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(k, μ)-CONTACT METRIC MANIFOLD WITH GENERALIZED SEMI-SYMMETRIC CONNECTION

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Abstract

In the present paper, we have studied (k, μ)-contact metric manifold with generalized semi-symmetric connection. The semi-symmetric linear connection on a differentiable manifold introduced by Friedmann and Schouten [4]. Hayden [5] introduced the idea of a metric connection on a Riemannian manifold. Yano investigated and studied the semi-symmetric metric connection on a Riemannian manifold. De and Sen-gupta [2], Verma and Prasad [6] defined and studied new type of semi-symmetric non-metric connections on a Riemannian manifold. Prasad, Verma and De [7] defined the most general form of semi-symmetric connection called generalized semi-symmetric connection. Generalized semi-symmetric connection studied by J. Upreti and S. K. Chanyal [8]. In this paper we have studied torsion tensor, Rimannian curvature and some of its properties on (k, μ)-contact metric manifold with generalized semi-symmetric connection and establised some results.

1. Introduction

Let M and $\overline{M}$ be two Riemannian manifolds with the metric tensors g and g respectively. If g and g are related by the equation

$$\bar{g}(X, Y) = e^{2\sigma} g(X, Y) \tag{1}$$

where σ is real function, the manifolds M and $\overline{M}$ are known as conformal transformation. A harmonic function is defined as a function whose Laplacian vanishes. A differentiable manifold M^{2n+1} is said to be contact manifold if it

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admits a global 1-form η, a unique vector field ξ, called the characteristic vector field such that

$$\eta(\xi) = 1$$

and

$$d\eta(\xi, X) = 0 \tag{2}$$

A Riemannian metric g on M^{2n+1} is said to be an associated metric if there exist a (1, 1) tensor field ϕ such that

$$d\eta(X, Y) = g(X, \phi Y)$$

and

$$\eta(X) = g(X, \xi) \tag{3}$$

$$\phi^2 = -I + \eta \otimes \xi. \tag{4}$$

From these equations, we have

$$\phi\xi = 0$$

$$\eta \circ \phi = 0$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{5}$$

A differentiable manifold M^{2n+1} equipped with the structure (ϕ, ξ, η, g) satisfying (5) is said to be a contact metric manifold and is denoted by $M = (M^{2n+1}, \phi, \xi, \eta, g)$. A contact metric manifold is called an η-Einstien manifold if the Ricci tensor S is of the form

$$S = ag + b\eta \otimes \eta,$$

where a, b are smooth functions on M^{2n+1} and if $b = 0$ then the manifold is called an Einstein manifold. In a contact metric manifold M, we define a (1,1) tensor field

$$h = L_\xi \phi$$

where L denotes the Lie differentiation. Then h is symmetric and

$$h\xi = 0, \ \eta \circ h = 0$$

and

$$h\phi = -\phi h$$

If ∇ denotes the Riemannian connection of g then the following relation holds

$$\nabla_X \xi = -\phi X - \phi h X \tag{6}$$

$$(\nabla_X \eta) Y = g(X, \phi Y) - g(\phi h X, Y). \tag{7}$$

An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi) Y = g(X, Y)\xi - \eta(Y) X, \tag{8}$$

for all vectors fields X, Y and r is the Levi-Civita connection of Riemannian metric g. A contact metric manifold for which ξ is killing vector field is said to be a k-contact manifold. It is well known that the tangent bundle of a at Riemannian manifold admits a contact metric structure satisfying

$$R(X, Y, \xi) = 0.$$

As a generalization of $R(X, Y, \xi) = 0$ and the Sasakian case, Blair, [3] considered the (k, μ)-nullity condition on a contact metric manifold. The (k, μ)-nullity distribution $N(k, \mu)$ Blair, [1] and Jun [3] of a contact metric manifold is defiend by

$$N(k, \mu) : p \to N_p(k, \mu)$$

$$= \{W \in T_p\} M / R(X, Y, W) = (kI + \mu h)[g(Y, W)X - g(X, W)Y]$$

for all $X, Y \in TM$, where $(k, \mu) \in R^2$. A contact metric manifold with $\xi \in N(k, \mu)$ is called a (k, μ)-contact metric manifold. For a (k, μ)-contact metric manifold, we have

$$R(X, Y, \xi) = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(XhY)). \tag{9}$$

On (k, μ)-contact metric manifold $k \leq 1$, for $k = 1$, the structure becomes Sasakian $(h = 0)$ and if $k < 1$, the (k, μ)-nullity condition completely deter-mines the manifold. The condition of being Sasakian manifold, a k-contact manifold, $k = 1$ and $h = 0$ are all equivalent in a (k, μ)-contact metric manifold.

2. Generalized Semi-Symmetric Connection

Let $(M; g)$ be a Riemannian manifold of dimension n. The generalized semi-symmetric connection is defined as

$$\overline{\nabla}_X Y = \nabla_X Y + u(Y)X + u(Y)X + a(X)Y + \eta(Y) - g(X, Y)\eta(\xi) \tag{10}$$

where X, Y are vectors fields, ∇ is Levi-Civita connection, u, a and η are 1-forms associated with the vectors fields U, A and ξ respectively and satisfied the following relations

$$g(X, U) = u(X) \tag{11}$$

$$g(X, A) = a(X) \tag{12}$$

$$g(X, \xi) = \eta(X). \tag{13}$$

The torsion tensor of $\overline{\nabla}$ is given by

$$\overline{T}(X, Y) = u(Y)X - u(X)Y + a(X)Y - a(Y)X + \eta(Y)X - \eta(X)Y. \tag{14}$$

The generalized semi-symmetric connection satisfies the following property

$$(\overline{\nabla}_{Xg})(Y, Z) = -u(Y)g(X, Z) - u(Z)g(X, Y) - 2a(X)g(Y, Z) - 2a(X)(Y, Z), \tag{15}$$

this shows that the generalized semi-symmetric connection is non-metric. The curvature tensor $\overline{R}(X, Y, Z)$ of ∇ is given by

$$\overline{R}(X, Y, Z) = R(X, Y, Z) - \alpha(Y, Z)X - \alpha(X, Z)Y - g(Y, Z)LX$$

$$+ g(Y, Z)LY - \beta(Y, Z)X + \beta(X, Z)Y + \gamma(Y, Z)X - \gamma(X, Z)Y - da(X, Y)Z, \tag{16}$$

where $R(X, Y, Z)$ is curvature tensor with respect to Levi-Civita connection ∇ and

$$\alpha(Y, Z) = g(KY, Z) = (\nabla_Y u)Z - u(Y)u(Z) + \frac{1}{2}u(\xi)g(Y, Z) \tag{17}$$

$$\beta(Y, Z) = g(LY, Z) = (\nabla_Y \eta)Z - \eta(Y)\eta(Z) + \frac{1}{2}u(\xi)g(Y, Z) \tag{18}$$

$$LX = \nabla_X \xi - \eta(X)\xi + \frac{1}{2}\eta(\xi)X \tag{19}$$

and

$$\gamma(Y, Z) = g(MY, Z) = \eta(Z)u(Y) - \eta(Y)u(Z) - \frac{1}{2}u(\xi)g(Y, Z) \tag{20}$$

Also we have,

$$\alpha(Y, Z) - \alpha(Z, Y) = du(Y, Z) \tag{21}$$

$$\beta(Y, Z) - \beta(Z, Y) = d\eta(Y, Z) \tag{22}$$

and

$$\gamma(Y, Z) - \alpha(Z, Y) = 0 \tag{23}$$

where

$$du(Y, Z) = (\nabla_Y u)Y[9]$$

3. Generalized Semi-Symmetric Connection on (k, μ)-Contact Metric Manifold

Let $M^{2n+1} = M$ is a (k, μ)-contact manifold. Let us define a generalized semi-symmetric connection $\overline{\nabla}$ on (k, μ)-contact metric manifold M. Then we have

The following relations on (k, μ)-contact metric manifold

$$\overline{\nabla}_X \xi = u(\xi)X + a(X)I + X + \xi \tag{24}$$

and generalized semi-symmetric connection $\overline{\nabla}$ is given by the equation

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)X + 2u(Y)X + a(X)\eta(X)\xi + u(Y)\xi$$

$$a(X)\eta(Y)\xi + \eta(X)\eta(Y)\xi + \eta(\nabla_X Y)\xi - g(X, Y)\xi, \tag{25}$$

with the help of equations (5) and (10), we have the equations (24) and (25).

Theorem 3.1. *In a* (k, μ)*-contact manifold with structure* $(M, g, \xi, A, U, \eta, a, u)$.

We have

$$(\overline{\nabla}_X \phi)Y = (\nabla_X \phi)Y + a(X)(\eta(X)\xi - \phi Y)$$

$$\eta(\nabla_X \phi)Y\xi - (u(Y) + \eta(Y))\phi X + \eta(\phi \nabla_X Y))\xi - g(X, \phi Y)\xi \tag{26}$$

and

$$(\overline{\nabla}_X \phi)Y = g(X, Y) - u(Y)\eta(X) - 2a(X)\eta(Y) + \eta(Y). \tag{27}$$

Proof. Using equations (4), (5), (6), (7) and (8), we have the equations (26) and (27).

Torsion tensor with respect to Riemannian connection on a Riemannian manifold is given by

$$S(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]. \tag{28}$$

If $\overline{S}(X, Y)$ is torsion tensor with respect to generalized semi-symmetric connection ∇, then

$$\overline{S}(X, Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y Y - [X, Y].$$

Also, we have

$$\overline{S}(X, Y) = S_1(X, Y) - 2S_2(X, Y) + a(X)\eta(X)\xi$$

$$-\alpha(Y)\eta(Y)\xi + [X, Y] + \eta([X, Y])\xi \tag{29}$$

If

$$S_1(X, Y) = \eta(Y)X - \eta(X)Y \tag{30}$$

$$S_2(X, Y) = u(Y)X - u(X)Y \tag{31}$$

$$\overline{S}(X, Y) = -\overline{S}(X, Y). \tag{32}$$

Theorem 3.2. *If M is a* (k, μ)*-contact metric manifold with generalized* *semi-symmetric connection* $\overline{\nabla}$, *then M is torsion free if*

$$\eta([X, Y])\xi + u(Y)\xi - u(X)\xi + \eta(Y)\xi)(aX - a(Y)) = S_1(X, Y) + 2S_2(X, Y) \quad (33)$$

and

$$\overline{\nabla}_X Y - \overline{\nabla}_Y X = [X, Y].$$

Proof. By equations (10), (28) and (29), we have the results.

4. Riemannian Curvature tensor *R* of (k, μ)-contact metric manifold with respect to generalized semi-symmetric connection $\overline{\nabla}$

If M is a (k, μ)-contact metric manifold and $\overline{\nabla}$ is a generalized semi-symmetric connection. $\overline{\nabla}$ be the Riemannian curvature tensor and $\overline{\nabla}$ be the Riemannian curvature tensor on M with respect to $\overline{\nabla}$, then

$$R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

Then using equations (7), (8), (10), (12), (20), (30) and (31), we have

$$\begin{aligned}\overline{R}(X, Y, Z) &= R(X, Y, Z) + u(\overline{\nabla}_Y Z)X - u(\overline{\nabla}_Y Z)Y \\ &+ (1 - \xi)[g(X, \overline{\nabla}_Y Z) - g(Y, \overline{\nabla}_X Z) - g([X, Y], Z)] \\ &+ g(X, \overline{\nabla}_Y U)Y - g(Y, \overline{\nabla}_Y U)X + g(X, Y)X - g(X, Y)Y \\ &+ [\tfrac{1}{2}u(\xi) + a(Y)\xi + u(Y) - 2a(Y) - 2a(Y) + \nabla_Y \xi + Y]g(X, Z) \\ &+ [u(\xi) - a(X)\xi - u(X) + 2a(X) - \nabla_X \xi + X]g(X, Z) \\ &+ g(Y, \overline{\nabla}_X A)Z - g(X, \overline{\nabla}_Y A)Z) + (\nabla_X \eta)Z)Y - ((\nabla_Y \eta)Z)X \end{aligned} \quad (34)$$

if

$$\begin{aligned}&\eta(Z)[S_1(X, Y) - 2S_2(X, Y) - S_3(X, Y) - u(\overline{\nabla}_X Z) - u(\overline{\nabla}_Y Z) - a([X, Y])\\ &= a(X)u(Y)Y - a(Y)u(X)X - u(X)\eta(Y)Y + \eta(Y)\eta(X)X + a(X)u(Z) \\ &- a(Y)u(Z) + \gamma(X, Z) + a(X)u(Y) - a(Y)u(X) + \eta(Y)a(X) - \eta(X)a(Y) \end{aligned} \quad (35)$$

where

$$S_3(X, Y) = a(X)Y - a(X)Y - a(Y)X.$$

Theorem 4.1. *On a (k, μ)-contact manifold M with structure $(M, g, \eta, a, u, \xi, A, U)$, and generalized semi-symmetric connection $\overline{\nabla}$, we have the following identity*

$$\overline{R}(X, Y, Z) + \overline{R}(Y, Z, X) + \overline{R}(Z, X, Z) = 0 \tag{36}$$

if

$$\Sigma u([X, Y])Z + \Sigma[((Y, \overline{\nabla}_X U)Y - g(X, \overline{\nabla}_Y U)Z]$$
$$+ \Sigma[g(Y, \overline{\nabla}_X A)Z - g(X, \overline{\nabla}_Y A)Z] + \Sigma[((\nabla_X \eta)Z)Y - (\nabla_Y \eta)Z)X]$$
$$= u(\xi)g(X, Y) - \frac{1}{2}\Sigma u(X)g(X, Z) - \Sigma[a(Y)u(X)X - a(X)u(Y)Y]$$
$$-\Sigma[u(X)\eta(Y)Y - u(Y)\eta(X)X] + \Sigma\gamma(Y, X) - \Sigma[\eta(X)a(Y) - \eta(Y)a(X)]$$
$$-\Sigma[S_1(X, Y)\eta(Z) - \eta(Z)S_3(X, Y)] + 2\Sigma[\eta(X)S_2(X, Y)]$$
$$-\Sigma u([X, Z]) + \Sigma a([X, Y])Y. \tag{37}$$

Proof. Using equations (34) and (35), we get equations (36) and (37).

Again in a (k, μ)-contact metric manifold we have,

Now we have the following theorem:

Theorem 4.2. *On a (k, μ)-contact metric manifold, with structure $(M, g, \eta, a, u, \xi, A, U)$ and generalized semi-symmetric connection $\overline{\nabla}$ then*

$$\overline{R}(X, Y, \xi) = k[\eta(X)Y - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY] \tag{38}$$

if

$$\eta(X)\xi - \eta(Y)\xi + a(Y)\eta(X)\xi + u(Y)\eta(X)$$
$$-a(X)\eta(Y)\xi - u(X)\eta(Y) + a(X)\eta(Y)Y + a(Y)u(X)X$$
$$+ u(X)\eta(Y)Y - u(Y)\eta(X)X - u(X) - a(X)u(Y) + a(Y)u(X)$$
$$= -g(Y, \overline{\nabla}_X U)Y + g(Y, \overline{\nabla}_Y U)X - g(X, Y)X - g(X, Y)Y$$

$$- \nabla_Y \xi \eta(X) + \overline{\nabla}_Y \eta \xi \eta(Y) - g(Y, \overline{\nabla}_X A)\xi + g(X, \overline{\nabla}_Y A)\xi$$

$$-((\nabla_X \eta)\xi)Y + ((\nabla_Y \eta)\xi)X) - \eta([X, Y])\xi(1 + \xi)$$

$$+ g(Y, \nabla_X \xi) + g(X, \overline{\nabla}_X \xi) + S_3(X, Y) + 2S_2(X, Y)$$

$$+ u(\overline{\nabla}_X \xi) - u(\overline{\nabla}_Y \xi) - a([X, Y]\xi). \tag{39}$$

Now if

$$R(X, Y, A) = k[a(Y)X - a(X)Y] + \mu[a(Y)hX - a(X)hY]$$

$$R(X, Y, U) = k[u(Y)X - u(X)Y] + \mu[a(Y)hX - u(X)hY],$$

then we have the following relations

$$\overline{R}(X, Y, U) = k[a(Y)X - u(X)Y] + \mu[a(Y)hX - a(X)hY], \tag{40}$$

if

$$g(X, \overline{\nabla}_Y A)\xi - g(Y, \overline{\nabla}_X A)\xi - g(Y, \overline{\nabla}_X U)Y + g(X, \overline{\nabla}_X U)X$$

$$- g(X, Y)(X - Y) - g(Y, \overline{\nabla}_X A) + g(X, \overline{\nabla}_Y A)A$$

$$+ g(Y, \overline{\nabla}_X A) - g(\overline{\nabla}_X A, X)$$

$$= a(X)Y - a(Y)X + \nabla_Y \xi a(X) - \nabla_X \xi a(Y)$$

$$+((\nabla_X \eta)A)Y - ((\nabla_Y \eta)A)X + a([X, Y])(\xi - 1)$$

$$+ a(Y)u(X)X - a(X)u(Y)Y + u(X)\eta(Y)Y - u(Y)\eta(X)X$$

$$+ \eta(X)a(Y) - \eta(Y)a(X) - u(\overline{\nabla}_X A) - u(\overline{\nabla}_X A) - a([X, Y])A$$

and

$$\overline{R}(X, Y, U) = k(u(Y)X - u(XY)) + \mu(u(Y)hX - u(X)hY) \tag{41}$$

if

$$(1 - \xi)[g(X, \overline{\nabla}_X U) - g(Y, \overline{\nabla}_X U) - u(X)a(Y) + a(X)u(Y) - u([X, Y])]$$

$$= g(X, \overline{\nabla}_X U)X - g(Y, \overline{\nabla}_X U)Y - g(X, Y)(X, Y)$$

$$+ g(X, \overline{\nabla}_X U)X - g(y, \overline{\nabla}_X A)U + u(Y)X - u(X)Y$$

$$+\nabla_X \xi u(Y) - \nabla_Y \xi u(X) + ((\nabla_X \eta)U)X - ((\nabla_X \eta)U)Y$$

$$a(X)u(Y)Y - a(Y)u(X)X + u(Y)u(X)X + u(Y)\eta(X)X - u(X)\eta(Y)Y$$

$$+ a(X) - a(Y) + \eta(X)$$

and

$$u(\xi) = 0,\ \eta(U) = 0,\ u(U) = 1.$$

Theorem 4.3. *On a* (k, μ)*-contact metric manifold with structure* $(M, g, \eta, u, a, \eta, U, A)$,

we have

$$\overline{R}(\xi, U, A) = R(\xi, U, A) + u(\overline{\nabla}_U A)\xi - \eta(\overline{\nabla}_U A)\xi$$

$$+ u(\overline{\nabla}_\xi A)\xi + u(\overline{\nabla}_\xi U)U - u(\overline{\nabla}_\xi A)A - \eta(\overline{\nabla}_U A)A$$

$$((\overline{\nabla}_{\xi A})A)U - ((\nabla_U \eta)A)\xi - a([\xi, U]) + \eta(\overline{\nabla}_U A)$$

$$- \xi + u(\overline{\nabla}_\xi A) - u(\overline{\nabla}_\xi A - a([\xi, U])A. \tag{42}$$

Proof. Using equations (3), (11), (12), (13), (20), (30), (32), (34) and (35), we have the equation (42).

Theorem 4.4. *If* $\overline{\nabla}$ *is generalized semi-symmetric connection on* (k, μ)*-contact metric manifold M with structure* $(M, g, \eta, a, u, \xi, A, U)$, *then we have the following relation*

$$\overline{\nabla}(X, Y, U) = -\overline{\nabla}(Y, X, U), \tag{43}$$

if

$$a(X)u(Y) + a(Y)u(X) = a(Y)u(X)X + a(X)u(Y)Y. \tag{44}$$

Proof. Using equations (39) and (40), we have the equations (43) and (44).

Theorem 4.5. *On* (k, μ)*-contact metric manifold with generalized Rieman-nian connection* $\overline{\nabla}$, *then Riemannian curvature tensor* $\overline{\nabla}$ *is skew-*

symmetric in first two slots with the condition

$$u(\xi) = 0 \tag{45}$$

or

$$g(X, Z) = -g(Y, Z) \tag{46}$$

Proof. By equations (34) and (35), we have the theorem.

Theorem 4.6. *On a* (k, μ)*-contact metric manifold M, we have the following relation*

$$(\overline{\nabla}_X \phi)Y = (g(X, Y) + g(X, hY))(1 + \xi)$$

$$-\eta(Y)(\eta(X) + (1 - h)X) + a(X)(\eta(X)\xi) - \phi(Y)). \tag{47}$$

Proof. By [70] we have

$$(\nabla_{\phi})Y = (g(X, Y) + g(X, hY))\xi - \eta(Y)(X + hX). \tag{48}$$

using equation (26), we have the equation (47).

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