Advances and Applications in Mathematical Sciences
Volume 23, Issue 4, February 2024, Pages 339-352

AN APPROXIMATE ANALYTICAL SOLUTION OF THE GENERALIZED ABEL INTEGRAL EQUATION OF SECOND KIND IN COUPLING OF VARIOUS TRANSFORM METHODS

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Abstract

In this paper, an approximate analytical solution of the generalized Abel integral equation of the second kind via Sumudu, Elzaki, Mohand and Sawi transform method using the homotopy perturbation transform method is introduced. This integral equation appears in the modeling of the numerous models in various fields of physical and applied sciences. The approach of the method is very simple and illustrates the accuracy, validity and stability of the numerical solution obtained by the proposed method in form of an exact solution.

1. Introduction

Abel integral equation [1] is one of the integral equations which is derived directly from a concrete problem of physics, without passing through a differential equation. This equation appears in several models in astrophysics, solid mechanics, physical sciences and applied sciences. Zeilon N. [2] in 1924, gave an idea of the solution of Abel integral equation on a finite segment. The various methods for solving Abel integral equation and

2020 Mathematics Subject Classification: 45E10, 44A05, 26A33.
Keywords: Generalized Abel integral equation, Sumudo transform, Elzaki transform, Mohand transform, Sawi transform, homotopy perturbation transform method.
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Received February 25, 2024; Accepted February 28, 2024

fractional differential equations are given in [4-5]. He [6-7] developed the homotopy perturbation transform method. Numerical solutions of Abel integral equation are given in [8, 28, 29] by using the different Wavelet methods. The homotopy perturbation transform method is used in [9-11, 19]. Bernstein polynomial method is discussed in [14]. Various transform methods are used in [3, 12, 13, 18, 21-24, 27]. Analytical solutions are given in [16, 17]. Fractional calculus is used in [20]. The Laplace decomposition method is discussed in [15]. The Taylor-Collocation method is used in [25]. The solution of Abel integral using the differential transform method is given in [26].

The main purpose of this paper is to produce an approximate analytical solution for the generalized Abel integral equation of the second kind in coupling of various transform methods with the homotopy perturbation transform method. The present method is coupling of homotopy perturbation transformation method with the Sumudo, Elzaki, Mohand and Sawi transform method on generalized Abel integral equation of the second kind.

2. Basic Definitions and Terminologies

Definition 2.1. The Sumudu transform [3, 12] over the set of functions $\{f(t) : \exists M > 0$ (finite), $\sigma_1, \sigma_2 > 0$ such that $|f(t)| < Me^{\frac{|t|}{\sigma_i}}$ if $t \in (-1)^i \times [0, \infty)\}$ is defined by the improper integral

$$S[f(t)] = \int_0^\infty e^{-t} f(st)dt; s \in (-\sigma_1, \sigma_2) \tag{1}$$

where S is the Sumudo operator.

In other words, the Sumudu transform of $f(t)$ can be defined as

$$\mathbb{S}(s) = \frac{1}{s}\int_0^\infty e^{\frac{-t}{s}} f(st)dt; s \in (-\sigma_1, \sigma_2)$$

Definition 2.2. The Elzaki transform [13, 18] over the set of functions $\{f(t) : \exists M > 0$ (finite), $\sigma_1, \sigma_2 > 0$ such that $|f(t)| < Me^{\frac{|t|}{\sigma_i}}$ if

$t \in (-1)^i \times [0, \infty)\}$ is defined by the improper integral

$$E[f(t)] = \mathcal{E}(s) = s\int_0^\infty e^{\frac{-t}{s}} f(t)dt; s \in (\sigma_1, \sigma_2) \tag{2}$$

where E is the Elzaki operator.

Definition 2.3. The Mohand transform [24] over the set of functions $\{f(t) : \exists M > 0$ (finite), $\sigma_1, \sigma_2 > 0$ such that $|f(t)| < Me^{\frac{|t|}{\sigma_i}}$ if $t \in (-1)^i \times [0, \infty)\}$ is defined by the improper integral

$$E[f(t)] = \mathcal{M}(s) = s^2\int_0^\infty e^{-st} f(t)dt; s \in [\sigma_1, \sigma_2] \tag{3}$$

where M is the Mohand operator.

Definition 2.4. The Sawi transform [27] over the set of functions $\{f(t) : \exists M > 0$ (finite), $\sigma_1, \sigma_2 > 0$ such that $|f(t)| < Me^{\frac{|t|}{\sigma_i}}$ if $t \in (-1)^i \times [0, \infty)\}$ is defined by the improper integral

$$S_w[f(t)] = \mathcal{S}_w(s) = \frac{1}{s^2}\int_0^\infty e^{\frac{-t}{s}} f(t)dt; s \in [\sigma_1, \sigma_2] \tag{4}$$

where S_w is the Sawi operator.

3. Description for an Approximate Analytical Solution for the Generalized Abel Integral Equation of the Second Kind

To demonstrate an idea of the analytical solution, we consider the following generalized Abel integral equation of the second kind,

$$f(t) = g(t) + \int_0^t \frac{f(u)}{(t-u)^\alpha} du; 0 \le t \le 1 \text{ and } 0 < \alpha < 1. \tag{5}$$

3.1. Coupling of Sumudu transform and homotopy perturbation transform method on the generalized Abel integral equation of the second kind

Operating the Sumudu transform on both sides of equation (5), we have

$$S[f(t)] = S[g(t)] + S\left\{\int_0^t \frac{f(u)}{(t-u)^{\alpha}} du\right\}. \tag{6}$$

By using convolution property of Sumudu transform in equation (6), we get

$$S[f(t)] = S[g(t)] + sS[f(t)]S[t^{-\alpha}].$$

$$= S[g(t)] + sS[f(t)]\Gamma(1-\alpha)s^{-\alpha}$$

$$\Rightarrow S[f(t)] = S[g(t)] + S[f(t)]\Gamma(1-\alpha)s^{1-\alpha}. \tag{7}$$

On operating the inverse Sumudu transform in equation (7), we have

$$f(t) = g(t) + S^{-1}\{\Gamma(1-\alpha)s^{1-\alpha}S[f(t)]\}. \tag{8}$$

Assume solution of the generalized Abel integral equation of second kind (5) in the series form as $\varphi(t) = \sum_{n=0}^{\infty} p^n \varphi_n(t)$, where $\varphi_n(t)$ are to be determined by the iterative scheme of HPTM.

Consider the following convex homotopy in order to solve the equation (5),

$$\sum_{n=0}^{\infty} p^n \varphi_n(t) = g(t) + p\left[S^{-1}\left\{\Gamma(1-\alpha)s^{1-\alpha}S\left(\sum_{n=0}^{\infty} p^n \varphi_n(t)\right)\right\}\right]. \tag{9}$$

This is a combination of the Sumudu transform and homotopy perturbation transform method. On equating the coefficients of same powers of p in (9) we obtained the following approximations;

$$\varphi_0(t) = g(t),\ \varphi_n(t) = S^{-1}\{\Gamma(1-\alpha)s^{1-\alpha}S[\varphi_{n-1}(t)]\};\ n \in \mathbb{N}. \tag{10}$$

The approximate analytical solution of the equation (5) is given by

$$f(t) = \lim_{p\to 1} \varphi(t) = \sum_{n=0}^{\infty} \varphi_n(t). \tag{11}$$

3.2. Coupling of Elzaki transform and homotopy perturbation transform method on the generalized Abel integral equation of the second kind

Operating the Elzaki transform on both sides of equation (5), we have

$$E[f(t)] = E[g(t)] + E\left\{\int_0^t \frac{f(u)}{(t-u)^{\alpha}}\,du\right\}. \tag{12}$$

By using convolution property of Elzaki transform in equation (12), we get

$$E[f(t)] = E[g(t)] + \frac{1}{s}E[f(t)]E[t^{-\alpha}]$$

$$= E[g(t)] + \frac{1}{s}E[f(t)]\Gamma(1-\alpha)s^{2-\alpha}.$$

$$\Rightarrow E[f(t)] = E[g(t)] + E[f(t)]\Gamma(1-\alpha)s^{1-\alpha}. \tag{13}$$

On operating the inverse Elzaki transform in equation (13), we have

$$f(t) = g(t) + E^{-1}\{\Gamma(1-\alpha)s^{1-\alpha}E[f(t)]\}. \tag{14}$$

Assume the solution of generalized Abel integral equation of second kind (5) in the series form as $\varphi(t) = \sum_{n=0}^{\infty} p^n \varphi_n(t)$, where $\varphi_n(t)$ are to be determined by the iterative scheme of HPTM. Consider the following convex homotopy in order to solve the equation (5),

$$\sum_{n=0}^{\infty} p^n \varphi_n(t) = g(t) + p\left[E^{-1}\left\{\Gamma(1-\alpha)s^{1-\alpha}E\left(\sum_{n=0}^{\infty} p^n \varphi_n(t)\right)\right\}\right]. \tag{15}$$

This is a combination of the Elzaki transform and homotopy perturbation transform method. On equating the coefficients of same powers of p in (15) we obtained the following approximations;

$$\varphi_0(t) = g(t),\ \varphi_n(t) = E^{-1}\{\Gamma(1-\alpha)s^{1-\alpha}E[\varphi_{n-1}(t)]\};\ n \in \mathbb{N}. \tag{16}$$

The approximate analytical solution of the equation (5) is given by

$$f(t) = \lim_{p\to 1}\varphi(t) = \sum_{n=0}^{\infty}\varphi_n(t). \tag{17}$$

3.3. Coupling of Mohand transform and homotopy perturbation transform method on the generalized Abel integral equation of the second kind

Operating the Mohand transform on both sides of equation (5), we have

$$M[f(t)] = M[g(t)] + M\left\{\int_0^t \frac{f(u)}{(t-u)^\alpha}\,du\right\}. \tag{18}$$

By using convolution property of Mohand transform in equation (18), we get

$$M[f(t)] = M[g(t)] + \frac{1}{s^2} M[f(t)]M[t^{-\alpha}].$$

$$= M[g(t)] + \frac{1}{s^2} M[f(t)]\Gamma(1-\alpha)s^{\alpha+1}$$

$$\Rightarrow M[f(t)] = M[g(t)] + M[f(t)]\Gamma(1-\alpha)s^{\alpha-1}. \tag{19}$$

On operating the inverse Mohand transform in equation (19), we have

$$f(t) = g(t) + M^{-1}\{\Gamma(1-\alpha)s^{\alpha-1}M[f(t)]\}. \tag{20}$$

Assume the solution of generalized Abel integral equation of second kind (5) in the series form as $\varphi(t) = \sum_{n=0}^{\infty} p^n \varphi_n(t)$, where $\varphi_n(t)$ are to be determined by the iterative scheme of HPTM.

Consider the following convex homotopy in order to solve the equation (5),

$$\sum_{n=0}^{\infty} p^n \varphi_n(t) = g(t) + p\left[M^{-1}\left\{\Gamma(1-\alpha)s^{1-\alpha}M\left(\sum_{n=0}^{\infty} p^n \varphi_n(t)\right)\right\}\right]. \tag{21}$$

This is a combination of the Mohand transform and homotopy perturbation transform method. On equating the coefficients of same powers of p in (21) we obtained the following approximations:

$$\varphi_0(t) = g(t),\ \varphi_n(t) = M^{-1}\{\Gamma(1-\alpha)s^{\alpha-1}M[\varphi_{n-1}(t)]\};\ n \in \mathbb{N}. \tag{22}$$

The approximate analytical solution of the equation (5) is given by

$$f(t) = \lim_{p\to 1} \varphi(t) = \sum_{n=0}^{\infty} \varphi_n(t). \tag{23}$$

3.4. Coupling of Sawi transform and homotopy perturbation transform method on the generalized Abel integral equation of the second kind

Operating the Sawi transform on both sides of equation (5), we have

$$S_w[f(t)] = S_w[g(t)] + S_w\left\{\int_0^t \frac{f(u)}{(t-u)^{\alpha}}\,du\right\}. \tag{24}$$

By using convolution property of the Sawi transform in, equation (24), we get

$$S_w[f(t)] = S_w[g(t)] + s^2 S_w[f(t)] S_w[t^{-\alpha}].$$

$$= S_w[g(t)] + s^2 S_w[f(t)]\Gamma(1-\alpha)s^{-1-\alpha}$$

$$\Rightarrow S_w[f(t)] = S_w[g(t)] + S_w[f(t)]\Gamma(1-\alpha)s^{1-\alpha}. \tag{25}$$

On operating the inverse Sawi transform in equation (25), we have

$$f(t) = g(t) + S_w^{-1}\{\Gamma(1-\alpha)s^{1-\alpha}S_w[f(t)]\}. \tag{26}$$

Assume the solution of generalized Abel integral equation of second kind (5) in the series form as $\varphi(t) = \sum_{n=0}^{\infty} p^n \varphi_n(t)$, where $\varphi_n(t)$ are to be determined by the iterative scheme of HPTM.

Consider the following convex homotopy in order to solve the equation (5),

$$\sum_{n=0}^{\infty} p^n \varphi_n(t) = g(t) + p\left[S_w^{-1}\left\{\Gamma(1-\alpha)s^{1-\alpha}S_w\left(\sum_{n=0}^{\infty} p^n \varphi_n(t)\right)\right\}\right]. \tag{27}$$

This is a combination of the Sawi transform and homotopy perturbation transform method. On equating the coefficients of same powers of p in (27) we obtained the following approximations;

$$\varphi_0(t) = g(t),\ \varphi_n(t) = S_w^{-1}\{\Gamma(1-\alpha)s^{1-\alpha}S_w[\varphi_{n-1}(t)]\};\ n \in \mathbb{N}. \tag{28}$$

The approximate analytical solution of the equation (5) is given by

$$f(t) = \lim_{p\to 1}\varphi(t) = \sum_{n=0}^{\infty} \varphi_n(t). \tag{29}$$

4. The Duality Relations of the Sumudo Transform with Elzaki, Mohand and Sawi Transforms

From equations (1) and (3), we have

$$s\mathcal{S}(s) = \frac{1}{s}\mathcal{E}(s) \Rightarrow \mathcal{S}(s) = \frac{1}{s^2}\mathcal{E}(s). \tag{30}$$

and from equations (1) and (4), we have

$$\mathcal{M}\left(\frac{1}{s}\right) = \frac{1}{s^2}\int_0^\infty e^{\frac{-t}{s}} f(t)dt = \frac{1}{s}\mathcal{S}(s) \Rightarrow \mathcal{S}(s) = s\mathcal{M}\left(\frac{1}{s}\right). \tag{31}$$

and so, from equations (1) and (5), we have

$$\mathcal{S}_w(s) = \frac{1}{s^2}\int_0^\infty e^{\frac{-t}{s}} f(t)dt = \frac{1}{s}\mathcal{S}(s) \Rightarrow \mathcal{S}(s) = s\mathcal{S}_w(s). \tag{32}$$

5. Numerical Implementation of the Method

The duality relations of the Sumudo transform with Elzaki, Mohand and Sawi transform implies that the solution obtained by coupling of HPTM and Sumudo transform method is same as by Elzaki, Mohand and Sawi transform method and so we prefer the Sumudo transform method.

Example 5.1. Consider the generalized Abel integral equation of second kind as follows [25]

$$f(t) = 1 - 2t - \frac{32}{21}t^{\frac{7}{4}} + \frac{4}{3}t^{\frac{3}{4}} - \int_0^t \frac{f(u)}{(t-u)^{\frac{1}{4}}}du; \text{ with exact solution } 1 - 2t.$$

By using the convex homotopy perturbation transform method, we have

$$\sum_{n=0}^{\infty} p^n \varphi_n(t) = 1 - 2t - \frac{32}{21}t^{\frac{7}{4}} + \frac{4}{3}t^{\frac{3}{4}} - p\left[S^{-1}\left\{\Gamma\left(1-\frac{1}{4}\right)s^{\frac{3}{4}}S\left(\sum_{n=0}^{\infty} p^n\varphi_n(t)\right)\right\}\right]$$

$$= 1 - 2t - \frac{32}{21}t^{\frac{7}{4}} + \frac{4}{3}t^{\frac{3}{4}} - p\left[S^{-1}\left\{\Gamma\left(\frac{3}{4}\right)s^{\frac{3}{4}}S\left(\sum_{n=0}^{\infty} p^n\varphi_n(t)\right)\right\}\right].$$

On equating the coefficients of corresponding powers of p on both sides in above equation, we have

$$\varphi_0(t) = 1 - 2t - \frac{32}{21}t^{\frac{7}{4}} + \frac{4}{3}t^{\frac{3}{4}},$$

$$\varphi_1(t) = -S^{-1}\left\{\Gamma\left(\frac{3}{4}\right)s^{\frac{3}{4}}S(\varphi_0(t))\right\}$$

$$= -\frac{4}{3}t^{\frac{3}{4}} + \frac{32}{21}t^{\frac{7}{4}} + \frac{16}{15\sqrt{\pi}}\left[\Gamma\left(\frac{3}{4}\right)\right]^2 t^{\frac{5}{2}} - \frac{4}{3\sqrt{\pi}}\left[\Gamma\left(\frac{3}{4}\right)\right]^2 t^{\frac{3}{2}},$$

$$\varphi_2(t) = -S^{-1}\left\{\Gamma\left(\frac{3}{4}\right)s^{\frac{3}{4}}S(\varphi_1(t))\right\}$$

$$= \frac{4}{3\sqrt{\pi}}\left[\Gamma\left(\frac{3}{4}\right)\right]^2 t^{\frac{3}{2}} - \frac{16}{15\sqrt{\pi}}\left[\Gamma\left(\frac{3}{4}\right)\right]^2 t^{\frac{5}{2}} - \frac{512}{585}\frac{\left[\Gamma\left(\frac{3}{4}\right)\right]^3}{\Gamma\left(\frac{1}{4}\right)}t^{\frac{13}{4}} + \frac{64}{45}\frac{\left[\Gamma\left(\frac{3}{4}\right)\right]^3}{\Gamma\left(\frac{1}{4}\right)}t^{\frac{9}{4}},$$

$$\varphi_3(t) = -S^{-1}\left\{\Gamma\left(\frac{3}{4}\right)s^{\frac{3}{4}}S(\varphi_2(t))\right\}$$

$$= -\frac{64}{45}\frac{\left[\Gamma\left(\frac{3}{4}\right)\right]^3}{\Gamma\left(\frac{1}{4}\right)}t^{\frac{9}{4}} + \frac{512}{585}\frac{\left[\Gamma\left(\frac{3}{4}\right)\right]^3}{\Gamma\left(\frac{1}{4}\right)}t^{\frac{13}{4}} + \frac{1}{12}\left[\Gamma\left(\frac{3}{4}\right)\right]^4 t^4 - \frac{1}{6}\left[\Gamma\left(\frac{3}{4}\right)\right]^4 t^3, \ldots\ldots\ldots$$

Finally, we approximate the solution $f(t)$ using the truncated series as

$f(t)$

$$= \sum_{n=0}^{\infty}\varphi_n(t) = (1 - 2t - \frac{32}{21}t^{\frac{7}{4}} + \frac{4}{3}t^{\frac{3}{4}}) + (-\frac{4}{3}t^{\frac{3}{4}} + \frac{32}{21}t^{\frac{7}{4}} + \frac{16}{15\sqrt{\pi}}\left[\Gamma\left(\frac{3}{4}\right)\right]^2 t^{\frac{5}{2}}$$

$$- \frac{4}{3\sqrt{\pi}}\left[\Gamma\left(\frac{3}{4}\right)\right]^2 t^{\frac{3}{2}}) + (\frac{4}{3\sqrt{\pi}}\left[\Gamma\left(\frac{3}{4}\right)\right]^2 t^{\frac{3}{2}} - \frac{16}{15\sqrt{\pi}}\left[\Gamma\left(\frac{3}{4}\right)\right]^2 t^{\frac{5}{2}} - \frac{512}{585}\frac{\left[\Gamma\left(\frac{3}{4}\right)\right]^3}{\Gamma\left(\frac{1}{4}\right)}t^{\frac{13}{4}}$$

$$+\frac{64}{45}\frac{\left[\Gamma\left(\frac{3}{4}\right)\right]^3}{\Gamma\left(\frac{1}{4}\right)}t^{\frac{9}{4}}) + (-\frac{64}{45}\frac{\left[\Gamma\left(\frac{3}{4}\right)\right]^3}{\Gamma\left(\frac{1}{4}\right)}t^{\frac{9}{4}} + \frac{512}{585}\frac{\left[\Gamma\left(\frac{3}{4}\right)\right]^3}{\Gamma\left(\frac{1}{4}\right)}t^{\frac{13}{4}} + \frac{1}{12}\left[\Gamma\left(\frac{3}{4}\right)\right]^4 t^4$$

$$-\frac{1}{6}\left[\Gamma\left(\frac{3}{4}\right)\right]^4 t^3) + \ldots\ldots\ldots\ldots$$

Hence $f(t) \to 1 - 2t$ as $n \to \infty$, which is the exact solution.

Example 5.2. Consider the Abel integral equation of second kind as follows [9, 17, 19]

$$f(t) = t + \frac{4}{3}t^{\frac{3}{2}} - \int_0^t \frac{f(u)}{(t-u)^{\frac{1}{2}}}du,\ 0 \le t \le 1 \text{ with exact solution } t.$$

By using the homotopy perturbation transform method, we get

$$\sum_{n=0}^{\infty} p^n \varphi_n(t) = t + \frac{4}{3}t^{\frac{3}{2}} - p\left[S^{-1}\left\{\sqrt{\pi s}S\left(\sum_{n=0}^{\infty} p^n \varphi_n(t)\right)\right\}\right].$$

Now, equating the coefficients of corresponding powers of p on both sides in above equation, we have

$$\varphi_0(t) = t + \frac{4}{3}t^{\frac{3}{2}},\ \varphi_1(t) = -S^{-1}\{\sqrt{\pi s}S(\varphi_0(t)\} = -\frac{4}{3}\pi t^{\frac{3}{2}} - \frac{1}{2}\pi t^2,$$

$$\varphi_2(t) = -S^{-1}\{\sqrt{\pi s}S(\varphi_1(t)\} = \frac{1}{2}\pi t^2 + \frac{8}{15}\pi t^{\frac{5}{2}},\ \varphi_3(t) = -S^{-1}\{\sqrt{\pi s}S(\varphi_2(t)\}$$

$$= -\frac{8}{15}\pi t^{\frac{5}{2}} - \frac{1}{6}\pi^2 t^3,\ \varphi_4(t) = -S^{-1}\{\sqrt{\pi s}S(\varphi_3(t)\} = \frac{1}{6}\pi^2 t^3 + \frac{16}{105}\pi^2 t^{\frac{7}{2}},\ldots\ldots$$

Finally, we approximate the solution $f(t)$ using the truncated series as

$$f(t) = \sum_{n=0}^{\infty}\varphi_n(t) = (t + \frac{4}{3}t^{\frac{3}{2}}) - (\frac{4}{3}\pi t^{\frac{3}{2}} + \frac{1}{2}\pi t^2) + (\frac{1}{2}\pi t^2 + \frac{8}{15}\pi t^{\frac{5}{2}})$$

$$-\left(\frac{8}{15}\pi t^{\frac{5}{2}}+\frac{1}{6}\pi^2 t^3\right)+\left(\frac{1}{6}\pi^2 t^3+\frac{16}{105}\pi^2 t^{\frac{7}{2}}\right)+\ldots\ldots\ldots$$

Hence $f(t) \to t$ as $n \to \infty$, which is the exact solution.

Example 5.3. Consider the Abel integral equation of second kind as follows [9, 17, 19]

$$f(t)=2\sqrt{t}-\int_0^t \frac{f(u)}{(t-u)^{\frac{1}{2}}}du, \quad \text{with exact solution} \quad f(t)=1-e^{\pi t}erfc(\sqrt{\pi t})$$

and the complementary error function $erfc(t)$ is defined as $erfc(t)=\frac{2}{\sqrt{\pi}}\int_t^\infty e^{-u^2}du.$

By using the convex homotopy perturbation transform method, we have

$$\sum_{n=0}^{\infty} p^n \varphi_n(t)=2\sqrt{t}-p\left[S^{-1}\left\{\sqrt{\frac{\pi}{s}}S\left(\sum_{n=0}^{\infty} p^n\varphi_n(t)\right)\right\}\right].$$

On equating the coefficients of corresponding powers of p on both sides in above equation, we have

$$\varphi_0(t)=2\sqrt{t},\ \varphi_1(t)=-S^{-1}\left\{\sqrt{\frac{\pi}{s}}S(\varphi_0(t))\right\}=-\pi t,\ \varphi_2(t)=-S^{-1}\left\{\sqrt{\frac{\pi}{s}}S(\varphi_1(t))\right\}$$

$$=\frac{4}{3}\pi t^{\frac{3}{2}},\ \varphi_3(t)=-S^{-1}\left\{\sqrt{\frac{\pi}{s}}S(\varphi_2(t))\right\}=-\frac{1}{2}\pi^2 t^2,\ \varphi_4(t)=-S^{-1}\left\{\sqrt{\frac{\pi}{s}}S(\varphi_3(t))\right\}$$

$$\frac{8}{15}\pi^2 t^{\frac{5}{2}},\ \ldots\ldots\ldots\ldots$$

Finally, we approximate the solution $f(t)$ using the truncated series as

$$f(t)=\lim_{p\to 1}\varphi(t)=\sum_{n=0}^{\infty}\varphi_n(t)=2\sqrt{t}-\pi t+\frac{4}{3}\pi t^{\frac{3}{2}}-\frac{1}{2}\pi^2 t^2+\frac{8}{15}\pi^2 t^{\frac{5}{2}}+\ldots$$

$$=1-\left(1-2\sqrt{t}+\pi t-\frac{4}{3}\pi t^{\frac{3}{2}}+\frac{1}{2}\pi^2 t^2-\frac{8}{15}\pi^2 t^{\frac{5}{2}}+\ldots\ldots\ldots\right).$$

Hence $f(t) \to 1-e^{\pi t}erfc(\sqrt{\pi t})$ as $n \to \infty$, which is the exact solution.

Example 5.4. Consider the generalized Abel integral equation of the second kind as

$$f(t) = t^4 + t^2 - 2t^3 - \frac{729}{15400} t^{\frac{14}{3}} + \frac{243}{2200} t^{\frac{11}{3}} - \frac{27}{400} t^{\frac{8}{3}} + \frac{1}{10} \int_0^t \frac{f(u)}{(t-u)^{\frac{1}{3}}} du,$$

with solution $t^2 - 2t^3 + t^4$.

By using the convex homotopy perturbation transform method, we have

$$\sum_{n=0}^{\infty} p^n \varphi_n(t) = t^4 + t^2 - 2t^3 - \frac{729}{15400} t^{\frac{14}{3}} + \frac{243}{2200} t^{\frac{11}{3}} - \frac{27}{400} t^{\frac{8}{3}}$$

$$+ \frac{1}{10} p \left[S^{-1} \left\{ \Gamma\left(1 - \frac{1}{3}\right) s^{\frac{2}{3}} S\left(\sum_{n=0}^{\infty} p^n \varphi_n(t) \right) \right\} \right].$$

$$= t^4 + t^2 - 2t^3 - \frac{729}{15400} t^{\frac{14}{3}} + \frac{243}{2200} t^{\frac{11}{3}} - \frac{27}{400} t^{\frac{8}{3}}$$

$$+ \frac{1}{10} p \left[S^{-1} \left\{ \Gamma\left(1 - \frac{1}{3}\right) s^{\frac{2}{3}} S\left(\sum_{n=0}^{\infty} p^n \varphi_n(t) \right) \right\} \right].$$

We have,

$f(t) \to t^2 - 2t^3 + t^4$ as $n \to \infty$, which is the exact solution.

6. Conclusion

We have drafted an approximate analytical solution for the generalized Abel integral equation of second kind in coupling of various integral transform methods with homotopy perturbation transform method. Abel integral equation of second kind is a particular case of the equation (5), with $\alpha = \frac{1}{2}$. The approach of the method is very simple and illustrates the accuracy and stability of the solution obtained by the proposed method in form of the exact solution of the generalized Abel integral equation of second kind.

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