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A MAGNETOGRAPHIC ANALYSIS OF MHD ORTHOGONAL ROTATING FLOW

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Abstract

This paper deals with the analytical study of a homogeneous, incompressible, viscous fluid having infinite electrical conductivity in a rotating reference frame flowing through porous medium under the presence of magnetic field which is mutually perpendicular to the velocity field. The governing non-liner partial differential equations of the flow are transformed into solvable form by application of the magnetograph transformation method. Two different forms of magnetic flux functions are considered and exact solutions of both of them are found out. Also the magnetic flux functions for each case are plotted.

1. Introduction

In this paper we apply magnetograph transformation method to find exact solution of non-liner partial differential equations which are the governing equations of the homogeneous, incompressible, viscous fluid having infinite electrical conductivity in a rotating reference frame flowing through

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porous media. The magnetic field and the velocity field are considered to be mutually perpendicular to each other in this work. Transformation techniques are often employed for solving the non-linear partial differential equations. If the magnetic field vectors are laid off from a fixed point, the extremities of these vectors trace out a curve, called the magnetograph [1]. Magnetograph transformation is the process of obtaining an equivalent linear system by interchanging the roles of dependent variables and independent variables. In other words, when transformations are employed to interchange the roles of the two independent variables and the two components of the magnetic field in the physical plane, these transformations are the magnetograph transformations. By applying this magnetograph transformation, the governing non-linear equations are transformed into linear equations in solvable form. Some researchers have applied magnetograph transformation in the study of MHD fluid flow and found exact solutions. S. N. Singh [1] introduced and employed magnetograph transformation to study orthogonal MHD flow. C. S. Bagewadi and Siddabasappa [2] studied variably inclined rotating MHD flow in magnetograph plane. Also M. Kumar and S Sil [3] studied aligned MHD flow in magnetograph plane and found exact solutions.

The study of rotating fluid and related aspects plays an important role due to its appearance in various natural phenomena and for its applications in many technological solutions which are directly governed by the action of coriolis force due to earth's rotation. The effect of coriolis force is found to be significant compared to the inertial and viscous forces in the equations of motion. The coriolis force exerts a strong influence on the hydromagnetic flow in the earth liquid core which plays an important role in the mean geomagnetic field [4]. The theory of rotating fluid is also important due to its place in solar physics involved in the sunspot development, the solar cycle and the structure of rotating magnetic stars. Many studies [5-13] have been made on the rotating fluid and several investigations have been carried out on various types of flows both non-MHD and MHD in a rotating system. Flow of viscous fluid in a porous medium is of great importance in the study of percolation through soils in hydrology, petroleum industry, in agricultural engineering and many other important areas. Many authors' [14-22] have studied fluid flows through porous media and found exact solution.

2. Basic Equations

The basic equations governing the motion of a steady, homogenous, infinite electrically conducting fluid flows through porous media is given by

$$\vec{\nabla} \cdot \vec{V} = 0, \tag{1}$$

$$\rho[(\vec{V} \cdot \vec{\nabla})\vec{V} + 2\vec{w} \times \vec{V} + \vec{w} \times (\vec{w} \times \vec{r})] + \vec{\nabla} p = \eta \nabla^2 \vec{V} + \mu(\vec{\nabla} \times \vec{H}) \times \vec{H} - \frac{\eta}{k}\vec{V}, \tag{2}$$

$$\vec{\nabla} \times (\vec{V} \times \vec{H}) = 0, \tag{3}$$

$$\vec{\nabla} \cdot \vec{H} = 0, \tag{4}$$

where $\vec{V}$ = velocity vector, $\vec{H}$ = magnetic field vector, p = fluid pressure, ρ = fluid density, $\vec{w}$ = angular velocity vector, $\vec{r}$ = the radius vector, η = coefficient of viscosity, μ = magnetic permeability, and k = permeability of the medium. As the magnetic field vector is perpendicular to the velocity vector, we get

$$uH_1 + vH_2 = 0, \tag{5}$$

where $\vec{V} = (u, v)$ and $\vec{H} = (H_1, H_2)$, from equation (3) we find that

$$uH_2 - vH_1 = f, \tag{6}$$

where *f* is an arbitrary non zero constant. Equation (5) and (6) yield

$$u = \frac{fH_2}{H_1^2 + H_2^2}, \; u = -\frac{fH_1}{H_1^2 + H_2^2},$$

or

$$u = \frac{fH_2}{H^2}, \; v = -\frac{fH_1}{H^2}, \tag{7}$$

where

$$H^2 = H_1^2 + H_2^2,$$

using equation (7) in (1) we get

$$(H_2^2 - H_1^2)\frac{\partial H_1}{\partial y} + 2H_1H_2\left(\frac{\partial H_1}{\partial x} - \frac{\partial H_2}{\partial y}\right) + (H_2^2 - H_1^2)\frac{\partial H_2}{\partial x} = 0, \tag{8}$$

from equation (4) we obtain

$$\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0. \tag{9}$$

Equations (8) and (9) are two nonlinear partial differential equations in H_1 and H_2. Now on application of magnetograph transformation with H_1 and H_2 as independent variable and x and y as functions of H_1 and H_2 equations (8) and (9) takes the form,

$$-(H_2^2 - H_1^2)\frac{\partial x}{\partial H_2} + 2H_1H_2\left(\frac{\partial y}{\partial H_2} - \frac{\partial x}{\partial H_1}\right) - (H_2^2 - H_1^2)\frac{\partial y}{\partial H_1} = 0, \tag{10}$$

$$\frac{\partial y}{\partial H_2} + \frac{\partial x}{\partial H_1} = 0, \tag{11}$$

provided that

$$\frac{\partial(H_1,\, H_2)}{\partial(x,\, y)} \neq 0,$$

Equation (11) implies in the existence of a function $\phi(H_1, H_2)$ such that

$$\frac{\partial \phi}{\partial H_1} = y,\ \frac{\partial \phi}{\partial H_2} = -x, \tag{12}$$

here $\phi(H_1, H_2)$ is called magnetic flux function.

Using equation (12) in (10) we get

$$(H_2^2 - H_1^2)\frac{\partial^2 \phi}{\partial H_2^2} + 4H_1H_2\frac{\partial^2 \phi}{\partial H_1 \partial H_2} - (H_2^2 - H_1^2)\frac{\partial^2 \phi}{\partial H_1^2} = 0. \tag{13}$$

Introducing polar coordinates (H, θ) in the (H_1, H_2) plane we get transformed equation (13) as

$$\frac{\partial^2 \phi}{\partial H^2} - \frac{1}{H^2}\frac{\partial^2 \phi}{\partial \theta^2} - \frac{1}{H}\frac{\partial \phi}{\partial H} = 0, \tag{14}$$

known solutions of (14) can be used to obtain some particular solutions for the plane orthogonal flows. Given a solution (H, θ) of (14), from (12) we have

$$x = -\frac{\partial \phi}{\partial H_2}, \; y = \frac{\partial \phi}{\partial H_1}, \tag{15}$$

having considered $x = x(H_1, H_2)$ and $y = y(H_1, H_2)$ provided that

$$\frac{\partial(x, y)}{\partial(H_1, H_2)} \neq 0.$$

In this the velocity field thus obtained must satisfy the momentum equation (2). Now two-dimensional flow equation (2) yield,

$$\frac{\partial \zeta}{\partial y} - \rho\zeta v - 2\rho w v + \mu j H_2 + \frac{\eta}{k} u = -\frac{\partial B}{\partial x}, \tag{16}$$

$$\eta \frac{\partial \zeta}{\partial x} - \rho\zeta u - 2\rho w u + \mu j H_1 - \frac{\eta}{k} v = \frac{\partial B}{\partial y}, \tag{17}$$

where $B = p' + \frac{1}{2}\rho v^2 + \frac{1}{2}\rho |\vec{w} \times \vec{r}\,|^2$ is Bernoulli function. $p' = p - \frac{1}{2}\rho |\vec{w} \times \vec{r}\,|^2$ is reduced pressure $j = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}$ is the current density function and $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is the vorticity function.

2. Exact Solutions

Let us consider solution of equation (14) be given by

$$\phi = \frac{1}{2} f_1 (H_1^2 + H_2^2) + f_2, \tag{18}$$

where f_1 and f_2 are arbitrary constants. In this case we have

$$x = -\frac{\partial \phi}{\partial H_2} = -f_1 H_2, \; y = \frac{\partial \phi}{\partial H_1} = f_1 H_1, \tag{19}$$

and their magnetic field is given by

$$H_1 = \frac{y}{f_1}, \; H_2 = -\frac{x}{f_1}, \tag{20}$$

these relation represent a circulatory flow.

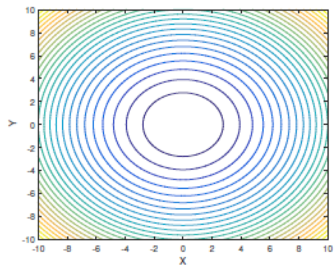

Figure 1. Circular Magnetic field lines.

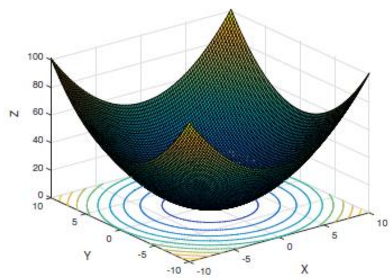

Figure 2. Magnetic field surface.

Further we have

$$u = -\frac{ff_1 x}{r^2},\; v = \frac{ff_1 y}{r^2},$$

where $r^2 = x^2 + y^2$,

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0, \tag{21}$$

$$j = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} = -\frac{2}{f_1},$$

Using (21) in (16) and (17) and from integrability conditions for B, we have

$$y\frac{\partial w}{\partial y} + x\frac{\partial w}{\partial x} = 0, \tag{22}$$

The most general solutions of (22) is given by

$$w =_2 F_1\left[a; b; c; \frac{y}{x}\right] = \sum \frac{a_n b_n y^n}{n!\, C_n x^n},\; x \neq 0 \tag{23}$$

where

$${}_2F_1[a; b; c; z] = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)z^2}{c(c+1)2!} +, \ldots,$$

is the Gauss hypergeometric function [23], a, b and c are constants $c \neq 0$ and

$$\alpha_n = \alpha(\alpha+1), \ldots, (\alpha+n-1), \ldots,$$

$$n! = n(n-1), \ldots, 1,$$

taking the most particular case when $n = 1$ we find that

$$w = \frac{c_3 y}{x},\ x \neq 0, \tag{24}$$

where c_3 is an arbitrary constant. Now equations (16) and (17) reduced respectively to

$$\frac{\partial B}{\partial x} = -2\rho f f_1 c_3 \frac{y^2}{x(x^2+y^2)} - \frac{2\mu x}{f_1^2} + \frac{\eta f f_1}{kr^2} x, \tag{25}$$

$$\frac{\partial B}{\partial x} = 2\rho f f_1 c_3 \frac{yx}{(x^2+y^2)} - \frac{2\mu y}{f_1^2} + \frac{\eta f f_1}{kr^2} y, \tag{26}$$

on simplifying above expression, we find the values of B,

$$B = c_4 - \rho f f_1 c_3 \ln\frac{x^2}{r^4} - \frac{\mu}{f_1^2} r^2 + \frac{\eta f f_1}{k} \ln r^2, \tag{27}$$

$$p = c_4 - \rho f f_1 c_3 \ln\frac{x^2}{r^2} - \frac{\mu}{f_1^2} r^2 + \frac{\eta f f_1}{k} \ln r^2 - \frac{\rho}{2}\frac{(f f_1)^2}{r^2}. \tag{28}$$

Further we take another solution of equation (14) as

$$\phi = A(H_1^2 + H_2^2) + B_1 H_1 + CH_2 + D, \tag{29}$$

where A, B_1, C, D are arbitrary constant.

In this case magnetic field is given by

$$H_1 = \frac{y - B_1}{2A},\ H_2 = -\frac{x + C}{2A},$$

these relations represent a circulatory flow.

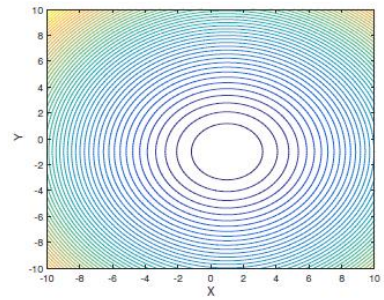

Figure 3. Circular Magnetic field lines.

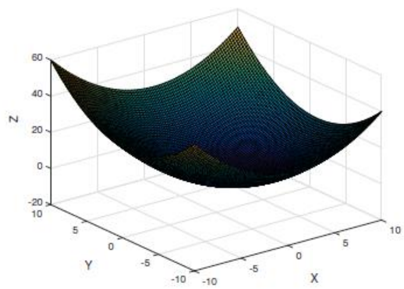

Figure 4. Magnetic field surface.

Further, we have

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0,\ u = \frac{-2Af(x + C)}{(y - B_1)^2 + (x + C)^2},\ v = \frac{-2Af(y - B_1)}{(y - B_1)^2 + (x + C)^2},$$

$$j = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} = 0, \tag{30}$$

Using integrability for B in equations (16) and (17) by use of equation (30), we get

$$(x + C)\frac{\partial w}{\partial x} + (y + B_1)\frac{\partial w}{\partial y} = 0, \tag{31}$$

solving above equation we have,

$$w = D_1 \ln(x + C) - D_1 \ln(y + B_1), \tag{32}$$

where D_1 is an arbitrary constant.

Now (16) and (17) reduce respectively to

$$\frac{\partial B}{\partial x} = -\frac{4\rho Af[D_1 \ln(x + C) - D_1 \ln(y + B_1)](y + B_1)}{(y + B_1)^2 + (x + C)^2} + \frac{2\eta Af(x + C)}{k(y + B_1)^2 + (x + C)^2}, \tag{33}$$

$$\frac{\partial B}{\partial y} = -\frac{4\rho Af[D_1 \ln(x + C) - D_1 \ln(y + B_1)](x - C)}{(y + B_1)^2 + (x + C)^2} + \frac{2\eta Af(y + B_1)}{k(y + B_1)^2 + (x + C)^2}, \tag{34}$$

which implies that

$$\begin{aligned} B = {} & -4Af\rho D_1(y + B_1)[\frac{\ln(y + B_1)}{2(y + B_1)} \\ & \tan^{-1}\frac{(x + C)}{(y + B_1)} - \frac{1}{2(y + B_1)}\tan^{-1}\frac{(x + C)}{(y + B_1)}\ln(x + C) \\ & - \frac{\ln(y + B_1)}{(y + B_1)}\tan^{-1}\frac{(x + C)}{(y + B_1)}] \\ & + 2Af\eta\frac{\ln[(y + B_1)^2 + (x + C)^2]}{k} + 4\rho AfD_1(x + C)[\frac{\ln(x + C)}{(x + C)}\tan^{-1}\frac{(y + B_1)}{(x + C)} \\ & - \frac{\ln(y + B_1)}{2(y + B_1)}\tan^{-1}\frac{(y + B_1)}{(x + C)} - \frac{\ln(y + B_1)}{2(x + C)}\tan^{-1}\frac{(y + B_1)}{x + C}]. \end{aligned} \tag{35}$$

and,

$$\begin{aligned} p = {} & -4Af\rho D_1(y + B_1)[\frac{\ln(y + B_1)}{2(y + B_1)}\tan^{-1}\frac{(x + C)}{(y + B_1)} \\ & - \frac{1}{2(y + B_1)}\tan^{-1}\frac{(x + C)}{(y + B_1)}\ln(x + C) \\ & - \frac{\ln(y + B_1)}{(y + B_1)}\tan^{-1}\frac{(x + C)}{(y + B_1)}] + 2Af\eta\frac{\ln[(y + B_1)^2 + (x + C)^2]}{k} \end{aligned}$$

$$+ 4\rho A f D_1 (x + C)[\frac{\ln(x + C)}{(x + C)} \tan^{-1} \frac{(y + B_1)}{(x + C)} - \frac{\ln(y + B_1)}{2(y + B_1)} \tan^{-1} \frac{(y + B_1)}{(x + C)}$$

$$- \frac{\ln(y + B_1)}{2(x + C)} \tan^{-1} \frac{(y + B_1)}{(x + C)}] - \frac{2\rho A^2 f^2 (y + B_1)^2}{[(y + B_1)^2 + (x + C)^2]^2}. \quad (36)$$

3. Conclusion

In this paper we have found exact solution of non-linear differential partial equations governing the flow of the homogeneous, incompressible, viscous fluid having infinite electrical conductivity in a rotating reference frame flowing through porous media by use of magnetograph transformation method. We have considered two possible solutions for the magnetic flux function satisfying the equation (14). In both the cases the solutions represent circulatory flow. The expressions for velocity field, magnetic field, vorticity function, current density function, angular velocity and pressure distribution are found out for these two different cases. Also the magnetic field lines and magnetic field surfaces are plotted for both the forms of the magnetic flux function. The present analysis is more general and for a non-porous medium i.e. $\frac{\eta}{k} \to 0$ the result of S. N. Singh [1] can be recovered for case of the first form of solution for magnetic flux function.

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