Advances and Applications in Mathematical Sciences
Volume 22, Issue 8, June 2023, Pages 1971-1976

ISSN 0974-6803

EXISTENCE AND UNIQUENESS OF SOLUTIONS OF FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we will discuss a unified approach to study the existence and uniqueness of solution of boundary value problems of fractional order subjected to non-local conditions. By the reduction of the problem to operator equation we establish the existence and uniqueness of solution. The approach used for the $\alpha + 1$ order nonlinear functional fractional differential equation can be applied to functional differential equations of any fractional orders.

1. Introduction

Some results on the problem of existence and uniqueness of solution of differential equations of fractional order have been discussed by some authors which can be found in [1, 2]. The purpose of this paper is to discuss a new approach to functional fractional differential equations, moreover this approach can be applied to functional differential equations of any fractional orders with nonlinear terms containing derivatives.

But for simplicity now we consider the following functional fractional differential equations of the form as:

$$\vartheta^{\alpha+2} = \phi(\eta, \vartheta(\eta), \vartheta(\varphi(\eta))), \eta \in [0, \sigma] \tag{1.1}$$

subjected to the general boundary conditions

2020 Mathematics Subject Classification: 26A33, 34A08, 34A12.

Keywords: fractional derivative and integral, fractional differential equation, existence and uniqueness of solutions.

Received January 28, 2022; Accepted February 22, 2022

$$B_1[\vartheta] = \alpha_1\vartheta(0) + \beta_1\vartheta^{\alpha}(0) + \gamma_1\vartheta^{\alpha-1}(0) = b_1,$$

$$B_2[\vartheta] = \alpha_2\vartheta(0) + \beta_2\vartheta^{\alpha}(0) + \gamma_2\vartheta^{\alpha-1}(0) = b_2,$$

$$B_3[\vartheta] = \alpha_3\vartheta(0) + \beta_3\vartheta^{\alpha}(0) + \gamma_3\vartheta^{\alpha-1}(0) = b_3, \tag{1.2}$$

such that

$$Rank\begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ \beta_1 & \beta_2 & 0 \\ \gamma_1 & \gamma_2 & 0 \\ 0 & 0 & \alpha_3 \\ 0 & 0 & \beta_3 \\ 0 & 0 & \gamma_3 \end{pmatrix} = 3$$

the function $\varphi(\eta)$ is assumed to be continuous and maps $[0, \sigma]$ into itself.

Definition 1.1 Riemann-Liouville definition[3, 4, 5]. For $\alpha \in [n-1, n)$ the α-derivative of f is

$$D_a^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n x}{dt^n} \int_a^{\alpha} \frac{f(x)}{(t-x)^{\alpha-n+1}}\, dx$$

Definition 1.2. Caputo definition[3, 4, 5]. For $\alpha \in [n-1, n)$ the α- derivative of f is

$${}_a^C D_a^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\, d\tau.$$

2. Existence and Uniqueness of Solution

To solve the problem (1.1)-(1.2) we introduce the nonlinear operator Ω defined in the space of continuous functions $C[0, \sigma]$ by the formula:

$$(\Omega\psi)(\eta) = \phi(\eta, u(\eta), u(\varphi(\eta))), \tag{2.1}$$

where $u(\eta)$ is the solution of the problem

$$\vartheta^{\alpha+2}(\eta) = \psi(\eta),\ 0 < \eta < 1$$

$$B_1[\vartheta] = b_1,$$

$$B_2[\vartheta] = b_2,$$

$$B_3[\vartheta] = b_3, \tag{2.2}$$

where $B_1[\vartheta], B_2[\vartheta], B_3[\vartheta]$ are defined by (1.2).

Proposition 2.1. *Suppose the function ψ is a fixed point of the operator Ω, i.e., ψ is the solution of the operator equation*

$$\Omega\psi = \psi, \tag{2.3}$$

where Ω is defined by (2.1)-(2.2) *then the function $\vartheta(\eta)$ determined from the BVP* (2.2) *is a solution of the BVP* (1.1)-(1.2). *Conversely, suppose the function $\vartheta(x)$ is the solution of the BVP* (1.1)-(1.2) *then the function*

$$\psi(\eta) = \phi(\eta, \vartheta(\eta), \vartheta(\varphi(\eta)))$$

satisfies the operator equation (2.3).

Now, let $\Phi(\eta, s)$ be the Green function of the problem (2.2). Then the solution of the problem can be represented in the form

$$\vartheta(\eta) = g(\eta) + \frac{1}{\Gamma(-\alpha - n)} \int_0^{\sigma} \frac{\Phi^{(n)}(\eta, s)\psi(s)ds}{(\sigma - s)^{-\alpha+1-n}}, \tag{2.4}$$

where $g(\eta)$ is the polynomial of $\alpha + 1$ degree satisfying the boundary conditions

$$B_1[\vartheta] = b_1,\ B_2[\vartheta] = b_2,\ B_3[\vartheta] = b_3, \tag{2.5}$$

Denote

$$\mathcal{M}_0 = \max_{0 \le \eta \le \sigma} \frac{1}{\Gamma(-\alpha - n)} \int_0^1 \frac{|\Phi^{(n)}(\eta, s)|}{|(\sigma - s)^{-\alpha+1-n}|}. \tag{2.6}$$

For any positive number M define the domain

$$\mathcal{D}_{\mathcal{M}} = \{(\eta, \vartheta, v) |\ 0 \le \eta \le \sigma; |\vartheta| \le \|g\| + \mathcal{M}_0\mathcal{M}; |v| \le \|g\| + \mathcal{M}_0\mathcal{M}\}, \tag{2.7}$$

where $\|g\| = \max_{0 \le \eta \le \sigma} |g(\eta)|$.

As usual, we denote by $B[0, \mathcal{M}]$ the closed ball of the radius $\mathcal{M}$ centered at 0 in the space of continuous functions $C[0, \sigma]$.

Theorem 2.2. *Suppose that*:

(i) *The function* $\varphi(\eta)$ *is a continuous map from* $[0, \sigma]$ *to* $[0, \sigma]$.

(ii) *The function* $\phi^{(n)}(\eta, \vartheta, v)$ *is continuous and bounded by* $\mathcal{M}$ *in the domain* $D_{\mathcal{M}}$, *that is,*

$$|\phi^{(n)}(\eta, \vartheta, v)| \leq \mathcal{M} \forall (\eta, \vartheta, v) \in D_{\mathcal{M}}. \tag{2.8}$$

(iii) *The function* $\phi^{(n)}(\eta, \vartheta, v)$ *satisfies the Lipschitz conditions in the variables* u, v *with the coefficients* $L_1, L_2 \geq 0$ *in* $D_{\mathcal{M}}$, *that is,*

$$|\phi^{(n)}(\eta, \vartheta_2, v_2) - \phi^{(n)}(\eta, \vartheta_1, v_1)| \leq L_1 |\vartheta_2 - \vartheta_1| + L_2 |v_2 - v_1|$$

$$\forall (\eta, \vartheta_i, v_i) \in D_{\mathcal{M}} (i = 1, 2) \tag{2.9}$$

(iv) $q = (L_1 + L_2)\mathcal{M}_0 < 1.$ (2.10)

The problem (1.1)-(1.2) *has a unique solution* $\vartheta(\eta) \in C^3[0, \sigma]$, *satisfying*

$$|\vartheta(\eta)| \leq \| g \| + \mathcal{M}_0 \mathcal{M} \forall \eta \in [0, \sigma]. \tag{2.11}$$

Proof. Claim 1. The operator Ω is a mapping $\mathcal{B}[0, \mathcal{M}] \to \mathcal{B}[0, \mathcal{M}]$.

Indeed, for any $\psi \in \mathcal{B}[0, \mathcal{M}]$, we have $\| \psi \| \leq \mathcal{M}$. Let $\vartheta(\eta)$ be the solution of the problem (2.2). From (2.4) it follows

$$|\vartheta(\eta)| \leq \| g \| + \mathcal{M}_0 \mathcal{M} \forall \eta \in [0, \sigma]. \tag{2.12}$$

Since $0 \leq \varphi(\eta) \leq a$, we have

$$|\vartheta(\varphi(\eta))| \leq \| g \| + \mathcal{M}_0 \mathcal{M} \forall t \in [0, \sigma].$$

Therefore, if $\eta \in [0, \sigma]$ then $(\eta, \vartheta(\eta), \vartheta(\varphi(\eta))) \in D_{\mathcal{M}}$. By the supposition (2.8) we have $|\phi(\eta, \vartheta(\eta), \vartheta(\varphi(\eta)))| \leq \mathcal{M} \forall \eta \in [0, \sigma]$. From (2.1) we have $|(\Omega\psi)(\eta)| \leq \mathcal{M} \forall \eta \in [0, \sigma]$. It means $|(\Omega\psi)| \mathcal{M}$ or $\Omega\psi \in B[0, B]$.

Claim 2. Ω is a contraction in $\mathcal{B}[0, \mathcal{M}]$.

If $\psi_1, \psi_2 \in \mathcal{B}[0, \mathcal{B}]$ and $\vartheta_1(\eta), \vartheta_2(\eta)$ is the solutions of the problem (2.2), respectively. Then from the supposition (2.9) we obtain

$$| \Omega\psi_2 - \Omega\psi_1 | \leq L_1 | \vartheta_2(t) - \vartheta_1(\eta) | + L_2 | \vartheta_2(\varphi(\eta)) - \vartheta_1(\varphi(\eta)) |. \tag{2.13}$$

From the representations

$$\vartheta_i(\eta) = g(\eta) + \frac{1}{\Gamma(-\alpha - n)} \int_0^\sigma \frac{\Phi^{(n)}(\eta, s)\psi_i(s)ds}{(\sigma - s)^{-\alpha+1-n}}, (i = 1, 2) \tag{2.14}$$

and (2.6) we have

$$| \vartheta_2(\eta) - \vartheta_1(\eta) | \leq \mathcal{M}_0 \| \psi_2 - \psi_1 \|,$$

$$| \vartheta_2(\varphi(\eta)) - \vartheta_1(\varphi(\eta)) | \leq \mathcal{M}_0 \| \psi_2 - \psi_1 \|$$

Together with the above estimates and (2.13), from the supposition (2.10) we get

$$\| \Omega\psi_2 - \Omega\psi_1 \| \leq q \| \psi_2 - \psi_1 \|, q < 1.$$

Thus, Ω is a contraction mapping in $\mathcal{B}[0, \mathcal{M}]$.

Therefore, the operator equation (2.3) has a unique solution $\psi \in \mathcal{B}[0, \mathcal{M}]$. From Proposition 2.1 the solution of the problem (2.2) for this right-hand side $\psi(\eta)$ is the solution of the original problem (1.1)-(1.2).

3. Conclusion

In this paper, we have discussed the existence and uniqueness of solution of boundary value problems of fractional order subjected to non-local conditions. By the reduction of the problem to operator equation we established the existence and uniqueness of solution. The approach used for the $\alpha + 1$ order nonlinear functional fractional differential equation can be applied to functional differential equations of any fractional orders. It also can be applied to fractional order integro-differential equations.

References

[1] A. D. Nagargoje, V. C. Borkar and R. A. Muneshwar, Existence and uniqueness of solution of fractional differential equation for the ocean flow in arctic gyres and mild solution of fractional Voltera integrodifferential equations, Novyi Mir Research Journal 6(6) (2021), 44-56.

[2] A. D. Nagargoje, V. C. Borkar and R. A. Muneshwar, Existence and uniqueness of solutions for neutral stochastic fractional integro-differential equations with impulses by a Rosenblatt process, Novyi Mir Research Journal 6(6) (2021), 88-97.

[3] Shantanu Das, Functional Fractional Calculus, Springer, (2011).

[4] I. Podlubny, Fractional Differential Equations, Academic Press USA, (1999).

[5] K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley and Sons Inc., New York, (1993).