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BOUNDS ON THE COVERING RADIUS OF SIMPLEX CODE AND MACDONALD CODE IN *R*

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Abstract

In this paper, the covering radius of codes over $R = \mathbb{Z}_2\mathbb{R}$, where $\mathbb{R} = \mathbb{Z}_2 + v\mathbb{Z}_2, v^2 = v$ with different weight are discussed. The block repetition codes over R is defined and the covering radius for block repetition codes, simplex code and macdonald code of type α and type β in R are obtained.

1. Introduction

Codes over finite commutative rings have been studied for almost 50 years. The main motivation of studying codes over rings is that they can be associated with codes over finite fields through the Gray map. Recently, coding theory over finite commutative non-chain rings is a hot research topic. Recently, there has been substantial interest in the class of additive codes. In [15, 16], Delsarte contributes to the algebraic theory of association scheme where the main idea is to characterize the subgroups of the underlying abelian group in a given association scheme. The covering radius is an important geometric parameter of codes. It not only indicates the maximum error correcting capability of codes, but also relates to some practical problems such as the data compression and transmission. Studying of the covering radius of codes has attracted many coding scientists for almost 30 years. The covering radius of linear codes over binary finite fields was studied in [13].

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Additive codes over $\mathbb{Z}_2\mathbb{Z}_4$ have been extensively studied in [1, 3, 4, 5]. Enormous results were made available on the simplex codes over finite fields and finite rings. A few of them are [6, 8, 9, 19, 21]. In [7, 10, 11], the authors, in particular, gave lower and upper bounds on the covering radius of codes over the ring $\mathbb{Z}_2 + u\mathbb{Z}_2$ where $u^2 = 0$ with respect to different distance and they explained the covering radius of various repetition codes, Simplex Codes and Macdonald Codes (Type α and Type β) The above results motivate us to work in this Paper.

2. Preliminaries

In $\mathbb{Z}_2$ and $\mathbb{R} = \mathbb{Z}_2 + v\mathbb{Z}_2$, $v^2 = v$ be the rings of integers modulo 2 and let $\mathbb{Z}_n^2$ and $\mathbb{R}^n$ denote the space of n-tuples over these rings. A ring $R = \mathbb{Z}_2\mathbb{R} = \{00, 01, 0v, 01 + v, 10, 11, 1v, 11 + v\}$, where $\mathbb{R} = \{0, 1, v, 1 + v\}$, $v^2 = v$ with integer modulo is 2. If C be a non-empty subset C of $\mathbb{Z}_n^2$ is called a code and if that subcode is a linear space, then C is said to be linear code.

In this section, some preliminary results are given [3, 5]. A non-empty set C is a Radditive code if it is a subgroup of $\mathbb{Z}_2^\gamma \times \mathbb{R}^\delta$. In this case, C is also isomorphic to an abelian structure $\mathbb{Z}_2^\gamma \times \mathbb{R}^\delta$ for some γ and δ and type of C is a $2^\gamma \mathbb{R}^\mu$ as a group. It pursue that it has $|C| = 2^{\gamma+2\delta}$ codewords and the number of order for two codewords in C is $|C| = 2^{\gamma+\delta}$. The Gray *map* : $\mu : \mathbb{R} \to \mathbb{Z}_2^2$ is defined as $\mu(0) = (00), \mu(1) = (01), \mu(v) = (11)$ and $\mu(1 + v) = (10)$ and the extension of the Gray map $\delta : \mathbb{Z}_2^\gamma \times \mathbb{R}^\delta \to \mathbb{Z}_n^2$, $\delta(u, w) = (u, \mu(w_1), \ldots, \mu(w_\delta)), \forall\, u \in \mathbb{Z}_2^\gamma$ and $(w_1, \ldots, w_\delta) \in \mathbb{R}^\delta$, with $n = \gamma + 2\delta$. Then the binary image of a R-additive code under the extended Gray map is called a R-linear code of length $n = \gamma + 2\delta$. The Hamming weight of u denoted by $w_H(u)$ and $w_L(w)$ and $w_E(w)$ the Lee and Euclidean weights of w respectively, where $u \in \mathbb{Z}_2^\gamma$ and $w \in \mathbb{R}^\delta$ are defined as $w_L(x_i) = 0$ if $x_i = 0, 1$ if $x_i = 1, (1 + v)$ and 2 if $x_i = v$ and $w_E(x_i) = 0$ if $x_i = 0, 1$ if $x_i = 1, (1 + v)$ and 4 if $x_i = v$. The Lee weight and Euclidean weight of x is defined as $w_L(x) = w_H(u) + w_L(w)$ and $w_E(x) = w_H(u) + w_E(w)$, where $x = (u, w) \in \mathbb{Z}_2^{\gamma} \times \mathbb{R}^{\delta}$, and $u = (u_1, \ldots, u_{\gamma}) \in \mathbb{Z}_2^{\gamma}$ and $w = (w_1, \ldots, w_{\delta}) \in \mathbb{R}^{\delta}$. The Gray map defined above is an isometry which transforms the Lee distance defined over $\mathbb{Z}_2^{\gamma} \times \mathbb{R}^{\delta}$ to the Hamming distance defined over $\mathbb{Z}_n^2$, with $n = \gamma + 2\delta$. In [12], the Bachoc weight of x is defined as $w_B(x_i) = 0$ if $x_i = 0, 1$ if $x_i = 1$ and 2 if $x_i = v, (1 + v)$.

Therefore, the Bachoc weight of x as $wt_B(x) = wt_H(u) + wt_B(w)$, where $x = (u, w) \in \mathbb{Z}_2^{\gamma} \times \mathbb{R}^{\delta}$, and $u = (u_1, \ldots, u_{\gamma}) \in \mathbb{Z}_2^{\gamma}$ and $w = (w_1, \ldots, w_{\delta}) \in \mathbb{R}^{\delta}$. The Chinese Euclidean weight of x is given as $wt_{CE}(x_i) = 0$ if $x_i = 0, 2$ if $x_i = 1, (1 + v)$ and 4 if $x_i = v$ [20]. Define, $w_{CE}(x) = wt_H(u) + wt_{CE}(w)$, where $x = (u, w) \in \mathbb{Z}_2^{\gamma} \times \mathbb{R}^{\delta}$ and $u = (u_1, \ldots, u_{\gamma}) \in \mathbb{Z}_2^{\gamma}$ and $w = (w_1, \ldots, w_{\delta}) \in \mathbb{R}^{\delta}$. If $c_1, c_2 \in C$, be any two distinct codewords of D distance is defined as $d_D(C) = \min\{d_D(c_1, c_2) \mid c_1 - c_2 \neq 0$ and $c_1, c_2 \in C\}$. The minimum D weight of C is $d_D(C) = \min\{d_D(c_1, c_2) \mid c_1 - c_2 \neq 0$ and $c_1, c_2 \in C\}$. In C is a linear code C, thus $d_D(C) = \min\{w_D(c) \mid c \neq 0 \in C\}$. Therefore, $d_D(c_1, c_2) = w_D(c_1, c_2)$. Let $C \subseteq R^n$ is a linear code, where n is a length of code, the number of codewords N and the minimum D distance d_D is said to be an (n, N, d_D) code in R, where $D = \{Lee(L)$, Euclidean (E), Bachoc (B), Chinese Euclidean (CE)$\}$.

3. The Covering Radius of the Block Repetition Codes over R

The covering radius of a code C is the smallest number r such that the spheres of radius r around the codewords cover $\mathbb{Z}_2^{\gamma} \times \mathbb{R}^{\delta} = R$ and thus the covering radius of a code C over R with respect to the different distance, such as (Lee, Euclidean, Bachoc, Chinese Euclidean) is given $r_d(C) = \max_{u \in R}\{\min_{c \in C} d(u, c)\}$.

In $F_q = \{0, 1, \beta_2, \ldots, \beta_{q-1}\}$ is a finite field. Let C be a q-ary repetition code C over F_q. That is $C = \{\overline{\beta} = (\beta\beta\ldots\beta) \mid \beta \in F_q\}$ and the repetition code C is an $[n, 1, n]$ code. Therefore, the covering radius of the code C is $\left\lceil \frac{n(q-1)}{q} \right\rceil$ this true for binary repetition code. In [7, 10, 11], the authors studied for different classes of repetition codes over $\mathbb{Z}_2 + u\mathbb{Z}_2, u^2 = 0$ and their covering radius has been obtained. Now, generalize those results for codes over $R = \mathbb{Z}_2\mathbb{R}, v^2 = v$. Consider the repetition codes over R. For a fixed $1 \le i \le 7$. For all $1 \le j \ne i \le 7, n_j = 0$, then the code $C^n = C^{n_i}$ is denoted by C_i. Therefore, the seven basic repetition codes are the following table,

Generator Matrix	Code	Parameters- $[n, k(N), d_i(d_j)]_{i,j=D}$
$G_1 = \overbrace{[01\ldots01]}^{n_1(3)} = G_3$	$C_{1(3)} = \{c_0, c_1, c_2, c_3\}$	$(n_{1(3)}, 4, n, n, n, 2n)$
$G_2 = \overbrace{[0v\ldots0v]}^{n_2}$	$C_2 = \{c_0, c_2\}$	$(n_{1(3)}, 4, n, n, n, 2n)$
$G_4 = \overbrace{[10\ldots10]}^{n_4}$	$C_4 = \{c_0, c_4\}$	$(n_{1(3)}, 4, n, n, n, 2n)$
$G_5 = \overbrace{[11\ldots11]}^{n_5(7)} = G_7$	$C_{5(7)} = \{c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7\}$	$[n_{5(7)}, 8, n], d_i = n$

here $c_0 = (00\ldots00), c_1(01\ldots01), c_2 = (0v\ldots0v), c_3 = (01+v\ldots01+v)$, $c_4 = (10\ldots10), c_5 = (11\ldots11), c_6 = (1v\ldots1v), c_7 = (11+v\ldots11+v)$.

Theorem 3.1. *Let* $C_{j, 1\le j\le 7}$, *be a code in* R. *Then*, $\frac{n}{2} \le r_L(C_1) = r_L(C_3) \le 2n, \frac{n}{2} \le r_L(C_2) \le 2n, \frac{n}{4} \le r_L(C_4) \le 2n, \frac{3n}{4} \le r_L(C_5)$ $= r_L(C_7) \le \frac{3n}{2}, \frac{3n}{4} \le r_L(C_6) \le \frac{3n}{2}$, *where* $r_L(C_j)$ *is a covering radius of* $C_{j,1\le j\le 7}$ *with Lee distance.*

Proof. For $c \in C_{j,1\le j\le 7}$ be a codeword of code C_j in R. Let $t_i(c)_{0\le i\le 7}$ is the number of occurrences of symbol i in the codeword c. Let $x \in R^n$ by

$(t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7)$, where $\sum_{j=0}^{7} t_j = n$, then $d_D(x, \overline{00}) = n - t_0 + t_2 + t_5 + 2t_6 + t_7$, $d_L(x, \overline{01}) = n - t_1 + t_3 + t_4 + t_6 + 2t_7, t_7$, $d_L(x, \overline{0v}) = n - t_2 + t_0 + 2t_4 + t_5 + t_7$, $d_L(x, \overline{01+v}) = n - t_3 + t_1 + t_4 + 2t_5 + t_6$, $d_L(x, \overline{10}) = n - t_4 + t_1 + 2t_2 + t_3 + t_6$, $d_L(x, \overline{11}) = n - t_5 + t_0 + t_2 + 2t_3 + t_7$, $d_L(x, \overline{1v}) = n - t_6 + 2t_0 + t_1 + t_3 + t_4$, $d_L(x, \overline{11+v}) = n - t_7 + t_0 + 2t_1 + t_2 + t_5$. In Code, $C_1 = C_3 \in R$, therefore, $d_L(x, C_1) = d_L(x, C_3) = \min\{d_L(x, \overline{00}), d_L(x, \overline{01}), d_L(x, \overline{0v}), d_L(x, \overline{01+v})\} \leq 2n$, then $r_L(C_1) = r_L(C_3) \leq 2n$.

If $x = \overbrace{00\ldots00}^{\frac{n}{4}}\overbrace{01\ldots01}^{\frac{n}{4}}\overbrace{0v\ldots0v}^{\frac{n}{4}}\overbrace{01+v\ldots01+v}^{\frac{n}{4}} \in R^n$, then $d_L(x, \overline{00}) = d_L(x, \overline{01}) = d_L(x, \overline{0v}) = d_L(x, \overline{01+v}) = \frac{n}{2}$. Thus $r_L(C_1) = r_L(C_3) \geq \frac{n}{2}$ and so $\frac{n}{2} \leq r_L(C_1) = r_L(C_3) \leq 2n$. In Code, $C_2 \in R$, $d_L(x, C_2) = \min\{d_L(x, \overline{00}), d_L(x, \overline{0v})\} \leq 2n$. Then $r_L(C_2) \leq 2n$. If $x = \overbrace{00\ldots00}^{\frac{n}{2}}\overbrace{0v\ldots0v}^{\frac{n}{4}} \in R^n$, then $d_L(x, \overline{00}) = d_L(x, \overline{0v}) = 2\left(\frac{n}{4}\right) = \frac{n}{2}$. Thus $r_L(C_2) \leq \frac{n}{2}$ and so $\frac{n}{2} \leq r_L(C_2) \leq 2n$. The remaining part of proof is follows from the code C_1 and C_2 for they Codes C_4, C_5, C_6. □

Theorem 3.2. *In Euclidean weight for the code* $C_{j, 1 \leq j \leq 7}$, *prove the* $\frac{3n}{4} \leq r_E(C_1) = r_E(C_3) \leq 2n$, $n \leq r_E(C_2) \leq 3n$, $\frac{n}{4} \leq r_E(C_4) \leq 4n$, $n \leq r_E(C_5) = r_E(C_7) \leq 2n$, $\frac{5n}{4} \leq r_E(C_6) \leq \frac{5n}{2}$.

Proof. In Code $C_{i, i=1 \text{ to } 7}$ with Euclidean weight is apply to theorem 3.1. □

Theorem 3.3. *Show that,* $\frac{5n}{8} \leq r_B(C_1) = r_B(C_3) \leq 2n$, $\frac{n}{2} \leq r_B(C_2)$

$\leq 2n$, $\frac{n}{4} \leq r_B(C_4) \leq \frac{5n}{2}$, $\frac{7n}{8} \leq r_B(C_5) = r_B(C_7) \leq \frac{7n}{4}$ *and* $\frac{3n}{4} \leq r_B(C_6)$ $\leq \frac{7n}{2}$, *here* $r_B(C_j)$ *be a covering radius of code* $C_{j, 1 \leq j \leq 7}$ *with Bachoc weight.*

Proof. To apply theorem 3.1 for Code $C_{i, i=1 \text{ to } 7}$ with Bachoc weight. □

Theorem 3.4. *In Chinese Euclidean weight of code of* $C_{j, 1 \leq j \leq 7}$, *to find* $n \leq r_{CE}(C_1) = r_{CE}(C_3) \leq \frac{11n}{4}$, $n \leq r_{CE}(C_2) \leq \frac{5n}{2}$, $\frac{n}{4} \leq r_{CE}(C_4) \leq 4n$, $\frac{5n}{4}$ $\leq r_{CE}(C_5) = r_{CE}(C_7) \leq \frac{5n}{2}$ *and* $\frac{5n}{4} \leq r_{CE}(C_6) \leq \frac{5n}{2}$.

Proof. In Code $C_{i, i=1 \text{ to } 7}$ with Chinese Euclidean weight is apply to theorem 3.1. □

Block repetition code in *R*

The block repetition code C^n over R is a R-additive code.

Let $G = [\overbrace{01 \ldots 01}^{n_1} \overbrace{0v \ldots 0v}^{n_2} \overbrace{01+v \ldots 01+v}^{n_3} \overbrace{10 \ldots 10}^{n_4} \overbrace{11 \ldots 11}^{n_5} \overbrace{1v \ldots 1v}^{n_6}$ $\overbrace{11+v1 \ldots 11+v1}^{n_1}]$ be a generator matrix with the parameters of $C^n : [n = \sum_{j=1}^{7} n_j, 8, d_L = \min\{n_4 + n_5 + n_6 + n_7, n_1 + 2n_2 + n_3 + n_5 + 2n_6$ $+ n_7\}, d_E = \min\{n_4 + n_5 + n_6 + n_7, n_1 + 4n_2 + n_3 + n_5 + 4n_6 + n_7\}, d_B =$ $\min\{n_4 + n_5 + n_6 + n_7, n_1 + 2n_2 + 2n_3 + n_5 + 2n_6 + 2n_7\}, d_{CE} = \min\{n_4 + n_5$ $n_6 + n_7)\}]$.

Theorem 3.5. *Let* C^n *be the block repetition code in* R *with length is* n. *Then the covering radius of block repetition code is*

1. $r_L(C^{7n}) = 2n$, *if* $n_1 = \ldots = n_7 = n$.

2. $\frac{3(n_1 + n_3) + n_4 + 4(n_2 + n_5 + n_7) + 5n_6}{4} \leq r_E(C^n)$

$\leq \frac{5(n_1 + n_3 + n_6) + 3n_2 + 9n_4 + 4(n_5 + n_7)}{2}$,

3. $\frac{5(n_1 + n_3) + 4n_2 + 2n_4 + 7(n_5 + n_7) + 6n_6}{8} \leq r_B(C^n)$

$\leq \frac{18(n_1 + n_3 + n_6) + 17n_2 + 15(n_4 + n_5 + n_6 + n_7)}{8}$ *and*

4. $\frac{4(n_1 + +n_2 + n_3) + n_4 + 5(n_5 + n_6 + n_7) + 6n_6}{4} \leq r_{CE}(C^n)$

$\leq \frac{6(n_1 + n_2 + n_3) + 8n_5 + 5(n_5 + n_6 + n_7)}{2}$.

Proof. For the Code, that $\phi(C^{7n})$ is the set given by

{000 … 000000 … 000000 … 000000 … 000000 … 000000 … 000000 … 000,

001 … 001011 … 011010 … 010100 … 100101 … 101111 … 111110 … 110,

001 … 001011 … 011010 … 010000 … 000001 … 001011 … 011010 … 010,

011 … 011000 … 000011 … 011000 … 000011 … 011000 … 000011 … 011,

010 … 010011 … 011001 … 001000 … 000010 … 010011 … 011001 … 001,

000 … 000000 … 000000 … 000100 … 100100 … 100100 … 100100 … 100,

011 … 011000 … 000011 … 011100 … 100111 … 111100 … 100111 … 111,

010 … 010011 … 011001 … 001100 … 100110 … 110111 … 111101 … 101}.

By Proposition [2], give $r_L(C^{7n}) = r(\phi(C^{7n})) = 2n$. Using Proposition [13], Theorem 3.2, 3.3 and 3.4, thus

- $\frac{3(n_1 + n_3) + n_4 + 4(n_2 + n_5 + n_7) + 5n_6}{4} \leq r_E(C^n)$,
- $\frac{5(n_1 + n_3) + 4n_2 + 2n_4 + 7(n_5 + n_7) + 6n_6}{8} \leq r_B(C^n)$ and
- $\frac{4(n_1 + +n_2 + n_3) + n_4 + 5(n_5 + n_6 + n_7) + 6n_6}{4} \leq r_{CE}(C^n)$.

Let $x = x_1x_2x_3x_4x_5x_6x_7 \in R^n$ with $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ is $(a_i), (b_i), (c_i), (d_i), (e_i), (f_i), (g_i)_{i=0,1,2,3,4,5,6,7}$ respectively such that n_1

$= \sum_{j=0}^{7} a_j, n_2 = \sum_{j=0}^{7} b_j, n_3 = \sum_{j=0}^{7} c_j, n_4 = \sum_{j=0}^{7} d_j, n_5 = \sum_{j=0}^{7} e_j,$ $n_6 = \sum_{j=0}^{7} f_j, n_7 = \sum_{j=0}^{7} g_j$. Then $d_E(x, \overline{y0}) = n_1 - a_0 + 3a_2 + a_5 + 4a_6 + a_7 + n_2 - b_0 + 3b_2 + b_5 + 4b_6 + b_7 + n_3 - c_0 + 3c_2 + c_5 + 4c_6 + c_7 + n_4 - d_0 + 3d_2 + d_5 + 4d_6 + d_7 + n_5 - e_0 + 3e_2 + e_5 + 4e_6 + e_7 + n_6 - f_0 + 3f_2 + f_5 + 4f_6 + f_7 + n_7 - g_0 + 3g_2 + g_5 + 4g_6 + g_7$, where $\overline{y0} = \overbrace{00\ldots00}^{n_1}\overbrace{00\ldots00}^{n_2}\overbrace{00\ldots00}^{n_3}\overbrace{00\ldots00}^{n_4}\overbrace{00\ldots00}^{n_5}\overbrace{00\ldots00}^{n_6}\overbrace{00\ldots00}^{n_7}$, is the first vector of C^n, where $n = n_1$.

$d_E(x, \overline{y_1}) = n_1 - a_1 + 3a_3 + a_4 + a_6 + 4a_7 + n_2 - b_2 + 3b_0 + 4b_4 + b_5 + b_7 + n_3 - c_3 + 3c_1 + 4c_5 + c_6 + n_4 - d_4 + d_1 + 4d_2 + d_3 + 3d_6 + n_5 - e_5 + e_0 + e_2 + 4e_3 + 3e_7 + n_6 - f_6 + 4f_0 + f_1 + f_3 + 3f_4 + n_7 - g_7 + g_0 + 4g_1 + g_2 + 3g_5$, where $\overline{y_1} = \overbrace{01\ldots01}^{n_1}\overbrace{0v\ldots0v}^{n_2}\overbrace{01+v\ldots01+v}^{n_3}\overbrace{10\ldots10}^{n_4}\overbrace{11\ldots11}^{n_5}\overbrace{1v\ldots1v}^{n_6}\overbrace{11+v\ldots11+v}^{n_7}$, is the second vector of C^n, where $n = n_2$.

$d_E(x, \overline{y_2}) = n_1 - a_1 + 3a_3 + a_4 + a_6 + 4a_7 + n_2 - b_2 + 3b_0 + 4b_4 + b_5 + b_7 + n_3 - c_3 + 3c_1 + 4c_5 + c_6 + n_4 - d_0 + 3d_2 + d_5 + 4d_6 + d_7 + n_5 - e_1 + 3e_3 + e_4 + e_6 + 4e_7 + n_6 - f_2 + 3f_0 + 4f_4 + f_5 + f_7 + n_7 - g_3 + 3g_1 + g_4 + 4g_5 + g_6$, where $\overline{y_2} = \overbrace{01\ldots01}^{n_1}\overbrace{0v\ldots0v}^{n_2}\overbrace{01+v\ldots01+v}^{n_3}\overbrace{00\ldots00}^{n_4}\overbrace{01\ldots01}^{n_5}\overbrace{0v\ldots0v}^{n_6}\overbrace{01+v\ldots01+v}^{n_7}$, is the third vector of C^n, where $n = n_3$.

$d_E(x, \overline{y_3}) = n_1 - a_2 + 3a_0 + 4a_4 + a_5 + a_7 + n_2 - b_0 + 3b_2 + 4b_6 + b_5 + b_7 + n_3 - c_2 + 3c_0 + 4c_4 + c_5 + c_7 + n_4 - d_0 + 3d_2 + d_5 + 4d_6 + d_7 + n_5 - e_2 + 3e_0 + 4e_4 + e_5 + e_7 + n_6 - f_0 + 3f_2 + f_5 + 4f_6 + f_7 + n_7 - g_2 + 3g_0 + 4g_4 + g_5 + g_7$, where $\overline{y_3} = \overbrace{0v\ldots0v}^{n_1}\overbrace{00\ldots00}^{n_2}\overbrace{0v\ldots0v}^{n_3}\overbrace{00\ldots00}^{n_4}\overbrace{0v\ldots0v}^{n_5}\overbrace{00\ldots00}^{n_6}\overbrace{0v\ldots0v}^{n_7}$ is the fourth vector of C^n, where $n = n_4$.

$d_E(x, \overline{y_4}) = n_1 - a_3 + 3a_1 + a_4 + 4a_5 + a_6 + a_7 + n_2 - b_2 + 3b_0 + 4b_4$ $+b_5 + b_7 + n_3 - c_1 + 3c_3 + c_4 + c_6 + 4c_7 + n_4 - d_0 + 3d_2 + d_5 + 4d_6 + d_7 + n_5$ $-e_3 + 3e_1 + e_4 + 4e_5 + e_6 + n_6 - f_2 + 3f_0 + 4f_4 + f_5 + f_7 + n_7 - g_1 + 3g_3 + g_4$ $+g_6 + 4g_7$, where $\overline{y_4} = \overbrace{01+v\ldots01+v}^{n_1}\overbrace{0v\ldots0v}^{n_2}\overbrace{01\ldots01}^{n_3}\overbrace{00\ldots00}^{n_4}$ $\overbrace{01+v\ldots01+v}^{n_5}\overbrace{0v\ldots0v}^{n_6}\overbrace{01+v\ldots01+v}^{n_7}$, is the fifth vector of C^n, where $n = n_5$.

$d_E(x, \overline{y_5}) = n_1 - a_0 + 3a_2 + a_5 + 4a_6 + a_7 + n_2 - b_0 + 3b_2 + b_5 + 4b_6 + b_7$ $+n_3 - c_0 + 3c_2 + c_5 + 4c_6 + c_7 + n_4 - d_4 + d_1 + 4d_2 + d_3 + 3d_6 + n_5 - e_4 + e_1$ $+4e_2 + e_3 + 3e_6 + n_6 - f_4 + f_1 + 4f_2 + f_3 + 3f_6 + n_7 - g_4 + g_1 + 4g_2 + g_3$ $+3g_6$, where $\overline{y_5} = \overbrace{00\ldots00}^{n_1}\overbrace{00\ldots00}^{n_2}\overbrace{00\ldots00}^{n_3}\overbrace{10\ldots10}^{n_4}\overbrace{10\ldots10}^{n_5}\overbrace{10\ldots10}^{n_6}\overbrace{10\ldots10}^{n_7}$, is the sixth vector of C^n, where $n = n_6$.

$d_E(x, \overline{y_6}) = n_1 - a_2 + 3a_0 + 4a_4 + a_7 + n_2 - b_0 + 3b_2 + b_5 + 4b_6 + b_7$ $+n_3 - c_2 + 3c_0 + 4c_4 + c_5 + c_7 + n_4 - d_4 + d_1 + 4d_2 + d_3 + 3d_6 + n_5 - e_6 + 4e_0$ $+e_1 + e_3 + 3e_4 + n_6 - f_4 + f_1 + 4f_2 + f_3 + 3f_6 + n_7 - g_6 + 4g_0 + g_1 + g_3 + 3g_4$, where $\overline{y_6} = \overbrace{0v\ldots0v}^{n_1}\overbrace{00\ldots00}^{n_2}\overbrace{0v\ldots0v}^{n_3}\overbrace{10\ldots10}^{n_4}\overbrace{1v\ldots1v}^{n_5}\overbrace{10\ldots10}^{n_6}\overbrace{1v\ldots1v}^{n_7}$, is the seventh vector of C^n, where $n = n_7$.

$d_E(x, \overline{y_7}) = n_1 - a_3 + 3a_1 + a_4 + 4a_5 + a_6 + n_2 - b_2 + 3b_0 + 4b_4 + b_5 + b_7$ $+n_3 - c_1 + 3c_3 + c_4 + c_6 + 4c_7 + n_4 - d_4 + d_1 + 4d_2 + d_3 + 3d_6 + n_5 - e_7 + e_0$ $+4e_1 + e_2 + 3e_5 + n_6 - f_6 + 4f_0 + f_1 + f_3 + 3f_4 + n_7 - g_5 + g_0 + g_2 + 4g_3 + 3g_7$, where $\overline{y_7} = \overbrace{01+v\ldots01+v}^{n_1}\overbrace{0v\ldots0v}^{n_2}\overbrace{01\ldots01}^{n_3}\overbrace{10\ldots10}^{n_4}\overbrace{11+v\ldots11+v}^{n_5}\overbrace{1v\ldots1v}^{n_6}$ $\overbrace{11\ldots11}^{n_7}$, is the eighth vector of C^n, where $n = n_8$.

Hence, $r_E(C^n) \le \frac{1}{2}[5(n_1 + n_3 + n_6) + 3n_2 + 9n_4] + 2(n_5 + n_7)$. The remaining part of proof is pursue for part 2 with Bachoc and Chinese Euclidean distance.

4. Simplex Codes of type α and type β in *R*

In this section, consider the construction of simplex codes of type α and type β over *R*.

Let $m_{2,k}^{\alpha}$ be the generator matrix of $S_{2,k}^{\alpha}$ of the binary simplex code of type α is defined as $\left[\frac{00\ldots0 \mid 11\ldots1}{m_{2,k-1}^{\alpha} \mid m_{2,k-1}^{\alpha}}\right]$, for $k \geq 2$, where $m_{2,1}^{\alpha} = [0, 1]$. In [6], the simplex codes $S_{4,k}^{\alpha}$ of type α over *R* were defined. The generator matrix $G_{\mathbb{R},k}^{\alpha}$ of $S_{\mathbb{R},k}^{\alpha}$ is $\left[\frac{00\ldots0}{G_{\mathbb{R},k-1}^{\alpha}}\middle|\frac{11\ldots1}{G_{\mathbb{R},k-1}^{\alpha}}\middle|\frac{vv\ldots v}{G_{\mathbb{R},k-1}^{\alpha}}\middle|\frac{1+v1+v\ldots1+v}{G_{\mathbb{R},k-1}^{\alpha}}\right]$, for $k \geq 2$, where $G_{\mathbb{R},k-1}^{\alpha} = [0\,1\,v\,1+v]$.

The generator matrix of S_k^{α}, the simplex code of type α over *R* is defined as the concatenation of 2^{2k} copies of the generator matrix of $S_{2,k}^{\alpha}$ and 2^k copies of the generator matrix of $S_{\mathbb{R},k}^{\alpha}$ given by

$$\Theta_k^{\alpha} = [m_{2,k}^{\alpha} \mid m_{2,k}^{\alpha} \mid \ldots \mid m_{2,k}^{\alpha} \mid G_{\mathbb{R},k}^{\alpha} \mid G_{\mathbb{R},k}^{\alpha} \mid \ldots \mid G_{\mathbb{R},k}^{\alpha}],\ k \geq 1. \tag{4.1}$$

The standard form of Θ_k^{α} of the generator matrix of S_k^{α} is

$$\Theta_k^{\alpha} = \left[\frac{00\,00\ldots00}{\Theta_{k-1}^{\alpha}}\middle|\frac{01\,01\ldots01}{\Theta_{k-1}^{\alpha}}\middle|\frac{\ldots}{\ldots}\middle|\frac{11+v11+v\ldots11+v}{\Theta_{k-1}^{\alpha}}\right],$$

for $k \geq 2$, where $\Theta_1^{\alpha} = [00\,01\,0v\,01+v\,10\,11\,1v\,11+v]$. The length of the simplex code of type α over *R* is equal to 2^{3k+1} and the number of code words is equal to $2^{k_0}\mathbb{R}^{k_1}$ for some k_0 and k_1. In the case where $k = 1$ with $k_0 = 0$ and $k_1 = 1$, n that all of the code words of the simplex code S_1^{α} are generated by Θ_1^{α} and are $\{00\,00\,00\,00\,00\,00\,00\,00,\ 00\,01\,0v\,01+v\,10\,11\,1v\,11+v,\ 00\,0v$

$00\,0v\,00\,0v,\ 00\,01 + v\,0v\,01\,10\,11 + v\,1v\,11\}$. The type β simplex code S_k^{β} is a punctured version of S_k^{α}. The number of codewords is $2^{k_0}\mathbb{R}^{k_1}$ for some k_0 and k_1 and its length is $2^k(2^{k-2}+1)(2^k-1)$. The generator matrix of S_k^{β} is the concatenation of 2^k copies of the generator matrix of $S_{2,k}^{\beta}$ and 2^{k-1} copies of the generator matrix of $S_{\mathbb{R},k}^{\beta}$ given by

$$\Theta_k^{\alpha} = [m_{2,k}^{\beta} \mid m_{2,k}^{\beta} \mid \dots \mid m_{2,k}^{\beta} \mid G_{\mathbb{R},k}^{\beta} \mid G_{\mathbb{R},k}^{\beta} \mid \dots \mid G_{\mathbb{R},k}^{\beta}], \text{ for } k \geq 2, \tag{4.2}$$

where $m_{2,k}^{\beta}$ is the generator matrix of the binary simplex code of type β is $\left[\begin{array}{c|c} 11\dots1 & 00\dots0 \\ \hline m_{2,k-1}^{\alpha} & m_{2,k-1}^{\beta} \end{array}\right]$, for $k \geq 3$, with $m_{2,2}^{\beta} = \left[\begin{array}{c|c} 11 & 0 \\ 01 & 1 \end{array}\right]$, and $G_{\mathbb{R},k}^{\beta}$ is a generator matrix of the simplex code over R of type β is defined as $\left[\begin{array}{c|c|c} 11\dots1 & 00\dots0 & vv\dots v \\ \hline G_{\mathbb{R},k-1}^{\beta} & G_{\mathbb{R},k-1}^{\beta} & G_{\mathbb{R},k-1}^{\beta} \end{array}\right]$, for $k \geq 3$, with $G_{\mathbb{R},2}^{\beta} = \left[\begin{array}{c|c|c} 1111 & 0 & v \\ \hline 01v1+v & 1 & 1 \end{array}\right]$. The following theorems provide upper bounds on the covering radius of simplex codes over R with respect to the different distance (D).

Theorem 4.1. *Prove that,* $r_L(S_k^{\alpha}) \leq 2^k(2^{2k-1}+2^{2k}+1)$, $r_L(S_k^{\alpha}) \leq \dfrac{2^k(3.2^{2k-1}+5(1+2^{2k}))}{3}$, $r_L(S_k^{\alpha}) \leq \dfrac{2^k(3.2^{2k-1}+2^{2k}-1)}{3}$ *and* $r_{CE}(S_k^{\alpha}) \leq 2^k(3.2^{2k-1}+2^{2k}+1)$, *here* $r_d(S_k^{\alpha})$ *be a covering radius of type α-simplex codes in R with different distance* (*D*).

Proof. In R-Simplex codes of type α have a Lee weight equal to 2^{3k} or 3.2^{k-1}. From the matrix (4.1), Proposition [13] and Theorem 3.5 with different distance (D), then

$$\begin{aligned} r_L(S_k^{\alpha}) &\leq r_L(2^{2k}S_{2,k}^{\alpha}) + r_L(2^{2k}S_{\mathbb{R},k}^{\alpha}) = 2^{2k}r_L(S_{2,k}^{\alpha}) + 2^k r_L(S_{\mathbb{R},k}^{\alpha}) \\ &\leq 2^{2k}r_H(S_{2,k}^{\alpha}) + 2^k r_L(S_{\mathbb{R},k}^{\alpha}) \\ &\leq 2^{2k}(2^{k-1}) + 2^k[(3.2^{2(k-1)} + 3.2^{2(k-2)} + \dots + 3.2^{2.1}) + r_L(S_{\mathbb{R},k}^{\alpha})] \end{aligned}$$

$r_L(S_k^{\alpha}) \le 2^k(2^{2k-1} + 2^{2k} + 1).$

The remaining part of proof is unification from part 1 but different distance (D). □

Theorem 4.2. *The covering radius of the R-Simplex codes of type* β *are given by*

$$r_L(S_k^{\beta}) \le 2^{k-1}(2^k + 2^{2k-1} - 2^{k-1} - 2),\ r_E(S_k^{\beta}) \le \frac{5.2^{3k-1} - 6.2^{k-1} - 2^{k+2}}{6},$$

$$r_B(S_k^{\beta}) \le \frac{2^{3k} + 3(2^{2k-1} + 2^{3(k-1)} - 3.2^{2k-3} - 2^{k-1})}{3} \text{ and}$$

$$r_{CE}(S_k^{\beta}) \le 2^{3k-1} - 8.2^{k-1}.$$

Proof. From (4.2), Proposition [13] and Theorem 3.5 with different distance(D), so

$$r_L(S_k^{\beta}) \le r_L(2^k S_{2,k}^{\beta}) + r_L(2^{2k} S_{\mathbb{R},k}^{\beta}) = 2^k r_L(S_{2,k}^{\beta}) + 2^{k-1} r_L(S_{\mathbb{R},k}^{\beta})$$

$$\le 2^k r_H(S_{2,k}^{\beta}) + 2^{k-1} r_L(S_{\mathbb{R},k}^{\beta}) = 2^k\left(\frac{2^k - 1}{2}\right) + 2^{k-1}[2^{k-1}(2^k - 1) - 1]$$

$$r_L(S_k^{\beta}) \le 2^{k-1}(2^k + 2^{2k-1} + 2^{k-1} + 2).$$

The Proof 2, 3 and 4 is use for 1 with apply different distance (D). □

5. MacDonald Codes of type α and type β in *R*

The q-ary MacDonald code $M_{k,t}(q)$ over the finite field $\mathbb{F}_q$ is a unique $\left[\frac{q^k - q^t}{q-1},\ k,\ q^{k-1} - q^{t-1}\right]$ linear code in which every non-zero codeword has weight either q^{k-1} or $q^{k-1} - q^{t-1}$ [17]. In [18], the author studied the covering radius of MacDonald codes over a finite field. In fact, the author has given many exact values for smaller dimension. In [14], authors have defined the MacDonald codes over a ring using the generator matrices of the Simplex codes. For $2 \le t \le k-1$, let $G_{k,t}^{\alpha}$ be the matrix obtained from G_k^{α} by deleting columns corresponding to the columns of G_t^{α}. That is,

$$G^{\alpha}_{k,t} = \left[G^{\alpha}_{k} \setminus \frac{0}{G^{\alpha}_{t}} \right] \quad (5.1)$$

and let $G^{\beta}_{k,t}$ be the matrix obtained from G^{β}_{k} by deleting columns corresponding to the columns of G^{β}_{t}. That is,

$$G^{\beta}_{k,t} = \left[G^{\beta}_{k} \setminus \frac{0}{G^{\beta}_{t}} \right] \quad (5.2)$$

where $[A \setminus B]$ denotes the matrix obtained from the matrix A by deleting the columns of the matrix B and 0 is a $(k-t) \times 2^{2t}((k-t) \times 2^{t-1}(2^{t}-1))$. The parameters in MacDonald codes of α-type and β-type is $[4^{k} - 4^{t}, k]$ and $[(2^{k-1} - 2^{t-1})(2^{k} + 2^{t-1}), k]$ code over R. Now, construct the MacDonald codes over $\mathbb{Z}_2\mathbb{R}$ of type α and type β by using the generator matrix of the $\mathbb{Z}_2\mathbb{R}$-simplex codes of type α and type β. If $1 \leq t \leq k-1$, let $\Theta^{\alpha}_{k,t}$ (resp., $\Theta^{\beta}_{k,t}$) be the matrix of MacDonald codes $M^{\alpha}_{k,t}$ (resp., $M^{\beta}_{k,t}$) with parametrs $[2^{3k+1} - 2^{k+1}(2^{k} - 2^{t})]$ (resp., $[2^{3k+1}(2^{2k-1}+1)(2^{k}-1) - 2^{k+t-1}$ $(2^{2t-3}+1)(2^{t}-1)]$ obtained from Θ^{α}_{k} (resp., Θ^{β}_{k}) by deleting columns corresponding to the columns of the matrix Θ^{α}_{t} and $0_{2^{2t}} \times (k-t)$ (resp., Θ^{β}_{t} and $0_{2^{2t}} \times (k-t)$). That is, for $k \geq 1$,

$$\Theta^{\alpha}_{k,t} = [m^{\alpha}_{k,t} \mid \ldots \mid m^{\alpha}_{k,t} \mid G^{\alpha}_{k,t} \mid \ldots \mid G^{\alpha}_{k,t}], \quad (5.3)$$

where $M^{\alpha}_{k,t}$ (resp., $G^{\alpha}_{k,t}$) repeat 2^{2k} (resp., 2^{k}) times in $\Theta^{\alpha}_{k,t}$ for $k \geq 3$,

$$\Theta^{\beta}_{k,t} = [m^{\beta}_{k,t} \mid \ldots \mid m^{\beta}_{k,t} \mid G^{\beta}_{k,t} \mid \ldots \mid G^{\beta}_{k,t}], \quad (5.4)$$

where $M^{\beta}_{k,t}$ (resp., $G^{\beta}_{k,t}$) repeat 2^{2k} (resp., 2^{k-1}) times in $\Theta^{\beta}_{k,t}$.

Theorem 5.1. *For $t \leq r \leq k$,*

1. $r_L(M^{\alpha}_{k,t}) \leq [2^{3k+1} - 2^{k+r}(2^r + 2^k)] + [2^{2.k} r_H(M^{\alpha,2}_{k,t}) + 2^k r_L(M^{\alpha,4}_{k,t})]$,

2. $r_E(M^{\alpha}_{k,t}) \leq \left[\dfrac{2^{3(k+1)} + 2^{k+r}(3.2^r + 5.2^k)}{3}\right]$

$$+ [2^{2.k} r_H(M^{\alpha,2}_{k,t}) + 2^k r_E(M^{\alpha,4}_{k,t})],$$

3. $r_B(M^{\alpha}_{k,t}) \leq \left[\dfrac{7.2^{3k} + 2^{k+r}(4.2^r + 3.2^k)}{3}\right]$

$$+ [2^{2.k} r_H(M^{\alpha,2}_{k,t}) + 2^k r_B(M^{\alpha,4}_{k,t})],$$

4. $r_{CE}(M^{\alpha}_{k,t}) \leq [3.2^{3k+1} - 2^{k+r}(2^r + 2.2^k)]$

$$+ [2^{2.k} r_H(M^{\alpha,2}_{k,t}) + 2^k r_{CE}(M^{\alpha,4}_{k,t})].$$

Proof. Use, the matrix (5.3), Proposition [13] and Theorem 3.5, thus

$$r_L(M^{\alpha}_{k,t}) \leq r_L(2^{2.k} M^{\alpha,2}_{k,t}) + r_L(2^{2.k} M^{\alpha,2}_{k,t}) = 2^{2.k} r_L(M^{\alpha,2}_{k,t}) + 2^k r_L(M^{\alpha,2}_{k,t}),$$

$$\leq 2^{2.k} r_H(M^{\alpha,2}_{k,t}) + 2^k r_L(M^{\alpha,2}_{k,t}),$$

$$\leq 2^{2.k}(2^k - 2^r) + 2^k(2^{2.k} - 2^{2.r}) + 2^{2.k} r_H(M^{\alpha,2}_{k,t}) + 2^k r_L(M^{\alpha,2}_{k,t}),$$

$$r_L(M^{\alpha,2}_{k,t}) \leq [2^{3k+1} - 2^{k+r}(2^k - 2^r)] + [2^{2.k} r_H(M^{\alpha,2}_{k,t}) + 2^k r_L(M^{\alpha,2}_{k,t})].$$

The remaining part of proof follows in part 1. □

Theorem 5.2. *For* $t \leq r \leq k$,

1. $r_L(M^{\beta}_{k,t}) \leq [2^{3k+1} - 2^{k+r-1}(2^{k+1} + 2^r - 1)]$

$$+ [2^{2.k} r_H(M^{\beta,2}_{k,t}) + 2^k r_L(M^{\beta,4}_{k,t})],$$

2. $r_E(M^{\beta}_{k,t}) \leq [6.2^{3k} - 2^{k+r}(2^k + 5.2^r + 6) - 6.2^{2k}]$

$$+ [2^{2.k} r_H(M^{\beta,2}_{k,t}) + 2^k r_E(M^{\beta,4}_{k,t})],$$

3. $r_B(M^{\beta}_{k,t})$

$$\leq \frac{6(2^{3k} - 2^{2k+r}) + 4(2^{3k} - 2^{k+2r}) + 3(2^{3k-2} - 2^{k+2(k-1)}) + 9(2^{k+r-1} - 2^{2k+1})}{6}$$

$$+ [2^{2.k} r_H(M^{\beta,2}_{k,t}) + 2^k r_B(M^{\beta,4}_{k,t})],$$

4. $r_{CE}(M^{\beta}_{k,t}) \leq [2^{3k+1} - 2^{k+r}(2^k + 2^r + 1)]$

$$+ [2^{2.k} r_H(M^{\beta,2}_{k,t}) + 2^k r_{CE}(M^{\beta,4}_{k,t})].$$

Proof. Use, the matrix (5.4), Proposition [13] and Theorem 3.5, so

$$r_L(M^{\beta}_{k,t}) \leq r_L(2^{2.k} M^{\beta,2}_{k,t}) + r_L(2^{2.k} M^{\beta,2}_{k,t})$$

$$\leq 2^{2.k} r_H(M^{\beta,2}_{k,t}) + 2^k r_L(M^{\beta,4}_{k,t}),$$

$$\leq 2^{2.k} r_L(M^{\beta,2}_{k,t}) + 2^k r_L(M^{\beta,4}_{k,t}),$$

$$\leq 2^{2.k}(2^k - 2^r) + 2^k[(2^{2.k}(2^k - 1) - 2^{2-1}(2^k - 1))]$$

$$+ 2^{2.k} r_H(M^{\beta,2}_{k,t}) + 2^k r_L(M^{\beta,4}_{k,t}),$$

$$r_L(M^{\beta,2}_{k,t}) \leq [2^{3k+1} - 2^{k+r}(2^{k+1} - 2^r - 1)] + [2^{2.k} r_H(M^{\beta,2}_{k,t}) + 2^k r_L(M^{\beta,4}_{k,t})].$$

The remaining part of proof is pursue in part 1. □

References

[1] T. Abualrub, I. Siap and N. Aydin, $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes, IEEE Trans. Inform. Theory, 60(3) (2014), 115-121.

[2] T. Aoki, P. Gaborit, M. Harada, M. Ozeki and P. Sole, On the covering radius of $\mathbb{Z}_4$-codes and their lattices, IEEE Trans. Inform. Theory 45(6) (1999), 2162-2168.

[3] M. Bilal, J. Borges, S. T. Dougherty and C. Fernandez-Cordoba, Maximum distance separable codes over $\mathbb{Z}_4$ and $\mathbb{Z}_2\mathbb{Z}_4$, Des. Codes Cryptogr. 61(1) (2011), 31-40.

[4] J. Borges, S. T. Dougherty and C. Fernandez-Cordoba, Characterization and constructions of self-dual codes over $\mathbb{Z}_2\mathbb{Z}_4$, Adv. Math. Commun. 6(3) (2012), 287-303.

[5] J. Borges, C. Fernandez-Cordoba, J. Pujol, J. Rifa and M. Villanueva, $\mathbb{Z}_2\mathbb{Z}_4$-linear codes: generator matrices and duality, Des. Codes Cryptogr. 54(2) (2010), 167-179.

[6] M. C. Bhandari, M. K. Gupta and A. K. Lal, On $\mathbb{Z}_4$ -Simplex codes and their grayimages, Applied Algebra, Algebraic Algorithms and Error Correcting Codes, AAECC-13, Lecture Notes in Computer Science 1719 (1999), 170-180.

[7] P. ChellaPandian and C. Durairajan, On the Covering radius of some Codes over $R = \mathbb{Z}_2 + u\mathbb{Z}_2$, where $u^2 = 0$, International Journal of Research in Applied, Natural and Social Sciences, Impact Journal 2(1) (2014), 61-70.

[8] P. Chella Pandian and C. Durairajan, On various parameters of $\mathbb{Z}_q$ -simplex codes for an even integer q, Journal of Algebra and Discrete Mathematics 19(2) (2015), 243-253.

[9] P. Chella Pandian and C. Durairajan, On $\mathbb{Z}_q$ -linear and $\mathbb{Z}_q$ -Simplex codes and its related parameters for q is a prime power, Journal of Discrete Mathematical Sciences and Cryptography 18(1, 2) (2015), 81-94.

[10] P. Chella Pandian, On covering radius of codes over $R = \mathbb{Z}_2 + u\mathbb{Z}_2$, where $u^2 = 0$, using chinese euclidean distance, Journal Discrete Mathematics, Algorithms and Applications 9(2) (2017), 1750017(1-8).

[11] P. Chella Pandian, On covering radius of codes over $R = \mathbb{Z}_2 + u\mathbb{Z}_2$, where $u^2 = 0$, using Bchoc distance, International Journal of Mathematics And its Applications 5(4C) (2017), 277-282.

[12] Christine Bachoc, Application of coding theory to the construction of modular lattices, Journal of Combin. Theory Ser. A78, 1997, 92-119.

[13] G. D. Cohen, M. G. Karpovsky, H. F. Mattson and J. R. Schatz, Covering radius-Survey and recent results, IEEE Trans. Inform. Theory 31(3) (1985), 328-343.

[14] C. J. Colbourn and M. K. Gupta, On quaternary MacDonald codes, Proc. Information Technology: Coding and Computing (ITCC), April (2003), 212-215.

[15] P. Delsarte, An algebraic approach to the association schemes of coding theory, Philips Research Rep. Suppl. 10 (1973).

[16] P. Delsarte and V. Levenshtein, Association schemes and coding theory, IEEE Trans. Inform. Theory 44(6) (1998), 2477-2504.

[17] S. Dodunekov and J. Simonis, Codes and projective multisets, The Electronic Journal of Communications 5 R37, (1998).

[18] C. Durairajan, On Covering Codes and Covering Radius of Some Optimal Codes. Ph.D. Thesis, Department of Mathematics, IIT Kanpur, (1996).

[19] M. K. Gupta, On some linear codes over $\mathbb{Z}_{2^s}$, Ph.D. Thesis, IIT Kanpur, (1999).

[20] M. K. Gupta, D. G. Glynn and T. Aaron Gulliver, On senary simplex codes, in International Symposium, on Applied Algebra, Algebraic Algorithms and Error-Correcting Codes (Springer, Berlin Heidelberg, 2001), pp. 112-121.

[21] M. K. Gupta and C. Durairajan, On the Covering Radius of some Modular Codes, Advances in Mathematics of Communications 8(2) (2014), 129-137.