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ON CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH POLYLOGARITHM FUNCTION

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Abstract

In this paper, we consider a new subclass $\mathcal{M}_c^{\delta}(\lambda, \alpha)$ of analytic functions involving an integral operator defined by polylogarithm function and obtain necessary and sufficient conditions for this class. Further, results on partial sums are investigated.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f defined on the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ with normalization $f(0) = f'(0) - 1 = 0$. Such a function has the Taylor series expansion about the origin in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \; z \in U \tag{1.1}$$

Denote by $\mathcal{S}$, the subclass of $\mathcal{A}$ consisting of functions that are univalent.

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Also, denote by T a subclass of $\mathcal{A}$ consisting functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ a_n \geq 0,\ z \in U \tag{1.2}$$

introduced and studied by Silverman [1].

For $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n; z \in U \tag{1.3}$$

Let $\Phi(a; z)$ denote the well-known generalisation of the Riemann zeta and polylogarithm functions or simply the δ^{th} order polylogarithm function given by

$$\Phi_{\delta}(c; z) = \sum_{k=1}^{\infty} \frac{z^k}{(k+c)^{\delta}} \tag{1.4}$$

where any term with $k + c = 0$ is excluded (see Lerch [2])

Using the definition of the Gamma function [[3], page 27], a simple transformation produces integral formula

$$\Phi_{\delta}(c; z) = \frac{1}{\Gamma(\delta)} \int_0^1 z \left(\log \frac{1}{t}\right)^{\delta-1} \frac{t^c}{1 - tz}\, dt \tag{1.5}$$

where $\mathrm{Re}(c) > -1$ and $\mathrm{Re}(\delta) > 1$.

More details about polylogarithm function can be seen in Ponnusamy [4] and Ponnusamy and Sabapathy [5].

Further, it is noted that $\Phi_{-1}(0; z) = \dfrac{z}{(1-z)^2}$ is Koebe function.

Now, for $f \in \mathcal{A}$ of the form (1.1), Al-Shaqsi [6] defined the following integral operator

$$\mathcal{J}_c^{\delta} f(z) = (1+c)^{\delta} \Phi_{\delta}(c, z) * f(z) = -\frac{(1+c)^{\delta}}{\Gamma(\delta)} \int_0^1 t^{c-1} \left(\log \frac{1}{t}\right)^{(\delta-1)} f(tz) dt \quad (1.6)$$

where $c > 0$, $\delta > 1$ and $z \in U$.

Also, in [6], Al-Shaqsi noted that the operator defined by (1.6) can be expressed by series expansion as below:

$$\mathcal{J}_c^{\delta} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+c}{k+c}\right)^{\delta} a_k z^k = z + \sum_{k=2}^{\infty} C_{k,\delta} a_k z^k \quad (1.7)$$

where

$$C_{k,\delta} = \left(\frac{1+c}{k+c}\right)^{\delta}. \quad (1.8)$$

We note that

$$z(\mathcal{J}_c^{\delta} f(z))' = (c+1)\mathcal{J}_c^{\delta-1} f(z) - c\mathcal{J}_c^{\delta} f(z)$$

and

$$z^2(\mathcal{J}_c^{\delta} f(z))'' = (c+1)^2 \mathcal{J}_c^{\delta-2} f(z) - (2c+1)(c+1)\mathcal{J}_c^{\delta-1} f(z) + c(c+1)\mathcal{J}_c^{\delta} f(z).$$

A class $UCD(\alpha)$, $\alpha \geq 0$ consisting of functions $f \in A$ satisfying

$$\operatorname{Re}[f'(z)] \geq \alpha |f''(z)|, \; z \in U$$

was introduced and investigated in [7].

Following the study of Rosy [8] and Sunil Verma et al. [9], we introduce a new subclass of $\mathcal{A}$ involving Al-Shaqsi operator [6] as below:

For $\alpha \geq 0$, $0 \leq \beta < 1$, $c > 0$, $\delta > 0$ let $\mathcal{M}_c^{\delta}(\lambda, \alpha)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) that satisfy the condition

$$\operatorname{Re}\left(\frac{\mathcal{J}_c^{\delta} f(z)}{z}\right) \geq \alpha \left| (\mathcal{J}_c^{\delta} f(z))' - \frac{\mathcal{J}_c^{\delta} f(z)}{z} \right| + \beta, \quad (1.9)$$

where $\mathcal{J}_{\mu, b} f(z)$ is given by (1.7).

We further let $T\mathcal{M}_c^\delta(\lambda, \alpha) = \mathcal{M}_c^\delta(\lambda, \alpha) \cap T$ For $\mu = 0; \beta = 0$, the class $\mathcal{M}_c^\delta(\lambda, \alpha)$ reduces to the class $SD(\alpha)$ studied by [9].

Motivated by the works of Sheil-Small [10], Silvia [11], Silverman [12], Owa et al. [13], Rosy et al. [14], Murugusundaramoorthy et al. [15], Soybas et al. [16], necessary and sufficient conditions are obtained for the class $\mathcal{M}_c^\delta(\lambda, \alpha)$. Further results on partial sums are investigated.

2. The Classes $\mathcal{M}_c^\delta(\lambda, \alpha)$ and $T\mathcal{M}_c^\delta(\lambda, \alpha)$

In this section, we obtain a sufficient condition for a function f given by (1.1) to be in the class $\mathcal{M}_c^\delta(\lambda, \alpha)$ and we prove that it is also a necessary condition for a function belonging to the class $T\mathcal{M}_c^\delta(\lambda, \alpha)$.

Theorem 2.1. *A function $f(z)$ be the form* (1.1) *is in $\mathcal{M}_c^\delta(\lambda, \alpha)$ if*

$$\sum_{k=2}^{\infty} [1 + \alpha(k-1)] C_{k,\delta} | a_k | \leq 1 - \beta, \tag{2.1}$$

where $\alpha \geq 0,\ 0 \leq \beta < 1$, *where* $C_{k,\delta}$ *is given by* (1.8).

Proof. Since $\alpha \geq 0,\ 0 \leq \beta < 1$. It suffices to show that

$$\alpha \left| (\mathcal{J}_c^\delta f(z))' - \frac{(\mathcal{J}_c^\delta f(z))}{z} \right| - \operatorname{Re}\left\{ \frac{(\mathcal{J}_c^\delta f(z))}{z} - 1 \right\} \leq 1 - \beta.$$

We have

$$\alpha \left| (\mathcal{J}_c^\delta f(z))' - \frac{(\mathcal{J}_c^\delta f(z))}{z} \right| - \operatorname{Re}\left\{ \frac{(\mathcal{J}_c^\delta f(z))}{z} - 1 \right\}$$

$$\leq \alpha \left| (\mathcal{J}_c^\delta f(z))' - \frac{(\mathcal{J}_c^\delta f(z))}{z} \right| + \left| \frac{(\mathcal{J}_c^\delta f(z))}{z} - 1 \right|$$

$$\leq \alpha \left| \frac{\sum_{k=2}^{\infty} (k-1) C_{k,\delta} a_k z^k}{z} \right| + \left| \frac{\sum_{k=2}^{\infty} C_{k,\delta} a_k z^k}{z} \right|$$

$$\leq \alpha \sum_{k=2}^{\infty} (k-1) C_{k,\delta} |a_k| + \sum_{k=2}^{\infty} C_{k,\delta} |a_k|$$

$$= \sum_{k=2}^{\infty} (1 + \alpha(k-1)) C_{k,\delta} |a_k|.$$

The last expression is bounded above by $(1 - \beta)$ if

$$\sum_{k=2}^{\infty} (1 + \alpha(k-1)) C_{k,\delta} |a_k| \leq 1 - \beta$$

and hence the proof. □

Theorem 2.2. *For* $\alpha \geq 0$, $0 \leq \beta < 1$, *a function* $f(z)$ *of the form* (1.2) *to be in the class* $T\mathcal{M}_c^{\delta}(\lambda, \alpha)$ *if and only if*

$$\sum_{k=2}^{\infty} [1 + \alpha(k-1)] C_{k,\delta} |a_k| \leq 1 - \beta$$

Proof. Suppose $f(z)$ of the form (1.2) is in the class $T\mathcal{M}_c^{\delta}(\lambda, \alpha)$. Then

$$\operatorname{Re}\left[\frac{T\mathcal{J}_c^{\delta} f(z)}{z}\right] - \alpha \left| (T\mathcal{J}_c^{\delta} f(z))' - \frac{(T\mathcal{J}_c^{\delta} f(z))}{z} \right| \geq \beta.$$

Equivalently,

$$\operatorname{Re}\left[1 - \sum_{k=2}^{\infty} C_{k,\delta} |a_k| z^{k-1}\right] - \alpha \left| \sum_{k=2}^{\infty} (k-1) C_{k,\delta} a_n z^{k-1} \right| \geq \beta.$$

Letting z to take real values and as $|z| \to 1$, we have

$$1 - \sum_{k=2}^{\infty} C_{k,\delta} |a_k| - \alpha \sum_{k=2}^{\infty} (k-1) C_{k,\delta} |a_n| \geq \beta,$$

which implies

$$\sum_{k=2}^{\infty} (1 + \alpha(k-1)) C_{k,\delta} |a_k| \leq 1 - \beta,$$

where $\alpha \geq 0$, $0 \leq \beta < 1$, $C_{k,\delta}$ is given by (1.8) and the sufficiency follows from Theorem 2.1. □

Corollary 2.1. *If* $f \in T\mathcal{M}_c^{\delta}(\lambda, \alpha)$, *then*

$$|a_k| \leq \frac{(1-\beta)}{(1+\alpha(k-1))C_{k,\delta}}, \tag{2.2}$$

where $k \geq 2$, $\alpha \geq 0$, $0 \leq \beta < 1$, $C_{k,\delta}$ *is given by* (1.8).

Equality holds for the function

$$f_k(z) = z - \frac{(1-\beta)}{(1+\alpha(k-1))C_{k,\delta}} z^k, \tag{2.3}$$

$\alpha \geq 0$, $0 \leq \beta < 1$, $C_{k,\delta}$ is given by (1.8).

3. Partial Sums of Functions in the Class $\mathcal{M}_c^{\delta}(\lambda, \alpha)$

For a function $f \in \mathcal{A}$ given by (1.1), Silverman [11] investigated the partial sums f_1 and fm defined by $f_1(z) = z$ and $f_m(z) = \sum_{k=2}^{m} a_k z^k$, $m = 2, 3, 4, \ldots$.

In this paper, we examine the ratio of the function of the form (1.7) to its sequence of partial sums $\mathcal{J}_c^{\delta} f_m(z) = z + \sum_{k=2}^{m} C_{k,\delta} a_k z^k$ when the coefficients of *f* are sufficiently small to satisfy the condition (2.1).

We determine sharp lower bounds for $\operatorname{Re}\left[\frac{\mathcal{J}_c^{\delta} f(z)}{\mathcal{J}_c^{\delta} f_m(z)}\right]$, $\operatorname{Re}\left[\frac{\mathcal{J}_c^{\delta} f_m(z)}{\mathcal{J}_c^{\delta} f(z)}\right]$, $\operatorname{Re}\left[\frac{\mathcal{J}_c^{\delta} f'(z)}{\mathcal{J}_c^{\delta} f'_m(z)}\right]$, $\operatorname{Re}\left[\frac{\mathcal{J}_c^{\delta} f'_m(z)}{\mathcal{J}_c^{\delta} f'(z)}\right]$. In the sequel we make frequent use of the well known result that $\operatorname{Re}\left[\frac{1+w(z)}{1-w(z)}\right] > 0$, $Z \in U$ if and only if $w(z) = \sum_{k=1}^{\infty} c_k z^k$ satisfy the inequality $|w(z)| \leq |z|$.

Theorem 3.1. *Let* $f(z)$ *of the form* (1.1) *belong to the class* $\mathcal{M}_c^{\delta}(\lambda, \alpha)$ *and* *satisfy* (2.1). *Then*

$$\operatorname{Re}\left[\frac{\mathcal{J}_c^{\delta} f(z)}{\mathcal{J}_c^{\delta} f_m(z)}\right] \geq 1 - \frac{1}{d_{m+1}},\ z \in U,\ m \in N \tag{3.1}$$

and

$$\operatorname{Re}\left[\frac{\mathcal{J}_c^{\delta} f_m(z)}{\mathcal{J}_c^{\delta} f(z)}\right] \geq \frac{d_{m+1}}{1 + d_{m+1}},\ z \in U,\ m \in N \tag{3.2}$$

where $d_k = \dfrac{(1 + \alpha(k-1))}{1 - \beta}$

The estimates in (3.1) *and* (3.2) *are sharp for every* m *with extremal function* $f(z) = z + \dfrac{1}{d_{m+1}} z^{m+1}$.

Proof. Clearly $d_{k+1} > d_k > 1,\ k = 2, 3, 4, \ldots$

Therefore, we have

$$\sum_{k=2}^{m} C_{k,\delta} |a_k| + d_{m+1} \sum_{k=2}^{m} C_{k,\delta} |a_k| \leq \sum_{k=2}^{m} d_k C_{k,\delta} |a_k| \leqq 1$$

Consider

$$g(z) = d_{m+1}\left[\frac{\mathcal{J}_c^{\delta} f(z)}{\mathcal{J}_c^{\delta} f_m(z)} - \left(1 - \frac{1}{d_{m+1}}\right)\right]$$

$$= \frac{1 + \sum_{k=2}^{m} C_{k,\delta} a_k z^{k-1} + d_{m+1} \sum_{k=2}^{m} C_{k,\delta} a_k z^{k-1}}{1 + \sum_{k=2}^{m} C_{k,\delta} a_k z^{k-1}} = \frac{1 + A(z)}{1 + B(z)}.$$

Set $\dfrac{1 + A(z)}{1 + B(z)} = \dfrac{1 + w(z)}{1 - w(z)}$, so that

$$w(z) = \frac{A(z) - B(z)}{2 + A(z) + B(z)}$$

$$= \frac{d_{m+1}\sum_{k=m+1}^{\infty} C_{k,\,\delta} a_k z^{k-1}}{2 + 2\sum_{k=2}^{m} C_{k,\delta} a_k z^{k-1} + d_{m+1}\sum_{k=m+1}^{\infty} C_{k,\delta} a_k z^{k-1}}$$

and $w(0) = 0$.

$$\text{Now, } |\, w(z)\,| \le \frac{d_{m+1}\sum_{k=m+1}^{\infty} C_{k,\delta} |\, a_k\,|}{2 - 2\sum_{k=2}^{m} C_{k,\delta} |\, a_k\,| - d_{m+1}\sum_{k=m+1}^{\infty} C_{k,\delta} |\, a_k\,|}$$

if

$$\sum_{k=2}^{m} C_{k,\delta} |\, a_k\,| + d_{m+1} \sum_{k=m+1}^{m} C_{k,\delta} |\, a_k\,| \le 1. \tag{3.3}$$

The LHS of (3.3) is bounded above by $\sum_{k=2}^{\infty} \frac{(1 + \alpha(k-1))}{(1-\beta)} C_{k,\,\delta} |\, a_k\,| \ge 0.$

$$\sum_{k=2}^{m} (d_k - 1) C_{k,\,\delta} |\, a_k\,| + \sum_{k=m+1}^{m} (d_k - d_{m+1}) C_{k,\,\delta} |\, a_k\,| \ge 0.$$

The above inequality holds because d_k is a non-decreasing sequence.

To see that the function $f(z) = z + \frac{1}{d_{m+1}} z^{m+1}$ gives the sharp result, we observe that for $z = re^{\frac{i\pi}{m}}$,

$$\frac{\mathcal{J}_c^{\delta} f(z)}{\mathcal{J}_c^{\delta} f_m(z)} = 1 + \frac{z^m}{d_{m+1}} \to 1 - \frac{1}{d_{m+1}} \text{ when } r \to 1^-,$$

and this completes the proof of (3.1). Similarly, if we set

$$h(z) = (1 + d_{m+1}) \left[\frac{\mathcal{J}_c^{\delta} f_m(z)}{\mathcal{J}_c^{\delta} f(z)} - \left(\frac{d_{M+1}}{1 + d_{m+1}} \right) \right]$$

the proof of (3.2) is similar to that of (3.1), and hence is omitted. □

Theorem 3.2. *Let* $f(z)$ *of the form* (1.1) *belong to the class* $\mathcal{M}_c^{\delta}(\lambda, \alpha)$ *and* *satisfy* (2.1). *Then*

$$\operatorname{Re}\left[\frac{\mathcal{J}_c^{\delta} f'(z)}{\mathcal{J}_c^{\delta} f_m'(z)}\right] \geq 1 - \frac{m+1}{d_{m+1}},\ z \in U,\ m \in N \tag{3.4}$$

and

$$\operatorname{Re}\left[\frac{\mathcal{J}_c^{\delta} f_m'(z)}{\mathcal{J}_c^{\delta} f'(z)}\right] \geq \frac{d_{m+1}}{m+1+d_{m+1}},\ z \in U,\ m \in N \tag{3.5}$$

where $d_k = \dfrac{(1+\alpha(k-1))}{1-\beta}$

The estimates in (3.4) *and* (3.5) *are sharp for every* m *with extremal function* $f(z) = z + \dfrac{1}{d_{m+1}} z^{m+1}$.

Proof. Clearly $d_{k+1} > d_k > 1,\ k = 2, 3, 4, \ldots$

Therefore, we have

$$\sum_{k=2}^{m} C_{k,\delta}|\,a_k\,| + d_{m+1}\sum_{k=2}^{m} C_{k,\delta}|\,a_k\,| \leq \sum_{k=2}^{m} d_k C_{k,\delta}|\,a_k\,| \leqq 1.$$

By setting,

$$g(z) = \frac{d_{m+1}}{m+1}\left[\frac{\mathcal{J}_c^{\delta} f'(z)}{\mathcal{J}_c^{\delta} f_m'(z)} - \left(1 - \frac{m+1}{d_{m+1}}\right)\right];\ z \in U$$

and

$$h(z) = (m+1+d_{m+1})\left[\frac{\mathcal{J}_c^{\delta} f_m'(z)}{\mathcal{J}_c^{\delta} f'(z)} - \left(\frac{d_{m+1}}{m+1+d_{m+1}}\right)\right];\ z \in U,$$

the proof is similar to that of Theorem 3.1 and hence we omit the details. □

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