the edges with one pebble, one more pebble has to be placed on e_1.

Hence the cover edge pebbling number of $J_{3,m}(m \geq 3)$ is

$$= 4+2(2^2)+2(2^3)+2(2^2)+2^3(m-2)+2(m-2)(2^4)$$

$$+[3m-7-2(m-2)]2^5+1$$

$$= 72m-139.$$

Theorem 2.10. *For* $J_{4,m}(m \geq 3)$, $CP_E(J_{4,m}) = 104m-167$.

Proof. $J_{4,m}(m \geq 3)$ has $4m+1$ vertices and $5m$ edges. For $J_{4,m}(m \geq 3)$ the edges of the cycle C_{4m} for which both the endpoints are not adjacent with the central vertex are the key edges. Choose any one of the key edges. Let it be e_1. Among the m edges which are incident with the central vertex, one edge can be reached by crossing exactly one edge and the remaining $m-1$ edges can be reached by crossing exactly two edges on finding the shortest path. Therefore, the minimum number of pebbles needed to reach the edges which are incident with the central vertex is $2^2+(m-1)2^3$. Now let us discuss the edges on the cycle C_{4m}. Two edges from the cycle are adjacent with the key edge. Another two edges can be reached by crossing exactly one edge. One pair of edge can be reached by crossing 2 edges and another one pair can be reached by crossing 3 edges. We know that m edges are incident with the central vertex. With each of these m edges two edges from the cycle C_{4m} are adjacent. Among these $2m$ edges from the cycle C_{4m} four edges are already dealt with. The remaining $2m - 4$ edges can be reached by crossing 3 edges. Now $9 + 2m - 4$ edges from the cycle C_{4m} are dealt with. Now, remaining $4m - (9 + 2m - 4)$ can be reached by crossing exactly 4 edges on finding the shortest path. Therefore, minimum number of pebbles needed in e_1 to reach the edges of the cycle is $(2 * 2) + (2^2 * 2) + (2^3 * 2) + (2^4 * 2)$ $+ (2m - 4)2^4 + (4m - (9 + 2(m - 2)))2^5$. After covering all the edges one more pebble has to be placed on e_1. Hence the cover edge pebbling number of $J_{4,m} (m \geq 3)$ is

$$2^2 + (m-1)2^3 + (2 * 2) + (2^2 * 2) + (2^3 * 2) + (2^4 * 2) + (2m - 4)2^4$$
$$+ (4m - (9 + 2(m - 2)))2^5 + 1$$
$$= 104m - 167.$$

Theorem 2.11. *For* $J_{5,m} (m \geq 3)$, $CP_E(J_{5,m}) = 336m - 659$.

Proof. $J_{5,m} (m \geq 3)$ has $5m + 1$ vertices and $6m$ edges. Let us label the vertices of the cycle as $v_1, v_2, v_3, \ldots, v_{5m}$ in a clockwise manner such that $\deg(v_1) = 3$. Now $v_{3+5i}v_{4+5i} \in E(C_{5m})$, the edge set of C_{5m} where $i = 0, 1, 2, \ldots, m - 1$ are the key edges. Totally there are m key edges. Without loss of generality, let us choose v_3v_4 to be the key edge. Two edges from the cycle C_{5m} are adjacent with v_3v_4 to reach those adjacent edges we need $2 * 2$ pebbles on v_3v_4. There are five pairs of edges on C_{5m} from which one pair can be reached by crossing one edge, another pair by crossing two edges, another one by three, next pair by four and another one pair by crossing five edges on finding the shortest path. Therefore, to reach these five pairs of edges we need $(2^2 * 2) + (2^3 * 2) + (2^4 * 2) + (2^5 * 2) + (2^6 * 2)$ pebbles on v_3v_4. Among m edges which are incident with the central vertex, two edges can be reached by crossing two edges. Therefore, we need $2(2^3)$ pebbles on v_3v_4. And the remaining $m - 2$ edges can be reached by crossing three edges and for that $(m-2)(2^4)$ pebbles are needed. Also, with each of these $m-2$ edges which are incident with the central vertex two edges from the cycle C_{5m} are adjacent. To reach these $2(m-2)$ edges of the cycle C_{5m} four edges have to be crossed on finding the shortest path. Therefore, we need $2(m-2)(2^5)$ pebbles on v_3v_4. The edges in the set $\{v_{3+5i}v_{4+5i} / 2 \le i \le m-2\}$ can be reached by crossing six edges from v_3v_4. Hence $2^7(m-3)$ are needed. Exactly two edges are adjacent to each of the edges of the set $\{v_{3+5i}v_{4+5i} / 2 \le i \le m-2\}$. To reach these adjacent edges we need $(2(m-3))2^6$ pebbles. All the edges except the chosen key edge v_3v_4 have been dealt with. After covering all the edges with one pebble, one more pebble has to be placed on v_3v_4. Hence the cover edge pebbling number for $J_{5,m}(m \ge 3)$ is $(2*2)+(2^2*2)+(2^3*2)+(2^4*2)+(2^5*2)+(2^6*2)+2(2^3)$ $+(m-2)(2^4)+2(m-2)(2^5)+2^7(m-3)+(2(m-3))2^6+1 = 336m-659.$

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