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ANALYSIS OF SCHWARZ ALGORITHMS FOR A SCALAR ELLIPTIC PROBLEM

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1. Abstract

In this paper, the main focus is on analysing the convergence properties of the Classical Schwarz and the Optimized Schwarz algorithms for the scalar elliptic equation $u_{xx}+u_{yy}+\eta u_x-\lambda u=f$,where $\lambda,\eta>0$. Schwarz algorithms have gained ground during the last decades aiming to solve partial differential equations efficiently by breaking a boundary value problem to a collection of smaller subproblems. Schwarz was a German analyst that pioneered these iterative methods that are known as Domain Decomposition methods. The analysis of these algorithms is carried out using Fourier analysis techniques at the continuous level in the case of two unbounded subdomains.

2. Introduction

Domain Decomposition methods have gained a great interest during the last decades by engineering and mathematical communities worldwide for solving partial differential equations numerically. The pioneer of these methods was Hermann Schwarz [7] who devised an iterative algorithm to solve the Laplace equation, where the domain is a union of a rectangle and a circle. This algorithm was used to close the gap that existed in Riemann's mapping theorem. The main spirit of this algorithm is to solve sequentially the problem on the rectangle and the disk and update the information at the interfaces [4], [5], [6]. This approach gives the opportunity to reduce the computational cost by breaking the global problem to a collection of smaller subproblems [8], [12], [13], [14]. Classical Schwarz algorithms require overlap

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to exist between the subdomains and they are the first overlapping domain decomposition methods. These methods are using Dirichlet transmission condition to pass the information from one subdomain to the other one. The main drawback is that Classical Schwarz algorithms are slow and stagnate without the existence of the overlap. Later on, Pierre Luis Lions [4] introduced the concept of modifying the interface conditions. Instead of using Dirichlet or Neumann conditions, the proposition was to impose Robin conditions. This means that the new interface conditions have a Neumann component plus a parameter times the Dirichlet component. By tuning this parameter, the convergence of the algorithm can be controlled appropriately. This modification of the transmission conditions led to the birth of Optimized Schwarz algorithms, which have better convergence properties than the Classical Schwarz methods and do not require an overlap to converge. Optimized Schwarz methods can work both in the case of overlapping and non overlapping domain decomposition, and usually are ideal candidates for heterogeneous or multi physics problems [1] where the nature of the problem changes from one subdomain to the other. A common approach to analyse the convergence of the previously mentioned algorithms is using Fourier analysis to obtain the appropriate reduction factors, and constitutes a strategy in the literature [2], [3]. In this paper, convergence analysis is carried out for Classical Schwarz and Optimized Schwarz algorithms using the Fourier transform, and reduction factors are obtained. This analysis occurs in the two subdomain case, where the two subdomains are unbounded. The equation of interest is $u_{xx}+u_{yy}+\eta u_x-\lambda u=f$ where $\lambda,\eta>0$.

3. Some Fourier Analysis

In this section, some basics will be provided with regards to the Fourier transform that will be used later on for the analysis of the Schwarz algorithms. The Fourier transform in the y direction, by definition, is given by $\hat{u}=\int_{-\infty}^{+\infty} u e^{-i\xi y}\, dy$.

Lemma 1. *The Fourier transformed Laplacian* Δu *is given by* $\widehat{\Delta u}=\frac{\partial^2 \hat{u}}{\partial x^2}-\xi^2 \hat{u}$ *, and the Fourier transformed* $\frac{\partial u}{\partial x}$ *by* $\frac{\partial \hat{u}}{\partial x}$.

Proof. By the definition of the Fourier transform it is obtained

$$\widehat{\Delta u}=\int_{-\infty}^{+\infty} \Delta u e^{-i\xi y} dy=\int_{-\infty}^{+\infty} \left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) e^{-i\xi y} dy$$

$$=\int_{-\infty}^{+\infty} \frac{\partial^2 u}{\partial x^2} e^{-i\xi y} dy+\int_{-\infty}^{+\infty} \frac{\partial^2 u}{\partial y^2} e^{-i\xi y} dy$$

$$=\frac{\partial^2}{\partial x^2}\int_{-\infty}^{+\infty} u e^{-i\xi y} dy+i\xi \int_{-\infty}^{+\infty} \frac{\partial u}{\partial x} e^{-i\xi y} dy$$

$$=\frac{\partial^2}{\partial x^2}\int_{-\infty}^{+\infty} u e^{-i\xi y} dy+(i\xi)^2\int_{-\infty}^{+\infty} u e^{-i\xi y} dy$$

$$=\frac{\partial^2 \hat{u}}{\partial x^2}-\xi^2 \hat{u},$$

$$\widehat{\frac{\partial u}{\partial x}}=\int_{-\infty}^{+\infty} \frac{\partial u}{\partial x} e^{-i\xi y} dy=\frac{\partial \hat{u}}{\partial x}.$$

□

4. Classical Schwarz Algorithm

In this section the Classical Schwarz algorithm will be analysed at the continuous level using Fourier analysis. The domain $\Omega=(-\infty,+\infty)\times(-\infty,+\infty)$ is decomposed into two subdomains $\Omega_1=(-\infty,\delta)\times(-\infty,+\infty)$ and $\Omega_2=(0,+\infty)\times(-\infty,+\infty)$ the left and right half plane, where $\delta>0$ is the size of the overlap between the two subdomains. The problem of interest is the scalar elliptic equation of the form $\Delta u+\eta\frac{\partial u}{\partial x}-\lambda u=f,\eta,\lambda>0$. The Classical Schwarz algorithm reads

$$\begin{array}{c} \Delta u_1^{(k)}+\eta\frac{\partial u_1^{(k)}}{\partial x}-\lambda u_1^{(k)}=f,(x,y)\in\Omega_1 \\ u_1^{(k)}=u_2^{(k-1)}, at\,\Gamma_1 \\ u_1^{(k)}: bounded\ at -\infty \end{array} \quad \text{and} \quad \begin{array}{c} \Delta u_2^{(k)}+\eta\frac{\partial u_2^{(k)}}{\partial x}-\lambda u_2^{(k)}=f,(x,y)\in\Omega_2 \\ u_2^{(k)}=u_1^{(k-1)}, at\,\Gamma_2 \\ u_2^{(k)}: bounded\ at +\infty \end{array} \tag{1}$$

where $\Gamma_1=\{\delta\}\times(-\infty,+\infty)$ and $\Gamma_2=\{0\}\times(-\infty,+\infty)$ are the two interfaces and (k) denotes the iterations. The initial guesses $u_1^{(0)}$ and $u_2^{(0)}$ are required to start the iterative process. The $\Delta(.)$ is the second order elliptic differential operator.

Theorem 1 (Convergence factor of the Classical Schwarz method)**.** *Given the initial guess* $u_1^{(0)}$ *and* $u_2^{(0)}$ *the reduction factor of the Classical Schwarz algorithm is given by the formula*

$$r_{csm}(\eta,\lambda,\delta,\xi)=\rho(\Psi_{csm})=max\{|v_1|,|v_2|\}=e^{\frac{-\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}\delta} \tag{2}$$

where $\Psi_{csm}=\begin{bmatrix} 0 & e^{-\sqrt{\eta^2+4(\lambda+\xi^2)}\delta} \\ 1 & 0 \end{bmatrix}$ *is the Schwarz iteration matrix,* $v_1=-e^{\frac{-\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}\delta}$, $v_2=e^{\frac{-\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}\delta}$ *are the eigenvalues of the iteration matrix,* $\rho(\Psi_{csm})$ *is the corresponding spectral radius, and* $\eta,\lambda>0$ *are the parameters that stem from the partial differential equation.*

Proof. The first step is to introduce the errors at each iteration (k) in each subdomain. The errors are defined as $e_1^{(k)}=u|_{\Omega_1}-u_1^{(k)}$ and $e_2^{(k)}=u|_{\Omega_2}-u_2^{(k)}$ where $u|_{\Omega_1}$ is the restriction of the solution to the subdomain Ω_1 and $u|_{\Omega_2}$ is the restriction of the solution to the subdomain Ω_2. Consequently, by linearity of the problems in the two subdomains in (1), the classical Schwarz algorithm becomes

$$\begin{array}{c} \Delta e_1^{(k)}+\eta\dfrac{\partial e_1^{(k)}}{\partial x}-\lambda e_1^{(k)}=0,(x,y)\in\Omega_1 \\ e_1^{(k)}=e_2^{(k-1)}, at\,\Gamma_1 \\ e_1^{(k)}: bounded\,at-\infty \end{array} \quad \text{and} \quad \begin{array}{c} \Delta e_2^{(k)}+\eta\dfrac{\partial e_2^{(k)}}{\partial x}-\lambda e_2^{(k)}=0,(x,y)\in\Omega_2 \\ e_2^{(k)}=e_1^{(k-1)}, at\,\Gamma_2 \\ e_2^{(k)}: bounded\,at+\infty \end{array}. \tag{3}$$

By taking the Fourier transform in the y direction for the two subproblems in (3), the algorithm becomes

$$\begin{array}{c} \dfrac{\partial^2\widehat{e_1^{(k)}}}{\partial x^2}+\eta\dfrac{\partial\widehat{e_1^{(k)}}}{\partial x}-(\lambda+\xi^2)\widehat{e_1^{(k)}}=0,\Omega_1 \\ \widehat{e_1^{(k)}}=\widehat{e_2^{(k-1)}}, at\,\Gamma_1 \\ \widehat{e_1^{(k)}}: bounded\,at-\infty \end{array}, \quad \begin{array}{c} \dfrac{\partial^2\widehat{e_2^{(k)}}}{\partial x^2}+\eta\dfrac{\partial\widehat{e_2^{(k)}}}{\partial x}-(\lambda+\xi^2)\widehat{e_2^{(k)}}=0,\Omega_2 \\ \widehat{e_2^{(k)}}=\widehat{e_1^{(k-1)}}, at\,\Gamma_2 \\ \widehat{e_2^{(k)}}: bounded\,at+\infty \end{array}. \tag{4}$$

It is observable that there are two ordinary differential equations which occur in (4), one in each subdomain. The solutions at the subdomains Ω_1 and Ω_2 can be given by explicit formulas. The characteristic polynomial is $P(\mu)=\mu^2+\eta\mu-(\lambda+\xi^2)$ and the roots of this polynomial are $\mu_1=\dfrac{-\eta+\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}$ and $\mu_2=\dfrac{-\eta-\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}$ .Consequently, the solutions in the subdomain Ω_1 and the subdomain Ω_2 are given by the explicit formulas $\widehat{e_1^{(k)}}=A_1^{(\kappa)}e^{\mu_1 x}+B_1^{(\kappa)}e^{\mu_2 x}$,

$\widehat{e_2^{(k)}}=A_2^{(\kappa)}e^{\mu_1 x}+B_2^{(\kappa)}e^{\mu_2 x}$ .The general solutions, by taking the boundedness assumptions for $\widehat{e_1^{(k)}}$ at $-\infty$ and for $\widehat{e_2^{(k)}}$ at $+\infty$, take the form $\widehat{e_1^{(k)}}=A_1^{(k)}e^{\mu_1 x}$, $\widehat{e_2^{(k)}}=B_2^{(k)}e^{\mu_2 x}$. By plugging in these solutions back to the interface conditions, it is obtained

$$A_1^{(k)}e^{\mu_1\delta}=B_2^{(k-1)}e^{\mu_2\delta},$$

$$B_2^{(k)}=A_1^{(k-1)}.$$

These relations can be written in matrix form

$$\begin{bmatrix} e^{\mu_1\delta} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A_1^{(k)} \\ B_2^{(k)} \end{bmatrix}=\begin{bmatrix} 0 & e^{\mu_2\delta} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} A_1^{(k-1)} \\ B_2^{(k-1)} \end{bmatrix}$$

and by doing some algebraic manipulations, it is obtained

$$\begin{bmatrix} A_1^{(k)} \\ B_2^{(k)} \end{bmatrix}=\begin{bmatrix} 0 & e^{(\mu_2-\mu_1)\delta} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} A_1^{(k-1)} \\ B_2^{(k-1)} \end{bmatrix}. \tag{5}$$

There is a stationary iteration arising (5) where $\Psi_{csm}=\begin{bmatrix} 0 & e^{-\sqrt{\eta^2+4(\lambda+\xi^2)}\delta} \\ 1 & 0 \end{bmatrix}$ is the Schwarz iteration matrix. The spectral properties of the Schwarz iteration matrix determine the convergence of the Classical Schwarz algorithm. The characteristic polynomial is given by the relation $C(v)=det(\Psi_{csm}-vI)=(v_1+\mathrm{e}^{\frac{-\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}\delta})(v_2-\mathrm{e}^{\frac{-\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}\delta})$ and the roots of the polynomial are $v_1=-\mathrm{e}^{\frac{-\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}\delta}$, $v_2=\mathrm{e}^{\frac{-\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}\delta}$ .The reduction factor of the Classical Schwarz algorithm is given by the formula $r_{csm}(\eta,\lambda,\delta,\xi)=\rho(\Psi_{csm})=max\{|v_1|,|v_2|\}=\mathrm{e}^{\frac{-\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}\delta}$, which depends on the parameters $\eta,\lambda>0$ that stem from the elliptic problem, the Fourier frequency ξ and the size of the overlap δ. This formula is identical to (2). □

Corollary 1. *The reduction factor of the Classical Schwarz algorithm satisfies the following properties*

$$r_{csm}(\eta,\lambda,\delta,\xi)=\begin{cases} 1, \delta=0 \\ 0, \delta\to+\infty \\ 0, |\xi|\to+\infty \\ <1, \xi\neq 0 \\ e^{\frac{-\sqrt{\eta^2+4\lambda}}{2}\delta}, \xi=0 \end{cases}. \tag{6}$$

Proof. With regards to the $(6)_1$ it is evident that choosing $\delta = 0$ the convergence factor becomes 1, which means that the method does stagnate. This is a common experience with the Classical Schwarz method where the overlap is required to be non zero for convergence. Moving to the $(6)_2$, taking a generous size of overlap means that the reduction factor becomes immediately zero. Therefore, the size of the overlap is of critical importance for the algorithm to converge. The $(6)_3$ means that the method behaves well for the high frequencies, and the convergence factor rapidly decays in the high frequency regime. With regards to the $(6)_4$ and $(6)_5$ it is evident that for $\xi \neq 0$ the reduction factor is less than 1 and for $\xi = 0$ an exponentially decaying quantity is obtained. In addition, the parameters η, λ that stem from the elliptic equation help with the convergence. □

5. Optimized Schwarz Method

In this section, convergence analysis of the Optimized Schwarz algorithm is carried out and the reduction factor is obtained. The configuration of the domain is identical to the one used in the analysis of Classical Schwarz algorithm. It is an unbounded domain decomposed into two half planes Ω_1 and Ω_2.The subdomain $\Omega_1 = (-\infty, \delta) \times (-\infty, +\infty)$ is the left half plane and $\Omega_2 = (0, +\infty) \times (-\infty, +\infty)$ the right half plane respectively. The Optimized Schwarz method uses Robin boundary conditions at the interfaces, therefore the algorithm reads

$$\begin{array}{cc} \Delta u_1^{(k)} + \eta \dfrac{\partial u_1^{(k)}}{\partial x} - \lambda u_1^{(k)} = f, (x, y) \in \Omega_1 & \Delta u_2^{(k)} + \eta \dfrac{\partial u_2^{(k)}}{\partial x} - \lambda u_2^{(k)} = f, (x, y) \in \Omega_2 \\ \nabla u_1^{(k)} \vec{n}_1 + \gamma u_1^{(k)} = \nabla u_2^{(k-1)} \vec{n}_1 + \gamma u_2^{(k-1)}, at\ \Gamma_1, & \nabla u_2^{(k)} \vec{n}_2 + \gamma u_2^{(k)} = \nabla u_1^{(k-1)} \vec{n}_2 + \gamma u_1^{(k-1)}, at\ \Gamma_2 \\ u_1^{(k)} : bounded\ at -\infty & u_2^{(k)} : bounded\ at + \infty \end{array}$$

(7)

where γ is the optimized parameter of the algorithm, $\vec{n}_1=(1,0)$ is the outward normal vector pointing outwards of the boundary of Ω_1 and $\vec{n}_2=(-1,0)$ the outward normal vector associated to the boundary of Ω_2. Consequently, the Optimized Schwarz method prescribed by (7) becomes

$$\begin{array}{ll} \Delta u_1^{(k)}+\eta\frac{\partial u_1^{(k)}}{\partial x}-\lambda u_1^{(k)}=f,(x,y)\in\Omega_1 & \Delta u_2^{(k)}+\eta\frac{\partial u_2^{(k)}}{\partial x}-\lambda u_2^{(k)}=f,(x,y)\in\Omega_2 \\ \frac{\partial u_1^{(k)}}{\partial x}+\gamma u_1^{(k)}=\frac{\partial u_2^{(k-1)}}{\partial x}+\gamma u_2^{(k-1)}, at\,\Gamma_1, & \frac{-\partial u_2^{(k)}}{\partial x}+\gamma u_2^{(k)}=\frac{-\partial u_1^{(k-1)}}{\partial x}+\gamma u_1^{(k-1)}, at\,\Gamma_2 \\ u_1^{(k)}: bounded\,at-\infty & u_2^{(k)}: bounded\,at+\infty \end{array} \quad (8)$$

Theorem 2 (Reduction factor of the Optimized Schwarz algorithm)**.** *Given the initial guesses* $u_1^{(0)}$, $u_2^{(0)}$ *the convergence factor of the Optimized Schwarz algorithm is given by the formula*

$$r_{osm}(\eta,\lambda,\delta,\xi,\gamma)=\sqrt{\frac{(\gamma-\frac{-\eta+\sqrt{\eta^2+4(\lambda+\xi^2)}}{2})}{(\gamma+\frac{-\eta+\sqrt{\eta^2+4(\lambda+\xi^2)}}{2})}\frac{(\gamma+\frac{-\eta-\sqrt{\eta^2+4(\lambda+\xi^2)}}{2})}{(\gamma-\frac{-\eta-\sqrt{\eta^2+4(\lambda+\xi^2)}}{2})}e^{\frac{-\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}\delta}}, \quad (9)$$

$$\gamma\in(-\infty,\frac{-\eta-\sqrt{\eta^2+4(\lambda+\xi^2)}}{2})\cup(-(\frac{-\eta+\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}),\frac{-\eta+\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}]\cup[-(\frac{-\eta-\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}),+\infty)$$

.

Proof. The errors $e_1^{(k)}=u|_{\Omega_1}-u_1^{(k)}$ and $e_2^{(k)}=u|_{\Omega_2}-u_2^{(k)}$ are introduced at each iteration (k) and by linearity of the two subproblems in (8), these errors satisfy the homogeneous counterparts and this gives

$$\begin{array}{ll} \Delta e_1^{(k)}+\eta\dfrac{\partial e_1^{(k)}}{\partial x}-\lambda e_1^{(k)}=0,(x,y)\in\Omega_1 & \Delta e_2^{(k)}+\eta\dfrac{\partial e_2^{(k)}}{\partial x}-\lambda e_2^{(k)}=0,(x,y)\in\Omega_2 \\ \dfrac{\partial e_1^{(k)}}{\partial x}+\gamma e_1^{(k)}=\dfrac{\partial e_2^{(k-1)}}{\partial x}+\gamma e_2^{(k-1)}, at\,\Gamma_1, & \dfrac{-\partial e_2^{(k)}}{\partial x}+\gamma e_2^{(k)}=\dfrac{-\partial e_1^{(k-1)}}{\partial x}+\gamma e_1^{(k-1)}, at\,\Gamma_2. \\ e_1^{(k)}: bounded\ at -\infty & e_2^{(k)}: bounded\ at +\infty \end{array} \tag{10}$$

Taking the Fourier transform towards the y direction in (10), the algorithm takes the form

$$\begin{array}{lcl} \dfrac{\partial^2 \widehat{e_1^{(k)}}}{\partial x^2}+\eta\dfrac{\partial \widehat{e_1^{(k)}}}{\partial x}-(\lambda+\xi^2)\widehat{e_1^{(k)}}=0,\Omega_1 & & \dfrac{\partial^2 \widehat{e_2^{(k)}}}{\partial x^2}+\eta\dfrac{\partial \widehat{e_2^{(k)}}}{\partial x}-(\lambda+\xi^2)\widehat{e_2^{(k)}}=0,\Omega_2 \\ \dfrac{\partial \widehat{e_1^{(k)}}}{\partial x}+\gamma\widehat{e_1^{(k)}}=\dfrac{\partial \widehat{e_2^{(k-1)}}}{\partial x}+\gamma\widehat{e_2^{(k-1)}}, at\,\Gamma_1 & \text{and} & \dfrac{-\partial \widehat{e_2^{(k)}}}{\partial x}+\gamma\widehat{e_2^{(k)}}=\dfrac{-\partial \widehat{e_1^{(k-1)}}}{\partial x}+\gamma\widehat{e_1^{(k-1)}}, at\,\Gamma_2. \\ \widehat{e_1^{(k)}}: bounded\ at -\infty & & \widehat{e_2^{(k)}}: bounded\ at +\infty \end{array}$$

(11)

The functions $\widehat{e_1^{(k)}}$ and $\widehat{e_2^{(k)}}$ are the solutions of two differential equations that arise in the two subdomains in (11). Taking the boundedness assumptions of the solutions in the two subdomains, the errors are given by the formulas $\widehat{e_1^{(k)}}=A_1^{(k)}e^{\mu_1 x}$, $\widehat{e_2^{(k)}}=B_2^{(k)}e^{\mu_2 x}$ where μ_1, μ_2 are the roots of the characteristic equation $P(\mu)=\mu^2+\eta\mu-(\lambda+\xi^2)$.

By substituting the two solutions back to the transmission conditions, it yields

$$A_1^{(k)}e^{\mu_1\delta}(\mu_1+\gamma)=B_2^{(k-1)}e^{\mu_2\delta}(\mu_2+\gamma),$$

$$B_2^{(k)}(\gamma-\mu_2)=A_1^{(k-1)}(\gamma-\mu_1).$$

The two equations are reformulated in matrix form

$$\begin{bmatrix} e^{\mu_1\delta}(\mu_1+\gamma) & 0 \\ 0 & (\gamma-\mu_2) \end{bmatrix}\begin{bmatrix} A_1^{(k)} \\ B_2^{(k)} \end{bmatrix}=\begin{bmatrix} 0 & e^{\mu_2\delta}(\mu_2+\gamma) \\ (\gamma-\mu_1) & 0 \end{bmatrix}\begin{bmatrix} A_1^{(k-1)} \\ B_2^{(k-1)} \end{bmatrix}.$$

The above relation takes the final form

$$\begin{bmatrix} A_1^{(k)} \\ B_2^{(k)} \end{bmatrix} = \begin{bmatrix} 0 & \frac{(\mu_2+\gamma)}{(\mu_1+\gamma)} e^{(\mu_2-\mu_1)\delta} \\ \frac{(\gamma-\mu_1)}{(\gamma-\mu_2)} & 0 \end{bmatrix} \begin{bmatrix} A_1^{(k-1)} \\ B_2^{(k-1)} \end{bmatrix}. \tag{12}$$

A stationary iteration naturally arises in (12), where $\Psi_{osm} = \begin{bmatrix} 0 & \frac{(\mu_2+\gamma)}{(\mu_1+\gamma)} e^{(\mu_2-\mu_1)\delta} \\ \frac{(\gamma-\mu_1)}{(\gamma-\mu_2)} & 0 \end{bmatrix}$ is the iteration matrix. The spectral properties of this matrix will characterize the convergence of the Optimized Schwarz algorithm. The characteristic polynomial associated with the iteration matrix is $C(v) = (v - \sqrt{\frac{(\gamma-\mu_1)(\gamma+\mu_2)}{(\gamma+\mu_1)(\gamma-\mu_2)}} e^{\frac{(\mu_2-\mu_1)\delta}{2}})(v + \sqrt{\frac{(\gamma-\mu_1)(\gamma+\mu_2)}{(\gamma+\mu_1)(\gamma-\mu_2)}} e^{\frac{(\mu_2-\mu_1)\delta}{2}})$ and the roots are

$$v_1 = \sqrt{\frac{(\gamma-\mu_1)(\gamma+\mu_2)}{(\gamma+\mu_1)(\gamma-\mu_2)}} e^{\frac{(\mu_2-\mu_1)\delta}{2}}, \; v_2 = -\sqrt{\frac{(\gamma-\mu_1)(\gamma+\mu_2)}{(\gamma+\mu_1)(\gamma-\mu_2)}} e^{\frac{(\mu_2-\mu_1)\delta}{2}}.$$

As a consequence the reduction factor of the Optimized Schwarz algorithm is given by the formula

$$r_{osm}(\eta,\lambda,\delta,\xi,\gamma) = \sqrt{\frac{(\gamma - \frac{-\eta+\sqrt{\eta^2+4(\lambda+\xi^2)}}{2})}{(\gamma + \frac{-\eta+\sqrt{\eta^2+4(\lambda+\xi^2)}}{2})} \frac{(\gamma + \frac{-\eta-\sqrt{\eta^2+4(\lambda+\xi^2)}}{2})}{(\gamma - \frac{-\eta-\sqrt{\eta^2+4(\lambda+\xi^2)}}{2})} e^{\frac{-\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}\delta}},$$

$$\gamma \in (-\infty, \frac{-\eta-\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}) \cup (-(\frac{-\eta+\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}), \frac{-\eta+\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}] \cup [-(\frac{-\eta-\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}), +\infty)$$

.The formula for the contraction factor is identical to (9). □

Corollary 2. *The convergence factor of the Optimized Schwarz algorithm satisfies the following properties*

$$r_{osm}(\eta,\lambda,\delta,\xi,\gamma)=\begin{cases} e^{\frac{-\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}\delta}, \gamma\to+\infty\,(Dirichlet\ IC) \\ e^{\frac{-\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}\delta}, \to 0\,(Neumann\ IC) \\ \sqrt{\dfrac{\left(\gamma-\frac{-\eta+\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}\right)}{\left(\gamma+\frac{-\eta+\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}\right)}\dfrac{\left(\gamma+\frac{-\eta-\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}\right)}{\left(\gamma-\frac{-\eta-\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}\right)}}, \delta=0 \\ 0, \gamma=\dfrac{-\eta+\sqrt{\eta^2+4(\lambda+\xi^2)}}{2} \\ 0, \gamma=\dfrac{\eta+\sqrt{\eta^2+4(\lambda+\xi^2)}}{2} \end{cases} \quad (13)$$

.

Proof. With regards to the $(13)_1$ this occurs by simply taking the limit of the reduction factor as $\gamma\to+\infty$ .This practically means that at the interfaces Γ_1 and Γ_2 Dirichlet boundary conditions are imposed. Moving to $(13)_2$, this result is obtained by taking $\gamma\to 0$ which means that the Dirichlet component vanishes ending up only with the Neumann conditions. In the third case $(13)_3$ for $\delta=0$, the non-overlapping Optimized Schwarz method occurs which is convergent. For the last two cases $((13)_4,(13)_5)$, by taking the Optimized parameter to be $\gamma=\frac{-\eta+\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}$ or $\gamma=\frac{\eta+\sqrt{\eta^2+4(\lambda+\xi^2)}}{2}$,the contraction factor becomes zero and very fast convergence is achieved (two iterations). □

6. Schwarz Algorithms-Experimental Behaviour of Contraction Factors

In this section there are some illustrations that exhibit the behaviour of the Classical Schwarz and Optimized Schwarz algorithm. It is evident that in figures 1, and 2 that the choice of the parameters that naturally arise from the elliptic problem, help to improve the convergence of the Classical Schwarz method. Moreover, choosing a larger overlap between the subdomains improves the convergence behaviour.

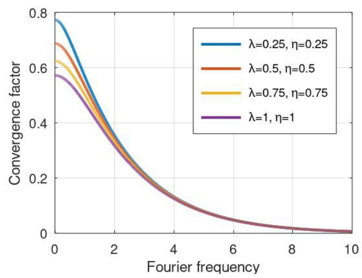

Figure 1. The reduction factor of the Classical Schwarz methods for δ=2h, h=0.25 (mesh size).

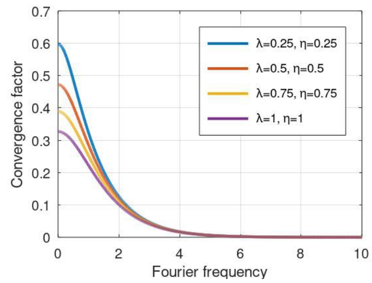

Figure 2. The reduction factor of the Classical Schwarz methods for δ=4h, h=0.25 (mesh size).

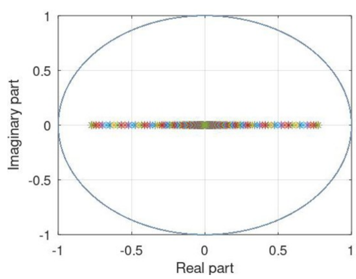

Figure 3. The spectrum of the Classical Schwarz iteration matrix for η=0.25, λ=0.25, δ=2h, h=0.25 (mesh size).

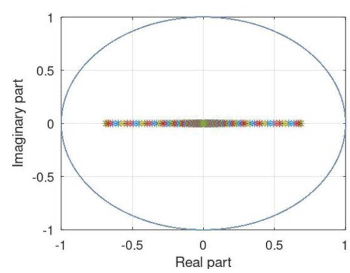

Figure 4. The spectrum of the Classical Schwarz iteration matrix for η=0.5 λ=0.5, δ=2h, h=0.25 (mesh size).

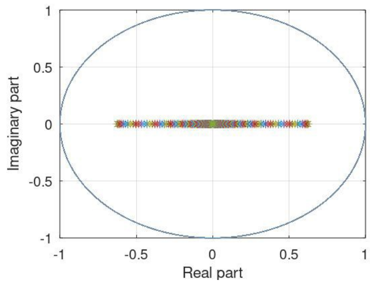

Figure 5. The spectrum of the Classical Schwarz iteration matrix for η=0.75, λ=0.75, δ=2h, h=0.25 (mesh size).

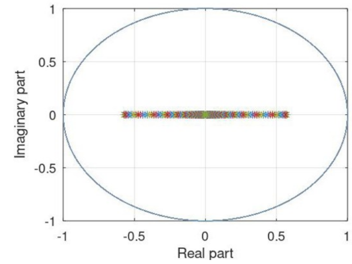

Figure 6. The spectrum of the Classical Schwarz iteration matrix for η=1, λ=1, δ=2h, h=0.25 (mesh size).

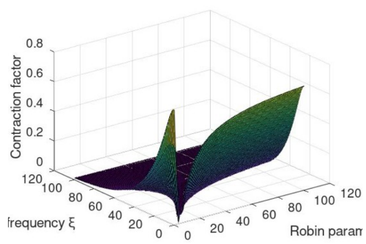

Figure 7. The contraction factor of the Optimized Schwarz algorithm in modulus for η=0.25, λ=0.25, δ=2h, h=0.25 (mesh size) as a function of the Fourier number ξ and the Robin parameter γ.

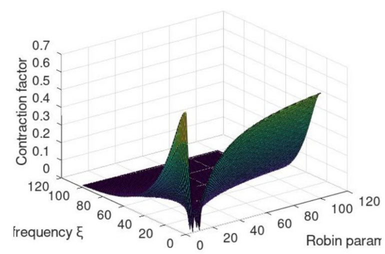

Fig. 8: The contraction factor of the Optimized Schwarz algorithm in modulus for η=0.5 λ=0.5, δ=2h, h=0.25 (mesh size) as a function of the Fourier number ξ and the Robin parameter γ.

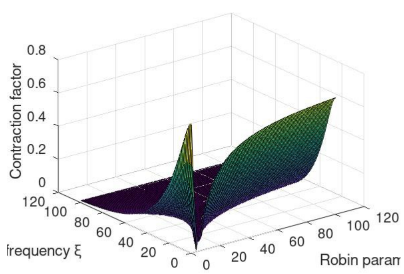

Figure 9. The contraction factor

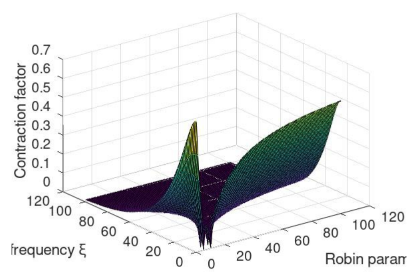

Figure 10. The contraction factor

of the Optimized Schwarz algorithm in modulus for η=0.75, λ=0.75, δ=2h, h=0.25 (mesh size) as a function of the Fourier number ξ and the Robin parameter γ.

of the Optimized Schwarz algorithm in modulus for η=1, λ=1, δ=2h, h=0.25 (mesh size) as a function of the Fourier number ξ and the Robin parameter γ.

Figures 3, 4, 5, 6 show the spectrum of the Classical Schwarz iteration matrix and it is noticeable that the choice of the physical parameters η, λ make the eigenvalues to contract even closer to the origin translating to a better convergence. Furthermore, all the eigenvalues are enclosed in the unit circle centered at the origin. With regards to the Optimized Schwarz algorithm, figures 7, 8, 9 and 10 display the contraction factor as a function of the Robin parameter and the Fourier number ξ for different physical parameters η, λ and fixed overlap.

7. Conclusions and Future Work

In this paper, convergence analysis was provided in the case of two unbounded subdomains for an elliptic problem using Fourier analysis techniques. More precisely, the Classical Schwarz method and the Optimized Schwarz algorithm have been analysed, obtaining explicit formulas for the reduction factors in both cases. The Optimized Schwarz method has better convergence properties than the Classical Schwarz method and converges even with the absence of overlap. The physical parameters that stem from the partial differential equation, are tuning the convergence factors, and affect the convergence behaviour of the methods. In addition, by choosing the appropriate Optimized parameter, it is obtained that the reduction factor becomes zero. Last but not least, it needs to be mentioned that there is space for more work in the future, answering what is the convergence behaviour in the case where there are more than two subdomains [9], [11]. Also further analysis can be carried out when the domains are bounded to observe how the Schwarz methods behave for this scalar elliptic problem [10]. Last but not least, it could be observed what happens when Ventcell interface conditions are used.

References

[1] M. J. Gander and T. Vanzan, Heterogeneous optimized Schwarz methods for coupling Helmholtz and Laplace equations, In Domain Decomposition Methods in Science and Engineering XXIV, LNCSE, Springer-Verlag, (2018), 311-320.

[2] M. J. Gander, Optimized Schwarz methods for Helmholtz problems, Proceedings of the 13th International Conference on Domain Decomposition, CIMNE, (2001), 245-252.

[3] M. J. Gander, L. Halpern and F. Nataf, Optimized Schwarz Methods, Proceedings of the 12th International Conference on Domain Decomposition, ddm.org, (2000), 15-27.

[4] P. L. Lions, On the Schwarz alternating method III: A variant for nonoverlapping subdomains, In: T. Chan, R. Glowinski, J. Periaux, O.B. Widlund (eds.) Third international symposium on domain decomposition methods for partial differential Equations, pp. 202-223. SIAM (1990).

[5] M. J. Gander and G. Wanner, The Origins of the Alternating Schwarz Method. Domain Decomposition Methods in Science and Engineering XXI, LNCSE, Springer-Verlag, pp. 487-496, 2014.

[6] Martin J. Gander, Schwarz methods over the course of time, ETNA, 31 (2008), 228-255.

[7] H. A. Schwarz, Uber Einen Grenzubergang durch alternierendes Verfahren, Vierteljahrsschrift der Naturforschenden Gesellschaft in Zurich, 15 272-286, Mai 1870.

[8] V. Dolean, P. Jolivet and F. Nataf, An Introduction to domain decomposition methods: Algorithms, Theory, and Parallel Implementation, SIAM 2016.

[9] V. Dolean, M. J. Gander and A. Kyriakis, Optimizing transmission conditions for multiple subdomains in the Magnetotelluric Approximation of Maxwell's equations, in Domain Decomposition Methods in Science and Engineering XXVI, LNCSE, Springer-Verlag, 2021.

[10] A. Kyriakis, Scalable Domain Decomposition Methods for time harmonic wave propagation problems, PhD thesis, University of Strathclyde, 2021.

[11] V. Dolean, M. J. Gander and A. Kyriakis, Closed Form optimized transmission conditions for complex diffusion with many subdomains, SIAM J. on Sci. Comput., in print, 2022.

[12] G. Ciaramella and M. J. Gander, Iterative Methods and Preconditioners for Systems of Linear equations. SIAM, 2022.

[13] B. Smith, P. Bjorstad and W. Gropp, Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations. Cambridge University Press, 1996.

[14] A. Quarteroni and A. Valli, Domain Decomposition Methods for Partial Differential Equations. Oxford Science Publications, 1999.