Advances and Applications in Mathematical Sciences **ISSN 0974-6803**
Volume 22, Issue 8, June 2023, Pages 1743-1753

SOME RESULTS ON RAINBOW CONNECTION NUMBER OF GRAPH

T. ARUNA and S. P. JEYAKOKILA

Department of Mathematics
Lady Doak College
Madurai, Tamilnadu, India
E-mail: arunat8910@gmail.com
jeyakokila@ldc.edu.in

Abstract

Let G be a connected graph on which an edge coloring $c : E(G) \to \{1, 2, \ldots, k\}, k \in N,$ is defined, where adjacent edges may be colored by the same color. The graph G is rainbow connected and the edge-coloring c is called rainbow coloring if there exists a rainbow path between every two vertices of G. Rainbow connection number of a graph G, denoted by $rc(G)$ is the minimum number of colors which are needed in order to make G rainbow connected. In this paper, the rainbow connection number for some graphs such as cocktail party graph, line and middle graph of wheel graph are determined and $rc(G) = \sum_{i=1}^{n} rc(B_i)$ if G is a connected graph with a cut vertex and has 'n' number of blocks B_i, is proved.

1. Introduction

Rainbow connection of graphs is a new developing concept in recent times. It was first introduced by G. Chartrand, G. L. Johns, K. A. McKeon and P. Zhang in [1]. It was developed to find the minimum number of distinct passwords assigned to the edges of a path between every two vertices in a network in order to transfer the high security information between the vertices.

All graphs considered in this paper are non-trivial, simple, finite and undirected graphs. Let G be a connected graph on which an edge coloring

2020 Mathematics Subject Classification: Primary 34E; Secondary 35K20, 68U20.
Keywords: rainbow connection number, cocktail party graph, line graph, middle graph, wheel graph.
Received March 31, 2022; Accepted April 19, 2022

$c : E(G) \rightarrow \{1, 2, \ldots, k\}$, is defined where adjacent edges may be colored by the same color. A path in G is a rainbow path if no two edges of the path are colored by the same color. The graph G is rainbow connected and the edge-coloring c is called rainbow k-coloring if every two vertices in G are connected by a rainbow path. The rainbow connection number of a graph G, denoted by $rc(G)$ is the minimum number of colors which are needed in order to make G a rainbow connected graph. For any connected graph G with size m and diameter $diam(G)$, then $diam(G) \leq rc(G) \leq m$ [1]. If G is rainbow k-colorable, then $rc(G) \leq k$.

The following results are proved by Chartrand et al. in [1]. Using these results, we will determine the rainbow connection number of certain graphs in this paper.

Proposition 1.1[1]. *Let G be a nontrivial connected graph of size m. Then*

(*a*) $rc(G) = 1$ *if and only if G is a complete graph*

(b) $rc(G) = m$ *if and only if G is a tree.*

Proposition 1.2[1]. *For each integer* $n > 4$, $rc(C_n) = \left\lceil \frac{n}{2} \right\rceil$.

2. Rainbow Connection Number of Cocktail Party Graph

In this section, the rainbow connection number of the cocktail party graph will be determined by using the results which were seen in the previous section.

Definition 2.1. Cocktail party graph [6]. Let $K_{n\times 2}$ denote the graph with $m = 2n$ vertices. Each vertex u_i is non-adjacent to u_{n+i} for $i = 1, 2, \ldots, n$ and all other pairs of vertices are adjacent. This unique $(m-2)$-regular graph on m vertices is called cocktail party graph. The diameter of $K_{n\times 2}$ is 2 when $n > 1$.

Theorem 2.1. *Let* $K_{n\times 2}$ *be the cocktail party graph of order* $2n$ *where* $n > 1$. *Then* $rc(K_{n\times 2}) = 2$.

Proof. Let $K_{n\times 2}$ be a cocktail party graph where $n > 1$. Let $m = 2n$.

Let $V(K_{n\times 2}) = \{u_1, u_2, u_3, \ldots, u_n, u_{n+1}, u_{n+2}, \ldots, u_{n+n}\}$ be a vertex set of $K_{n\times 2}$.

Let $E(K_{n\times 2}) = \{u_i u_j \in V(K_{n\times 2}) \times V(K_{n\times 2}) \mid i \neq j, i + n \neq j$ and $1 \leq i, j \leq m\}$.

Since $diam(K_{n\times 2}) = 2$, $rc(K_{n\times 2}) \geq 2$. It remains to prove $rc(K_{n\times 2}) \leq 2$. Since $K_{n\times 2}$ is $(m-2)$ regular graph, it contains a cycle C_m of length m formed by the vertices $u_1, u_2, \ldots, u_n, u_{n+1}, u_{n+2}, \ldots u_{n+n}$. Define the coloring $c : E(K_{n\times 2}) \to \{1, 2\}$ as follows

$c(e_i) = 1, \forall e_i = u_i u_{i+1} \in E(C_m)$, where $i = 1, 2, \ldots, m$ and $u_m u_{m+1} = u_m u_1$

$c(e) = 2, \forall e \in E(K_{n\times 2})$ and $e \notin E(C_m)$

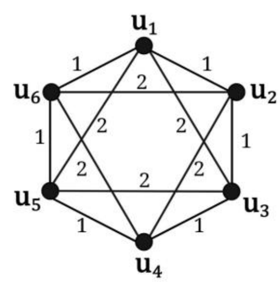


Figure 2.1. Coloring of the graph $K_{3\times 2}$ by the definition of c.

So that the edges between adjacent vertices are colored by the color 1 or 2 whereas the edges of the path between any two non-adjacent vertices are colored by the colors 1 and 2 since diameter of $K_{n\times 2}$ is 2. c is a rainbow 2-coloring which implies $rc(K_{n\times 2}) \geq 2$. Therefore $rc(K_{n\times 2}) = 2$.

3. Rainbow Connection Number of Line and Middle Graph of Wheel Graph

In this section, the rainbow connection number of line and middle graph of wheel graph will be determined by using the known results.

Definition 3.1 Line Graph. A line graph $L(G)$ of a simple graph G is a graph whose vertices are edges of G and two vertices in $L(G)$ are adjacent if and only if the corresponding edges of G have a vertex in common. The number of vertices in $L(G)$ is $|V(L(G))| = |E(G)|$ and the number of edges in $L(G)$ is $|E(L(G))| = \frac{1}{2}(\sum_{i=1}^{|V(G)|} d_1^2) - |E(G)|$ where d_i is a degree of a vertex v_i.

Theorem 3.1. *Let $L(W_n)$ be a line graph of a wheel graph W_n. Then* $rc(L(W_n)) = \begin{cases} 2 \; if \; n = 3, 4 \\ 3 \; if \; n \geq 5. \end{cases}$

Proof. Let $V(L(W_n)) = \{u_1, u_2, u_3, \ldots u_n, v_1, v_2, \ldots, v_n\}$ be a vertex set of $L(W_n)$. $L(W_n)$ contains a complete graph K_n with vertices $V(K_n) = \{u_1, u_2, u_3, \ldots, u_n\}$ and a cycle C_n with vertices $V(C_n) = \{v_1, v_2, v_3, \ldots, v_n\}$. Let $E(K_n) = \{e_{ij}, u_i, u_j \mid i, j \ldots, 1, 2, n$ and $i < j\}$. Let $E(C_n) = \{e_1, v_i v_{i+1} \mid i = 1, 2, 3, \ldots v_n$ and $v_n v_{n+1} = v_n v_1\}$. Let $M = E(L(W_n))/\{E(K_n) \cup E(C_n)\} = \{a_{ij} = v_i u_j \mid i = 1, 2, 3, \ldots, n$ and $j = i$ and $j = i + 1, v_n u_{n+1} = v_n u_1\}$.

Therefore, $E(L(W_n)) = E(K_n) \cup E(C_n) \cup M$.

Case 1. $n = 3, 4$

$diam(L(W_n)) = 2$, where $n = 3, 4$. Therefore $rc(L(W_n)) \geq 2$. It remains to show that $rc(L(W_n)) \geq 2$. Define a coloring $c : E(L(W_n)) \to \{1, 2\}$ as follows

$c(e_i) = 1, \forall \; e_i \in E(C_n)$ and i is odd

$c(e_i) = 2, \forall \; e_i \in E(C_n)$ and i is even

$c(e_{ij}) = 2, \forall \; e_{ij} \in E(K_n)$

$c(a_i) = 1, \forall \; a_{ij} \in M$

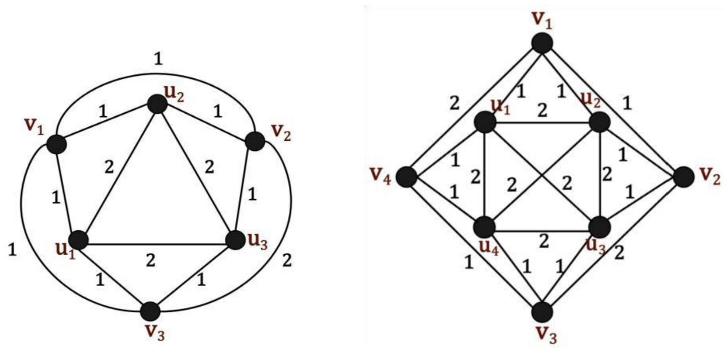


Figure 3.1. Coloring of graphs $L(W_3)$ and $L(W_4)$ by the definition of c.

By the definition of c, there exists a rainbow path between every pair of vertices of $L(W_n)$. Therefore c is rainbow 2-coloring which implies that $rc(L(W_n)) \geq 2$. Therefore $rc(L(W_n)) = 2$, when $n = 3, 4$.

Case 2. Let $n \geq 5$.

Define a coloring $c_1 : E(L(W_n) \to \{1, 2\}$ as follows

$c_1(e_i) = 1, \forall e_i \in E(C_n)$ and i is odd

$c_1(e_i) = 2, \forall e_i \in E(C_n)$ and i is even

$c_1(e_{ij}) = 2, \forall e_{ij} \in E(K_n)$

$c_1(a_{ii}) = 1, \forall a_{ii} \in v_i u_i \in M$

$c_1(a_{ij}) = 3, \forall a_{ij} \in M$ and $j = i + 1$.

By the definition of c_1, there exists a rainbow path between every pair of vertices of $L(W_n)$. Therefore, c is rainbow 3-coloring which implies that

$$rc(L(W_n)) \leq 3 \tag{1}$$

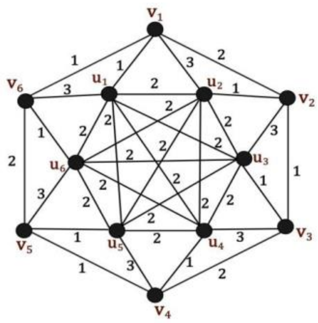


Figure 3.2. Coloring of the graph $L(W_n)$ by the definition of c_1.

Sub case (i). $n > 5$

Since

$$diam(L(W_n) = 3, rc(L(W_n)) \geq 3 \qquad (2).$$

Therefore by (1) and (2) $rc(L(W_n)) = 3$ for $n > 5$.

Sub case (ii). $n = 5$

Since $(L(W_5) = 2, rc(L(W_5)) \geq 2$. Suppose that $rc(L(W_5) = 2$. $L(W_5)$ contains a cycle C_5. Then $rc(C_n) = 3$ by the proposition 1.2.

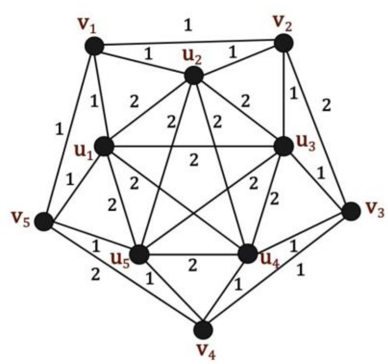


Figure 3.3. Coloring of $L(W_5)$ by definition of c.

If two distinct colors are assigned by the definition of c which is shown in the figure 3.5, then there is no rainbow path between the vertices v_2 and v_5 which is a contradiction to the assumption that $rc(L(W_5) = 2$. Therefore $rc(L(W_5) = 3$ for $n = 5$.

Definition 3.2. Middle Graph. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The middle graph $M(G)$ of G is a graph whose vertex set is the union of $V(G)$ and $E(G)$ and two vertices are adjacent if and only if

(i) They are adjacent edges of G or

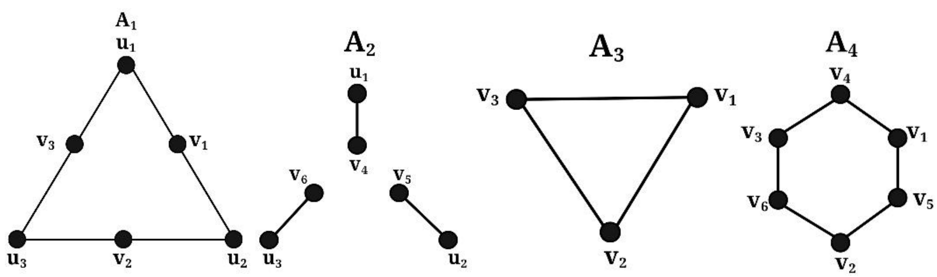
(ii) One is a vertex and the other is an edge incident with it.

Theorem 3.2. *Let G be the middle graph of Wheel graph W_n with $3n+1$ vertices then the diameter of $M(W_n)$ is* $\begin{cases} 2 & if\ n = 3 \\ 3 & if\ n > 3. \end{cases}$

Proof.

Case 1. Let $n = 3$.

Let $V(G) = \{u_1, u_2, u_3, u, v_1, v_2, \ldots, v_6\}$ be a vertex set of G. Let A_1 be a subgraph of G with vertex set $E(A_1) = \{u_1, u_2, u_3, v_1, v_2, v_3\}$ and edges set $E(A_1) = \{u_i v_i, v_i u_{i+1} \mid i = 1, 2, 3$ and $v_3 u_4 = v_3 u_1\}$. Let A_2 be a subgraph of G with vertex set $V(A_2) = \{u_1, u_2, u_3, v_4, v_5, v_6\}$ and edges set $E(A_2) = \{u_i v_{i+3} \mid i = 1, 2, 3\}$. Let A_3 be a subgraph of G with vertex set $V(A_3) = \{v_1, v_2, v_3\}$ and edges set $E(A_3) = \{v_i v_j \mid i, j, = 1, 2, 3$ and $i < j\}$. Let $E(A_1) = \{u_i v_i, v_i u_{i+1} \mid i = 1, 2, 3$ be a subgraph of G with vertex set $V(A_4) = \{v_1, v_2, \ldots, v_6\}$ and edges set $V(A_4) = \{v_i v_{i+3}, v_i v_{i+4} \mid i, j = 1, 2, 3$ and $v_3 v_7 = v_3 v_4\}$. Let A_5 be a subgraph of G with vertex set $V(A_5) = \{u, v_1, v_2, \ldots, v_3\}$ and edges set $E(A_4) = \{v_i v_{i+3}, v_i v_{i+4} \mid i, j = 1, 2, 3$ and $v_3 v_7 = v_3 v_4\}$. Let $E(G) = E(E(G) = E(A_1) \cup E(A_2) \cup E(A_3) \cup E(A_4)$ $\cup E(A_5)$



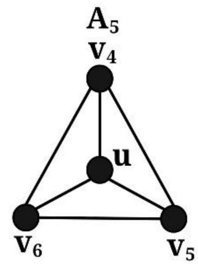


Figure 3.4. Subgraphs of the graph G.

$rc(G) \geq 2$. Since $diam(G) = 2$, it remains to show that $rc(G) \leq 2$. So define a coloring $c : E(G) \to \{1, 2\}$ as follows

$c(e) = 1, \forall e \in E(A_1)$ and $e = u_i v_i$, where $i = 1, 2, 3$

$c(e) = 2, \forall e \in E(A_1)$ and $e \neq u_i v_i$, where $i = 1, 2, 3$

$c(e) = 2, \forall e \in E(A_2)$

$c(e) = 2, \forall e \in E(A_3)$

$c(e) = 1, \forall e \in E(A_4)$ and $e = v_i v_{i+3}$, where $i = 1, 2, 3$

$c(e) = 2, \forall e \in E(A_4)$ and $e \neq v_i v_{i+3}$, where $i = 1, 2, 3$

$c(e) = 1, \forall e \in E(A_5)$

So that there exists a rainbow path between every pair of vertices of G. Therefore, c is rainbow-2-coloring which implies that $rc(G) \leq 2$. Therefore $rc(G) = 2$ when $n = 3$.

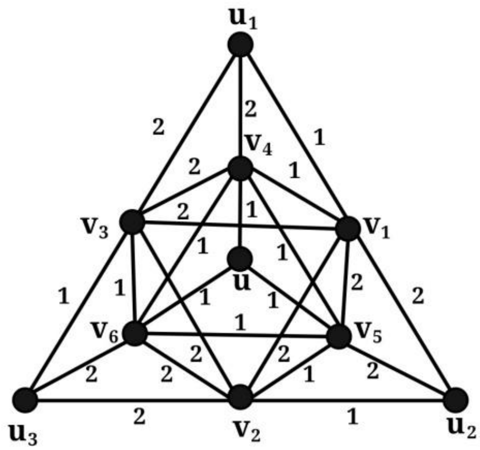


Figure 3.5. Coloring of the graph $M(W_4)$ by the defining of c.

Case 2. Let $n > 3$

Let $V(G) = \{u_1, u_2, \ldots, u_n, u, v_1, v_2, \ldots, v_{2n}\}$ be a vertex set of G. Let B_1 be a subgraph of G with vertex set $V(B_1) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and edges set $E(B_1) = \{u_i v_i, v_i u_{i+1} \mid i = 1, 2, \ldots, n$ and $v_n u_{n+1} = v_n u_1\}$. Let B_2 be a subgraph of G with vertex set $V(B_2) = \{u_1, u_2, \ldots, u_n, v_{n+1}, v_{n+2}, \ldots, v_{2n}\}$ and edges set $E(B_2) = \{e_i = u_i v_{i+1} \mid i = 1, 2, \ldots, n\}$. Let B_3 be a subgraph of G with vertex set $V(B_3) = \{v_1, v_2, \ldots, v_{2n}\}$ and edges set $E(B_3) = \{e_i = v_i v_{i+n}, v_i v_k \mid i, j = 1, 2, \ldots, n$ and $k = n + 1 + i$ and $v_n v_{2n+1} = v_n v_{n+1}\}$. Let B_4 be a subgraph of G with vertex set $V(B_4) = \{u, v_1, v_2, v\}$ and edges set $E(B_4) = \{uv_i, v_i v_j \mid i, j = n + 1, n + 2, \ldots, 2n$ and $i < j\} \cup \{v_i v_{i+1} \mid i = 1, 2, \ldots, n$ and $v_n v_{n+1} = v_n v_1\}$. Let $E(G) = E(B_1) \cup E(B_2) \cup E(B_3) \cup E(B_4)$.

$rc(G) \geq 3$. Since $diam(G) = 3$. It remains to show that $rc(G) \leq 3$. So define a coloring $c_1 : E(G) \to \{1, 2, 3\}$ as follows

$c_1(e) = 2, \forall\, e \in E(B_1)$ and $e = u_i v_i$, where $i = 1, 2, \ldots, n$

$c_1(e) = 2, \forall\, e \in E(B_1)$ and $e \neq u_i v_i$, where $i = 1, 2, \ldots, n$

$c_1(e_i) = 1, \forall\, e_i \in E(B_2)$ and i is odd

$c_1(e_i) = 1, \forall e_i \in E(B_2)$ and i is even

$c_1(e) = 2, \forall e \in E(B_3)$ and $e = v_i v_{i+n}$, where $i = 1, 2, \ldots, n$

$c_1(e) = 1, \forall e \in E(B_3)$ and $e \neq v_i v_{i+n}$, where $i = 1, 2, \ldots, n$

$c_1(e) = 3, \forall e \in E(B_4)$

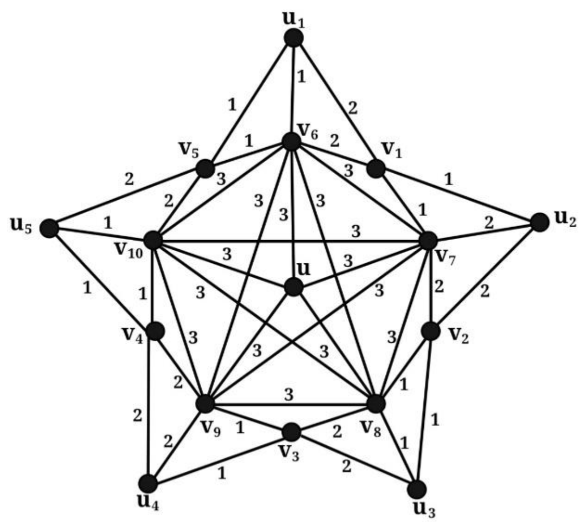


Figure 3.6. Coloring of the graph $M(W_5)$ by the defining of c_1.

So that there exists a rainbow path between every pair of vertices of G. Therefore c_1 is rainbow-3-coloring which implies that $rc(G) \leq 3$. Therefore $rc(G) = 3$ when $n > 3$. Hence proved. □

4. Conclusion

In this paper, we have determined the rainbow connection number of graphs such as cocktail party graph line and middle graph of wheel graph.

References

[1] G. Chartrand, G. L. Johns, K. A. McKeon and P. Zhang, Rainbow connection in graphs, Math. Bohem. 133 (2008), 85-98.

[2] G. Chartrand and P. Zhang, Chromatic Graph Theory, Chapman and Hall/CRC Press, Boca Raton, FL, 2009.

[3] Sara Kischnick (M.Sc.), Rainbow partitions and rainbow polynomial, 47th South-Eastern International Conference on Combinatorics, Graph Theory and Computing, Boca Raton, FL, USA, 2016.

[4] S. Arumugam and S. Ramachandran, Invitation to Graph Theory, 2001, Scitech Publications India Pvt. Ltd., 7/3C, Madley Road, T. Nagar, Chennai – 17.

[5] Syafrizal Sy, G. H. Medika and L. Yulianti, The rainbow connection of fan and sun, Appl. Math. Sci. 7(64) (2013), 3155-3159.

[6] Eric W. Weisstein, Cocktail party graph, From Math World-A Wolfram Web Resource, https://mathworld.wolfram.com/CocktailPartyGraph.html

[7] X. Li and Y. Sun, Rainbow Connections of Graphs, Springer Briefs in Mathematics, 2012.

[8] X. Li and S. Liu, Sharp upper bound for the rainbow connection numbers of 2-connected graphs, arXiv:1105.4210v2 [math.CO] 2011.