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MULTIPLY DIVISOR CORDIAL LABELING OF SPECIAL GRAPHS

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Abstract

Multiply divisor cordial labeling of a graph G^* having set of node V^* is a bijective h from $V(G^*)$ to $\{1, 2, \ldots, |V(G^*)|\}$ such that an edge xy is assigned the label 1 if 2 divides $(h(x) \cdot h(y))$ and 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph having multiply divisor cordial labeling is said to be a multiply divisor cordial graph.

1. Introduction

Graph theory is concerned with the use of edges to connect nodes. In computer science, the theory is quite useful. One of the unique concepts of graph labeling is demonstrated in this article using Multiply Divisor Cordial Labeling. All graphs included here are without loops and parallel edges, having no orientation, finite and connected. We follow the basic notations and terminologies of graph theory as in [8]. A graph labeling is a mapping that carries the graph components to the set of numbers, usually to the set of natural numbers. If the domain is the set of nodes the labeling is called node labeling. If the domain is the set of edges, then we speak about edge labeling. If the labels are assigned to both nodes and edges then the labeling is called total labeling. For a dynamic survey of various graph labeling, we refer to [6].

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The concept of multiply divisor cordial labeling was introduced by J. T. Gondalia and A. H. Rokad. J. T. Gondalia and A. H. Rokad et al. [1], [2] proved that cycle, cycle with one chord, cycle with twin chord, cycle with triangle, path graph, star graph, jelly fish and coconut tree are multiply divisor cordial graph. Further they proved that ring sum of star with cycle, ring sum of star with cycle having one chord, ring sum of star with cycle having twin chords, ring sum of star with cycle having triangle, ring sum of star with double fan, ring sum of star with double wheel and ring sum of star with helm graph are multiply divisor cordial labeling.

2. Preliminaries

Definition 2.1. The Petersen graph is 3-regular undirected graph with 10 vertices and 15 edges.

Definition 2.2. The fan graph is denoted by F_n and described as $F_n = P_n + K_1$, where P_n indicates the path graph with n vertices.

Definition 2.3. The helm H_n is the graph obtained from a wheel graph W_n by attaching a pendant vertex through an edge to each rim vertex of W_n.

Definition 2.4. The flower Fl_n is the graph obtained from a helm H_n by joining each pendant vertex of the helm to the apex vertex. Here the pendant vertices of helm H_n are referred as external vertices of Fl_n.

3. Main Results

Theorem 1. *The graph G obtained by joining two copies of Petersen graph by a path of arbitrary length is multiply divisor cordial.*

Proof. Let G be the graph obtained by joining two copies of Petersen graph by a path P_k of length $k-1$. Let $u_1, u_2, \ldots, u_5$ and $u_6, u_7, \ldots, u_{10}$ be external and internal vertices of first copy of Petersen graph respectively. Here each u_i is adjacent to $u_{i+5}, i = 1, 2, 3, 4, 5$. Similarly let $w_1, w_2, \ldots, w_5$ and $w_6, w_7, \ldots, w_{10}$ be external and internal vertices of second copy of Petersen graph respectively. Here each w_i is adjacent to $w_{i+5}, i = 1, 2, 3, 4, 5$. Let $v_1, v_2, \ldots, v_k$ be successive vertices of path P_k with $v_1 = u_1$ and $v_k = w_1$.

We define a labeling function $f : V(G) \to \{1, 2, \ldots, k+18\}$ as follows.

$$f(u_i) = 4i - 3;\ 1 \le i \le 5$$

$$= 4(i-5) - 1;\ 6 \le i \le 10,$$

$$f(v_j) = 2j + 17;\ 2 \le j \le \left\lceil \frac{k}{2} \right\rceil,$$

$$= 2j - \left\lceil \frac{k}{2} \right\rceil + 20;\ \left(\left\lceil \frac{k}{2} \right\rceil + 1\right) \le j \le k - 1,$$

$$f(w_i) = 4i - 2;\ 1 \le i \le 5,$$

$$= 4(i-5);\ 6 \le i \le 10.$$

The labeling defined above satisfies the conditions of multiply divisor cordial labeling and hence the graph under consideration is a multiply divisor cordial graph.

Illustration 1. A multiply divisor cordial labeling of the graph obtained by joining two copies of the Petersen graph by a path P_7 is shown in Figure 1.

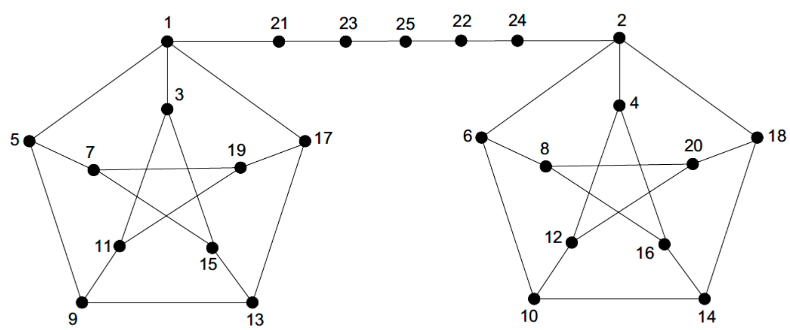

Figure 1. Petersen graph P_7.

Theorem 2. *The graph G obtained by joining two copies of cycle with one chord by a path of arbitrary length is multiply divisor cordial.*

Proof. Let G be the graph obtained by joining two copies of cycle C_n with one chord by path P_k. Let $u_1, u_2, \ldots, u_n$ be the vertices of first copy of cycle with one chord, $v_1, v_2, \ldots, v_k$ be the vertices of second copy of cycle with one chord and $w_1, w_2, \ldots, w_n$ be the vertices of path P_k with $w_1 = u_1$ and $w_k = v_1$. Let $e = u_1u_3$ be the chord in first copy of cycle C_n and $e' = v_1v_3$ be the chord in second copy of cycle C_n. To define labeling function $f : V(G) \to \{1, 2, \ldots, 2n + k - 2\}$ we consider following cases.

Case 1. $n = 4$, k is odd. Let $k = 2t + 1$, $t \in N$.

$f(u_1) = 3$, $f(u_2) = 1$, $f(u_3) = 5$, $f(u_4) = 7$

$f(w_1) = f(u_1) = 3$;

$f(w_i) = 2n + 2i - 3$; $2 \le i \le t + 1$.

$= 2[i - (t + 1)]$; $t + 2 \le i \le 2t + 1$.

$f(v_i) = 2t + 2(i - 1)$; $1 \le i \le n$.

Case 2. $n = 4, k$ is even.

Let $k = 2t$, $t \in N$.

$f(u_1) = 3$, $f(u_2) = 1$, $f(u_3) = 5$, $f(u_4) = 7$

$f(w_1) = f(u_1) = 3$;

$f(w_i) = 2n + 2i - 3$; $2 \le i \le t$.

$= 2(i - t)$; $t + 1 \le i \le 2t$.

$f(v_i) = 2t + 2(i - 1)$; $1 \le i \le n$.

Case 3. $n \ge 5, k$ is odd.

Let $k = 2t + 1$, $t \in N$.

$f(u_1) = 1$, $f(u_2) = 3$, $f(u_3) = 9$

$f(u_i) = 2i - 3$; if $4 \le i \le 5$

$= 2i - 1$; if $6 \le i \le n$.

$f(w_1) = f(u_1) = 1$;

$f(w_i) = 2n + 2i - 3;$ if $2 \leq i \leq t + 1.$

$= 2\{i - (t + 1)\};$ if $t + 2 \leq i \leq 2t + 1.$

$f(v_i) = 2t + 2(i - 1);$ if $1 \leq i \leq n.$

Case 4. $n \geq 5$, k is even.

Let $k = 2t$, $t \in N$.

$f(u_1) = 1$, $f(u_2) = 3$, $f(u_3) = 9$

$f(u_i) = 2i - 3;$ if $4 \leq i \leq 5$

$= 2i - 1;$ if $6 \leq i \leq n.$

$f(w_1) = f(u_1) = 1;$

$f(w_i) = 2n + 2i - 3;$ if $2 \leq i \leq t.$

$= 2\{i - t\};$ if $t + 1 \leq i \leq 2t.$

$f(v_i) = 2t + 2(i - 1);$ if $1 \leq i \leq n.$

One can observe that in each case the labeling defined above satisfies the conditions of multiply divisor cordial labeling and the graph under consideration is multiply divisor cordial graph.

Illustration 2. For better understanding of above defined labeling pattern the multiply divisor cordial labeling of graph obtained by joining two copies of C_6 with one chord by path P_5 is shown in Figure 2.

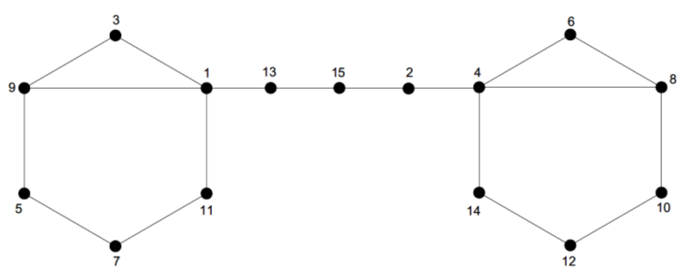

Figure 2. Join of two copies of C_6 with one chord by path P_5.

Theorem 3. *The graph G obtained by joining two copies of cycle with twin chords by a path of arbitrary length is multiply divisor cordial.*

Proof. Let G be the graph obtained by joining two copies of cycle C_n with twin chords by path P_k. Let $u_1, u_2, \ldots, u_n$ be the vertices of first copy of cycle with twin chords, $v_1, v_2, \ldots, v_k$ be the vertices of second copy of cycle with twin chords and $w_1, w_2, \ldots, w_n$ be the vertices of path P_k with $v_1 = u_1$ and $w_k = v_1$. Let $e_1 = u_1u_3$ and $e_2 = u_1u_4$ be the chords in first copy of cycle C_n and $e_1' = v_1v_3$ and $e_1' = v_1v_4$ be the chords in second copy of cycle C_n.

We define labeling function $f : V(G) \to \{1, 2, \ldots, 2n + k - 2\}$ as follows.

Case 1. k is odd. Let $k = 2t + 1,\ t \in N$.

$f(u_1) = 1,\ f(u_2) = 3,\ f(u_3) = 9$

$f(u_i) = 2i - 3;$ if $4 \leq i \leq 5$

$= 2i - 1;$ if $6 \leq i \leq n.$

$f(w_1) = f(u_1) = 1;$

$f(w_i) = 2n + 2i - 3;$ if $2 \leq i \leq t + 1.$

$= 2\{i - (t + 1)\};$ if $t + 2 \leq i \leq 2t + 1.$

$f(v_i) = 2t + 2(i - 1);$ if $1 \leq i \leq n.$

Case 2. k is even.

Let $k = 2t,\ t \in N.$

$f(u_1) = 1,\ f(u_2) = 3,\ f(u_3) = 9$

$f(u_i) = 2i - 3;$ if $4 \leq i \leq 5$

$= 2i - 1;$ if $6 \leq i \leq n.$

$f(w_1) = f(u_1) = 1;$

$f(w_i) = 2n + 2i - 3;$ if $2 \leq i \leq t.$

$= 2(i - t);$ if $t + 1 \leq i \leq 2t.$

$f(v_i) = 2t + 2(i - 1);$ if $1 \leq i \leq n.$

One can observe that in each case the labeling defined above satisfies the conditions of multiply divisor cordial labeling and the graph under consideration is multiply divisor cordial graph.

Illustration 3. For well understanding of above defined labeling pattern the multiply divisor cordial labeling of graph obtained by joining two copies of C_6 with twin chords by path P_6 is shown in Figure 3.

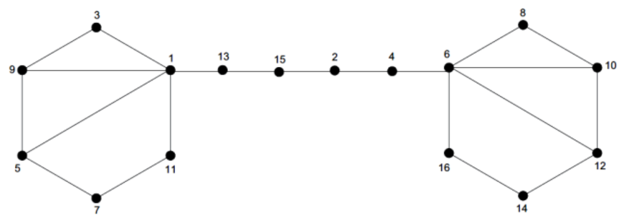

Figure 3. Join of two copies of C_6 with twin chords by path P_6.

Theorem 4. *The graph G obtained by joining two copies of cycle C_n with triangle by a path of arbitrary length is multiply divisor cordial.*

Proof. Let G be the graph obtained by joining two copies of cycle C_n with triangle by path P_k of length $k-1$. Let $u_1, u_2, \ldots, u_n$ be the vertices of first copy of cycle with triangle. Let $w_1, w_2, \ldots, w_n$ be the vertices of second copy of cycle with triangle. Let $v_1, v_2, \ldots, v_k$ be the vertices of path P_k with $u_1 = v_1$ and $v_k = w_1$. Let $e_1 = u_1u_3, e_2 = u_3u_5, e_3 = u_5u_1$ be the chords in first copy of cycle C_n and $e_1' = w_1w_3, e_2' = w_3w_5$ and $e_3' = w_5w_1$ be the chord in second copy of cycle C_n.

We define labeling function $f : V(G) \to \{1, 2, \ldots, 2n + k - 2\}$ as follows.

Case 1. k is even.

In this case define f as:

$f(u_1) = f(v_1) = 1,$

$f(w_1) = f(v_k) = k,$

$f(u_2) = 3, f(u_3) = 9, f(u_4) = 5, f(u_5) = 7,$

$f(u_i) = 2i - 1; 6 \leq i \leq n,$

$$f(v_j) = 2n + 2j - 3;\ 2 \le i \le \frac{k}{2}$$

$$= 2j - k;\ \frac{k}{2} + 1 \le i \le k,$$

$$f(w_i) = k + 2(i - 1);\ 1 \le i \le n.$$

Case 2. k is odd.

In this case define f as:

$$f(u_1) = f(v_1) = 1,$$

$$f(w_1) = f(v_k) = k - 1,$$

$$f(u_2) = 3,\ f(u_3) = 9,\ f(u_4) = 5,\ f(u_5) = 7,$$

$$f(u_i) = 2i - 1;\ 6 \le i \le n,$$

$$f(v_j) = 2n + 2j - 3;\ 2 \le i \le \frac{k+1}{2}$$

$$= 2j - (k + 1);\ \frac{k+3}{2} \le i \le k,$$

$$f(w_i) = k + 2i - 3;\ 1 \le i \le n.$$

One can observe that in each case the labeling defined above satisfies the conditions of multiply divisor cordial labeling and the graph under consideration is a multiply divisor cordial graph.

Illustration 4. The multiply divisor cordial labeling of the graph obtained by joining two copies of C_7 with triangle by a path P_4 is shown in Figure 4. It is the case related to k is even.

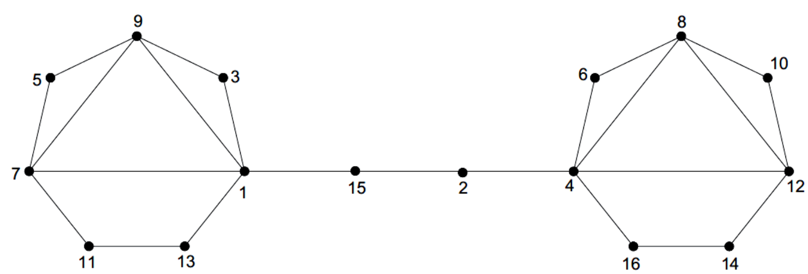

Figure 4. Join of two copies of C_7 with triangle by path P_4.

Theorem 5. *The graph G obtained by joining two copies of fan graph* F_n *by a path of arbitrary length is multiply divisor cordial.*

Proof. Let G be the graph obtained by joining two copies of fan graph F_n by a path P_k of length $k-1$. Let us denote the successive vertices of first copy of fan graph by $u_1, u_2, \ldots, u_{n+1}$ and the successive vertices of second copy of fan graph by $w_1, w_2, \ldots, w_{n+1}$. Let $v_1, v_2, \ldots, v_k$ be the vertices of path P_k with $v_1 = u_1$ and $v_k = w_1$. Here we consider the case for $n \geq 3$. We define a labeling function $f : V(G) \to \{1, 2, \ldots, 2n+k-2\}$ as follows.

Case 1. k is even.

In this case define f as:

$$f(u_1) = f(v_1) = 2,\ f(w_1) = f(v_k) = 1,$$

$$f(u_2) = 4,\ f(v_{\frac{k}{2}}) = 6,$$

$$f(u_i) = k + 2(i-1);\ 3 \leq i \leq n,$$

$$f(v_j) = 6 + 2(j-1);\ 2 \leq j \leq \frac{k}{2} - 1$$

$$= 2j - k + 1;\ \frac{k}{2} + 1 \leq j \leq k-1,$$

$$f(w_i) = k + 2i - 3;\ 1 \leq i \leq n.$$

Case 2. k is odd.

In this case define f as:

$$f(u_1) = f(v_1) = 2,\ f(w_1) = f(v_k) = 1,$$

$$f(u_2) = 4,\ f(v_{\frac{k-1}{2}}) = 6,$$

$$f(u_i) = k + 2i - 3;\ 3 \leq i \leq n,$$

$$f(v_j) = 6 + 2(j-1);\ 2 \leq j \leq \frac{k-3}{2}$$

$$= 2(j+1) - k; \frac{k+1}{2} \le j \le k-1,$$

$$f(w_i) = k + 2(i-1); 1 \le i \le n.$$

In each case f satisfies the conditions of multiply divisor cordial labeling and hence the graph under consideration is a multiply divisor cordial graph.

Illustration 5. Multiply divisor cordial labeling of the graph obtained by joining two copies of F_8 by a path P_5 is shown in Figure 5.

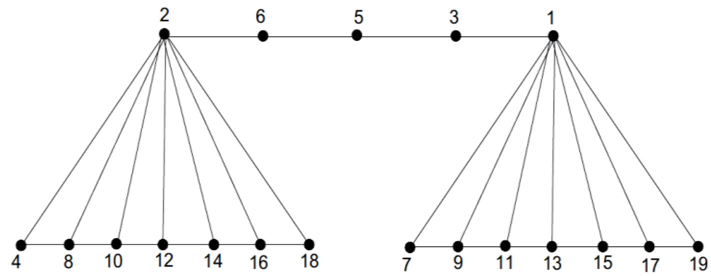

Figure 5. Join of two copies of F_8 by path P_5.

Theorem 6. *The graph G obtained by joining two copies of flower graph Fl_n by a path of arbitrary length is multiply divisor cordial.*

Proof. Let G be the graph obtained by joining two copies of flower graph Fl_n by a path P_k of length $k-1$. Let u_0 be the apex vertex, $u_1, u_2, \ldots, u_n$ be the rim vertices and $u_1^J, u_2^J, \ldots, u_n^J$ be the external vertices of first copy of flower Fl_n. Let w_0 be the apex vertex, $w_1, w_2, \ldots, w_n$ be the rim vertices and $w'_1, w'_2, w'_3, \ldots, w'_n$ be the external vertices of second copy of flower Fl_n. Let $v_1, v_2, \ldots, v_k$ be the vertices of path P_k with $v_1 = u_1$ and $v_k = w_1$.

We define a labeling function $f : V(G) \to \{1, 2, \ldots, 2n + k - 2\}$ as follows.

Case 1. $k = 2$.

In this case define f as:

$$f(u_1) = f(v_1) = 4,$$

$$f(w_1) = f(v_2) = 3,$$

$f(u_0) = 2,\ f(w_0) = 1,$

$f(w_1') = 7,$

$f(u_i) = 2(i+1);\ 1 \le i \le n,$

$f(u_i') = 2(n+i+1);\ 1 \le i \le n,$

$$f(w_i) = 8i - 1;\ 2 \le i \le \left\lceil \frac{n}{2} \right\rceil$$

$$= 8(n-i) + 9;\ \left(\left\lceil \frac{n}{2} \right\rceil + 1\right) \le i \le n,$$

$$f(w_i') = 8i - 5,\ 2 \le i \le \left\lceil \frac{n}{2} \right\rceil$$

$$= 8(n-i) + 5;\ \left(\left\lceil \frac{n}{2} \right\rceil + 1\right) \le i \le n.$$

Case 2. $k = 3$.

In this case define f as:

$f(u_1) = f(v_1) = 4,$

$f(v_2) = 4n + 3,$

$f(w_1) = f(v_3) = 3,$

$f(u_0) = 2,\ f(w_0) = 1,$

$f(w_1') = 7,$

$f(u_i) = 2(i+1);\ 1 \le i \le n,$

$f(u_i') = 2(n+i+1);\ 1 \le i \le n,$

$$f(w_i) = 8i - 1;\ 2 \le i \le \left\lceil \frac{n}{2} \right\rceil$$

$$= 8(n-i) + 9;\ \left(\left\lceil \frac{n}{2} \right\rceil + 1\right) \le i \le n,$$

$$f(w_i') = 8i - 5,\ 2 \le i \le \left\lceil \frac{n}{2} \right\rceil$$

$$= 8(n - i) + 5;\ \left(\left\lceil \frac{n}{2} \right\rceil + 1\right) \le i \le n.$$

Case 3. $k \ge 3$.

In this case define f as:

$f(u_0) = 2,\ f(w_0) = 1,$

$f(w_1) = f(v_k) = 3,$

$f(w_1') = 7,$

$f\left(v_{\left\lceil \frac{k}{2} \right\rceil}\right) = 4,$

$f(u_i) = 2(i + 2);\ 1 \le i \le n,$

$f(u_i') = 2(n + i + 2);\ 1 \le i \le n,$

$$f(w_i) = 8i - 1;\ 2 \le i \le \left\lceil \frac{n}{2} \right\rceil$$

$$= 8(n - i) + 9;\ \left(\left\lceil \frac{n}{2} \right\rceil + 1\right) \le i \le n,$$

$$f(w_i') = 8i - 5,\ 2 \le i \le \left\lceil \frac{n}{2} \right\rceil$$

$$= 8(n - i) + 5;\ \left(\left\lceil \frac{n}{2} \right\rceil + 1\right) \le i \le n.$$

$$f(v_j) = 4n + 2(j + 1);\ 2 \le j \le \left\lfloor \frac{k}{2} \right\rfloor - 1,$$

$$= 4n + 2\left(j - \left\lfloor \frac{k}{2} \right\rfloor\right) + 1;\ \left\lceil \frac{k}{2} \right\rceil \le j \le k.$$

One can observe that in each case the labeling defined above satisfies the conditions of multiply divisor cordial labeling and the graph under consideration is a multiply divisor cordial graph.

Illustration 6. The multiply divisor cordial labeling of the graph obtained by joining two copies of Fl_6 by path P_7 is shown in Figure 6.

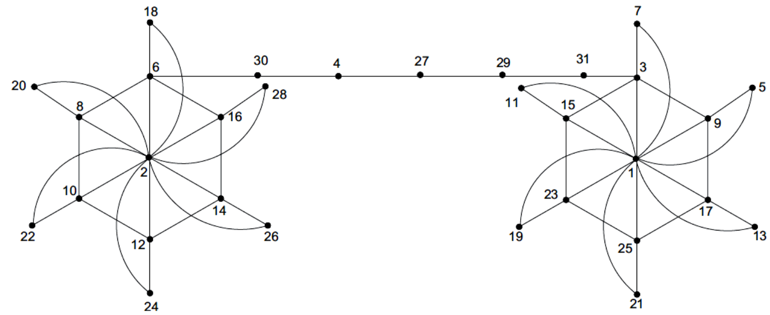

Figure 6. Join of two copies of Fl_6 by path P_7.

5. Conclusions

Multiply divisor cordial labelling is variant of divisor cordial labelling. Because all graphs do not allow multiply divisor cordial labelling, it is particularly fascinating to explore graphs or graph families that are multiply divisor cordial. This will give the study work in the area of graph labelling, number theory, and network algorithms in computer engineering a new level. Here, I looked through six novel graph families those allow multiple divisor cordial labelling.

Acknowledgement

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