



FIXED POINT THEOREMS IN CONVEX G -METRIC SPACES

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Abstract

The aim of this paper is to introduce convex structure G -metric spaces and extended Mann's iteration algorithm to these spaces. By using Mann's iteration scheme, a series of fixed point results and Mann's iteration algorithm was generalized. Also, strong convergence theorems for contraction mappings in convex G -metric spaces were developed. Furthermore, the problem of T -stability of the Mann's iteration procedure for the mappings in complete convex G -metric spaces was considered.

1. Introduction

The fixed point theorems, convex metric spaces and convex structure have been well discussed in literature (Alnafe'i et al. [1], Hadzic [5], Hamaizia [5], Saha et al. [12], Shimizu [14]). The properties of fixed point of the b -metric spaces and E -metric spaces have been established by different authors and references (Chen et al. [3], Goswami et al. [4], Haokip and Goswami [4], Mehmood et al. [8]). The T -stable is one of the important requirements for a fixed point iteration to be valuable and applicable from a numerical point of view (Rani and Jyoti [12]).

In Mustafa and Sims [11], the authors introduced a new concept of generalized metric space called G -metric spaces. After that, many authors have proven several fixed point results in these spaces (Modi and Bhatt [9], Mustafa et al. [10]). This paper introduced the concept of the convex G -metric space by the convex structure. Moreover, Mann's iteration algorithm was

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extended to G -metric space. Additionally, by means of Mann's iteration scheme, strong convergence theorems for two types of contraction mapping in convex G -metric spaces were obtained.

2. Preliminaries

In this section, definitions of general metric, properties and the other result that are needed in the sequel are explained.

Definition 1. [9.11]. Assume that X is a nonempty set, and let $G : X \times X \times X \rightarrow R^+$ be a function satisfying the following properties:

- (i) $G(u, v, w) = 0$ if $u = v = w$;
- (ii) $0 < G(u, u, v), \forall u, v \in X$ with $u \neq v$;
- (iii) $G(u, u, v) \leq G(u, v, w), \forall u, v, w \in X$ with $u \neq w$;
- (iv) $G(u, v, w) = G(v, u, w) = G(w, v, u) = \dots$ (all permutations of u, v, w), (symmetry in three variables);
- (v) $G(u, v, w) \leq G(u, a, a) + G(a, v, w) \forall u, v, w, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric or a G -metric on X and the pair (G, X) is called a G -metric space.

Definition 2. [9.11]. Let (G, X) be a G -metric and let $\{u_n\}$ be a sequence of points of X , a point $\{u_n\}$ is said to be limit of the sequence $\{u_n\}$ if $\lim_{n, m \rightarrow \infty} G(u, u_n, u_m) = 0$ and the sequence $\{u_n\}$ is said to be G -convergent to u . Thus, if $u_n \rightarrow u$ in a G -metric space (G, X) then for any $\varepsilon > 0$, there exists a positive integer N such that

$$G(u, u_n, u_m) < \varepsilon \forall n, m \geq N.$$

Definition 3. [9.11]. Let (G, X) be a G -metric space. A sequence $\{u_n\}$ in X is called G -Cauchy if for every $\varepsilon > 0$, there is a positive integer N such that $G(u_n, u_m, u_l) < \varepsilon, \forall n, m, l \geq N$, that is, if $G(u_n, u_m, u_l) \rightarrow 0$, as $n, m, l \rightarrow \infty$.

Definition 4. [9]. Let (G, X) be a G -metric space. A mapping $W : X \times X \times X \times (0, 1] \rightarrow X$ is said to be convex structure on (G, X) if for each $(u, v, w, \lambda) \in X \times X \times X \times (0, 1]$ and for all $x, y \in X$ the condition

$$G(x, y, W(u, v, w, \lambda)) \leq \frac{\lambda}{3} \{G(u, v, x) + G(u, v, y) + G(u, v, z)\}$$

holds. If W is convex structure on a G -metric space (G, X) then the triplet (X, G, W) is called a convex G -metric space.

Definition 5. [9.11]. A G -Metric space (G, X) is said to be G -complete (or complete G -metric space) if every Cauchy sequence in (G, X) is convergent in X .

Definition 6. [3]. Assuming that T is a self-map on a complete G -metric space (G, X) . Let $u_{n+1} = f(T, u_n)$ be an iteration sequence that yields the sequence u_n of the points from X . So, the iteration procedure $u_{n+1} = f(T, u_n)$ is going to be weakly T -stable if $\{u_n\}$ converges to a fixed point u^* of T , and if $\{v_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} G(v_{n+1}, f(Tv_n), a) = 0$ and sequence $\{G(v_n, Tv_n, a)\}$ is bounded, then $\lim_{n \rightarrow \infty} v_n = u^*$.

Lemma 1. [3]. Let $\{k_n\}, \{l_n\}$ be a non-negative sequence that satisfies $k_{n+1} \leq hk_n + l_n \forall n \in N, 0 \leq h < 1, \lim_{n \rightarrow \infty} l_n = 0$, then $\lim_{n \rightarrow \infty} k_n = 0$.

3. Main Results

In this section, we prove complete convex generalized metric version of Banach’s contraction principle (Banach [2]) by means of Mann’s iteration algorithm, complete convex generalized metric of Kannan type fixed point theorem (Chen et al. [3]), and the weak T -stability procedure respectively.

Theorem 1. Suppose that (X, G, W) is a complete convex G -metric space and $T : X \rightarrow X$ is a contraction mapping; that is, there exists $B \in [0, 1)$ such that

$$G(Tu, Tv, \alpha) \leq BG(u, v, \alpha) \forall u, v \in X.$$

Let us choose $u_0 \in X$ in such a way that $G(u_0, Tu_0, \alpha) = M < \infty$ and define $u_n = W(u_{n-1}, Tu_{n-1}; \alpha_{n-1})$ where $0 \leq \alpha_{n-1} < 1$ and $n \in N$. If $B < 1$

and $0 < \alpha_{n-1} < \frac{1-B}{1-B}$ for each $n \in N$; then T has a unique fixed point in X .

Proof. For any $n \in N$, it is said that

$$G(u_n, u_{n+1}, \alpha) = G(u_n W(u_n, Tu_n; \alpha_n), \alpha) \leq (1 - \alpha_n)G(u_n, Tu_n, \alpha)$$

and

$$\begin{aligned} G(u_n, Tu_n, \alpha) &\leq G(u_n, Tu_n; Tu_{n-1}) + G(Tu_{n-1}, Tu_n, \alpha) \\ &\leq G(W(u_{n-1}, Tu_{n-1}; \alpha_{n-1}), Tu_{n-1}, Tu_{n-1}) + BG(u_{n-1}, u_n, \alpha) \\ &\leq \alpha_{n-1}G(u_{n-1}, Tu_{n-1}, Tu_{n-1}) + B(1 - \alpha_{n-1})G(u_{n-1}, Tu_{n-1}, \alpha). \end{aligned}$$

From definition (1) (iii) and (iv) we have

$$\leq (\alpha_{n-1} + B(1 - \alpha_{n-1}))G(u_{n-1}, Tu_{n-1}, \alpha)$$

$$\therefore G(u_n, Tu_n, \alpha) \leq (\alpha_{n-1} + B(1 - \alpha_{n-1}))G(u_{n-1}, Tu_{n-1}, \alpha)$$

Let $\lambda_{n-1} = \alpha_{n-1} + B(1 - \alpha_{n-1})$. By gathering this and the above inequality with the assumptions $B < 1$ and $0 < \alpha_{n-1} < \frac{1-B}{1-B} \forall n \in N$, we get

$$G(u_n, Tu_n, \alpha) \leq \lambda_{n-1}G(u_{n-1}, Tu_{n-1}, \alpha) < 1^{\frac{1-B}{4}} BG(u_{n-1}, Tu_{n-1}, \alpha). \quad (1)$$

This means that $\{G(u_n, Tu_n, \alpha)\}$ is a decreasing sequence of non-negative reals. Therefore, $\exists \eta \geq 0$ Such that

$$\lim_{n \rightarrow \infty} G(u_n, Tu_n, \alpha) = \eta.$$

We will show that $\eta = 0$. Suppose that $\eta > 0$. Letting $n \rightarrow \infty$ in inequality (1), we obtain

$$\eta \leq B.1^{\frac{1}{4}-B} \eta < \eta,$$

a contradiction. Hence we get $\eta = 0$. Furthermore, we have

$$G(u_n, u_{n+1}, \alpha) \leq (1 - \alpha_n)G(u_n, Tu_n, \alpha) < G(u_n, Tu_n, \alpha),$$

which shows that $\lim_{n \rightarrow \infty} G(u_n, u_{n+1}, \alpha) = 0$. Now we will show that $\{u_n\}$ is a Cauchy sequence.

In fact, if $\{u_n\}$ is not a Cauchy sequence, so there exists $\varepsilon_0 > 0$ and the subsequences $\{u_{\theta(k)}\}$ and $\{u_{\tau(k)}\}$ of $\{u_n\}$, such that $\theta(k)$ is the smallest natural index with $\theta(k) > \tau(k) > k$,

$$G(u_{\theta(k)}, u_{\tau(k)}, \alpha) \geq \varepsilon_0$$

and

$$G(u_{\theta(k)-1}, u_{\tau(k)}, \alpha) < \varepsilon_0$$

Therefore, it is concluded that

$$\varepsilon_0 \leq G(u_{\theta(k)}, u_{\tau(k)}, \alpha) \leq G(u_{\theta(k)}, v, v) + G(v, u_{\tau(k)}, \alpha)$$

which implies that

$$\varepsilon_0 \leq \limsup_{k \rightarrow \infty} G(u_{\theta(k)}, u_{\tau(k)+1}, v).$$

Noticing that

$$\begin{aligned} G(u_{\theta(k)}, u_{\tau(k)+1}, \alpha) &= G(W(u_{\theta(k)-1}, Tu_{\theta(k)-1}; \alpha_{\theta(k)-1}), u_{\tau(k)+1}, \alpha) \\ &\leq \alpha_{\theta(k)-1}G(u_{\theta(k)-1}, u_{\tau(k)+1}, u_{\tau(k)+1}) + (1 - \alpha_{\theta(k)-1})G(Tu_{\theta(k)-1}, u_{\tau(k)+1}, \alpha) \\ &\leq \alpha_{\theta(k)-1}G(u_{\theta(k)-1}, u_{\tau(k)+1}, u_{\tau(k)+1}) + (1 - \alpha_{\theta(k)-1})\{G(Tu_{\theta(k)-1}, Tu_{\tau(k)+1}, \\ &Tu_{\tau(k)+1}) + G(Tu_{\tau(k)+1}, u_{\tau(k)+1}, \alpha)\} \\ &\leq \alpha_{\theta(k)-1}G(u_{\theta(k)-1}, u_{\tau(k)+1}, u_{\tau(k)+1}) + (1 - \alpha_{\theta(k)-1})\{BG(u_{\theta(k)-1}, u_{\tau(k)+1}, u_{\tau(k)+1}) \\ &+ G(Tu_{\tau(k)+1}, u_{\tau(k)+1}, \alpha)\} \end{aligned}$$

$$\begin{aligned}
&\leq (\alpha_{\theta(k)-1} + (1 - \alpha_{\theta(k)-1})B)G(u_{\theta(k)-1}, u_{\tau(k)+1}, u_{\tau(k)+1}) \\
&+ (1 - \alpha_{\theta(k)-1})G(Tu_{\tau(k)+1}, u_{\tau(k)+1}, a) \\
&\leq (\alpha_{\theta(k)-1} + (1 - \alpha_{\theta(k)-1})B)\{G(u_{\theta(k)-1}, Tu_{\tau(k)+1}, Tu_{\tau(k)+1}) \\
&+ G(Tu_{\tau(k)+1}, u_{\tau(k)+1}, u_{\tau(k)+1})\} + (1 - \alpha_{\theta(k)-1})G(Tu_{\tau(k)+1}, u_{\tau(k)+1}, a)
\end{aligned}$$

We obtain

$$\varepsilon_0 \leq \limsup_{k \rightarrow \infty} G(u_{\theta(k)}, u_{\tau(k)+1}, a) \leq \varepsilon_0,$$

which is a contradiction. Hence, $\{u_n\}$ is a Cauchy sequence in X . By the completeness of X , there exists $u^* \in X$ Such that $\lim_{n \rightarrow \infty} G(u_n, u^*, a) = 0$.

Next, we will verify that u^* is a fixed points of T . Note that

$$G(u^*, Tu^*, a) \leq G(u^*, u_n, u_n) + G(u_n, Tu^*, a).$$

Letting $n \rightarrow \infty$ we assume that $G(u^*, Tu^*, a) = 0$ which implies that $Tu^* = u^*$. Hence, u^* is a fixed point of T . Now we explain that T has a unique fixed point. Suppose that $v \in X$ is another fixed point, that is $Tv = v$. Then,

$$G(u^*, v, a) = G(Tu^*, Tv, a) \leq BG(u^*, v, a).$$

For some $B \in [0, 1)$, a contradiction. Hence $u^* = v$, which completes the proof.

Theorem 2. *Suppose that (X, G, W) is a complete convex G -metric space, and the mapping $T : X \rightarrow X$ be defined as*

$$G(Tu, Tv, a) \leq k(G(u, Tu, a) + G(v, Tv, a)) \forall u, v \in X, \quad (2)$$

and for some $k \in \left[0, \frac{1}{2}\right)$. Let us choose $u_0 \in X$ in such a way that $G(u_0, Tu_0, a) = M < \infty$ and we define $u_n = W(u_{n-1}Tu_{n-1}; \alpha_{n-1})$ for $n \in \mathbb{N}$ and $\alpha_{n-1} \in \left(0, \frac{1}{2}\right]$. If $k \in \left[0, \frac{1}{4}\right]$, then T has a unique fixed point in X .

Proof. For any $n \in N$, we have

$$G(u_n, u_{n+1}, a) = G(u_n W(u_n, Tu_n; \alpha_n), a) \leq (1 - \alpha_n)G(u_n, Tu_n, a) \quad (3)$$

and

$$\begin{aligned} G(u_n, Tu_n, a) &= G(W(u_{n-1}, Tu_{n-1}; \alpha_{n-1}), Tu_n, a) \\ &\leq \alpha_{n-1}G(u_{n-1}, Tu_n, a) + (1 - \alpha_{n-1})G(Tu_{n-1}, Tu_n, a) \\ &\leq \alpha_{n-1}(G(u_{n-1}, Tu_{n-1}, Tu_{n-1}) + G(Tu_{n-1}, Tu_n, a)) + G(Tu_{n-1}, Tu_n, a) \\ &\leq \alpha_{n-1}(G(u_{n-1}, Tu_{n-1}, Tu_{n-1}) + (\alpha_{n-1} + 1)G(Tu_{n-1}, Tu_n, a)) \\ &\leq \alpha_{n-1}G(u_{n-1}, Tu_{n-1}, Tu_{n-1}) + k(\alpha_{n-1} + 1)(G(u_{n-1}, Tu_n, a) \\ &\quad + G(u_n, Tu_n, a)) \end{aligned}$$

Let $Tu_{n-1} = a$

$$\leq (\alpha_{n-1} + k\alpha_{n-1} + k)G(u_{n-1}, Tu_{n-1}, a) + (k\alpha_{n-1} + k)G(u_n, Tu_n, a)$$

i.e.,

$$(1 - (k\alpha_{n-1} + k))G(u_n, Tu_n, a) \leq (\alpha_{n-1} + k\alpha_{n-1} + k)G(u_{n-1}, Tu_{n-1}, a)$$

Since

$$k\alpha_{n-1} + k \leq \frac{5}{4}k < \frac{5}{16} < 1,$$

then

$$G(u_n, Tu_n, a) \leq \frac{\alpha_{n-1} + k\alpha_{n-1} + k}{1 - (k\alpha_{n-1} + k)} G(u_{n-1}, Tu_{n-1}, a). \quad (4)$$

Denote $\lambda_{n-1} = \frac{\alpha_{n-1} + k\alpha_{n-1} + k}{1 - (k\alpha_{n-1} + k)}$ for $n \in N$. We assume that

$$\lambda_{n-1} = \frac{\alpha_{n-1} + k\alpha_{n-1} + k}{1 - (k\alpha_{n-1} + k)} < \frac{\frac{5}{4}}{1 - \frac{5}{16}} < \frac{\frac{5}{4}}{1 - \frac{5}{16}} - 1 = \frac{9}{11}.$$

From the above inequality, as well as inequality (4) with the assumptions of the theorem, we get

$$G(u_n, Tu_n, a) \leq \lambda_{n-1} G(u_{n-1}, Tu_{n-1}, a) < \frac{9}{11} G(u_{n-1}, Tu_{n-1}, a), \quad (5)$$

which implies that $\{G(u_n, Tu_n, a)\}$ is a decreasing sequence of non-negative reals. Hence, there exists $\eta \geq 0$ Such that

$$\lim_{n \rightarrow \infty} G(u_n, Tu_n, a) = \eta.$$

We will show that $\eta = 0$. suppose that $\eta > 0$. Letting $n \rightarrow \infty$ in inequality (5), we obtain that $\eta \leq \frac{9}{11} \eta < \eta$, a contradiction. Hence we get that $\eta > 0$; i.e.,

$$\lim_{n \rightarrow \infty} G(u_n, Tu_n, a) = 0.$$

Moreover, by inequality (3) we obtain

$$G(u_n, u_{n+1}, a) \leq (1 - \alpha_n) G(u_n, Tu_n, a) < G(u_n, Tu_n, a),$$

which implies that $\lim_{n \rightarrow \infty} G(u_n, u_{n+1}, a) = 0$. Now it will be shown that $\{u_n\}$ is a Cauchy sequence. In fact, if $\{u_n\}$ is not a Cauchy sequence, then there exist $\varepsilon_0 > 0$ and the subsequences $\{u_{\theta(l)}\}$ and $\{u_{\tau(l)}\}$ of $\{u_n\}$ such that $\theta(l)$ is the smallest natural index with $\theta(l) > \tau(l) > l$,

$$G(u_{\theta(l)}, u_{\tau(l)}, a) \geq \varepsilon_0$$

and

$$G(u_{\theta(l)-1}, u_{\tau(l)}, a) < \varepsilon_0$$

Therefore, it is concluded that

$$\varepsilon_0 \leq G(u_{\theta(l)}, u_{\tau(l)}, a) \leq G(u_{\theta(l)}, u_{\tau(l)+1}, u_{\tau(l)+1}) + G(u_{\theta(l)+1}, u_{\tau(l)}, a),$$

which implies that

$$\varepsilon_0 \leq \limsup_{l \rightarrow \infty} G(u_{\theta(l)}, u_{\tau(l)+1}, a)$$

Noticing that

$$\begin{aligned}
 G(u_{\theta(l)}, u_{\tau(l)+1}, a) &= G(W(u_{\theta(l)-1}, Tu_{\theta(l)-1}; \alpha_{\theta(l)-1}), u_{\tau(l)+1}, a) \\
 &\leq \alpha_{\theta(l)-1}G(u_{\theta(l)-1}, u_{\tau(l)+1}, a) + (1 - \alpha_{\theta(l)-1})G(Tu_{\theta(l)-1}, u_{\tau(l)+1}, a) \\
 &\leq \alpha_{\theta(k)-1}G(u_{\theta(l)-1}, u_{\tau(l)+1}, a) + (1 - \alpha_{\theta(l)-1})\{G(Tu_{\theta(l)-1}, Tu_{\tau(l)+1}, Tu_{\tau(l)+1}) \\
 &\quad + G(Tu_{\tau(l)+1}, u_{\tau(l)+1}, a)\} \leq \alpha_{\theta(l)-1}G(u_{\theta(l)-1}, u_{\tau(l)+1}, a) \\
 &\quad + (1 - \alpha_{\theta(l)-1})\{kG(u_{\theta(l)-1}Tu_{\theta(l)-1}, a) + kG(u_{\tau(l)+1}, Tu_{(l)+1}a) \\
 &\quad + G(Tu_{\tau(l)+1}, u_{\tau(l)+1}a)\} \\
 &\leq \alpha_{\theta(l)-1}G(u_{\theta(l)-1}, u_{\tau(k)+1}, a) + (1 - \alpha_{\theta(l)-1})(kG(u_{\theta(l)-1}, Tu_{\theta(l)-1}, a) \\
 &\quad + (k + 1)G(u_{\theta(l)+1}, Tu_{\theta(l)+1}, a)) \text{ (for some } k \in \left[0, \frac{1}{2}\right) \text{ satisfying (2))} \\
 &\leq \alpha_{\theta(l)-1}\{G(u_{\theta(l)-1}, u_{\tau(l)}, u_{\tau(l)}) + G(u_{\tau(l)}, u_{\tau(l)+1}, a)\} \\
 &\quad + (1 - \alpha_{\theta(l)-1})\{kG(u_{\theta(l)-1}, Tu_{\theta(l)-1}, a) + (k + 1)G(u_{\tau(l)+1}, Tu_{\tau(l)+1}, a)\}
 \end{aligned}$$

We obtain

$$\limsup_{l \rightarrow \infty} G(u_{\theta(l)}, u_{\tau(l)+1}, a) \leq \frac{\varepsilon_0}{4} < \varepsilon_0$$

which is a contradiction. Thus $\{u_n\}$ is a Cauchy sequence in X . By the completeness of X , it follows that there exists $u^* \in X$. Such that

$$\lim_{l \rightarrow \infty} G(u_n, u^*, a) = 0.$$

Now we will show that u^* is a fixed point of T . Since

$$\begin{aligned}
 G(u^*, Tu^*, a) &\leq G(u^*, u_n, u_n) + G(u, Tu^*, a) \\
 &\leq G(u^*, u_n, u_n) + G(u_n, Tu_n, Tu_n) + G(Tu_n, Tu^*, a) \\
 &\leq G(u^*, u_n, u_n) + G(u_n, Tu_n, Tu_n) + k\{G(u_n, Tu_n, a) + G(u^*, Tu^*, a)\}
 \end{aligned}$$

We conclude that

$$\begin{aligned}
 (1 - k)G(u^*, Tu^*, a) &\leq G(u^*, u_n, u_n) + (1 + k)\left(\frac{9}{11}\right)^n G(u_n, Tu_n, a) \\
 &\quad \text{(for some } Tu_n = a)
 \end{aligned}$$

Consequently, we get that $G(u^*, Tu^*, \alpha) = 0$, so u^* is a fixed point of T .

In order to illustrate the uniqueness of the fixed point, suppose that $q \in X$, $q \neq u^*$, is another fixed point of T . Then $Tq = q$. However,

$$0 < G(u^*, q, \alpha) = G(Tu^*, Tq, \alpha) \leq kG(u^*, Tu^*, \alpha) + kG(q, Tq, \alpha) = 0,$$

a contradiction. Hence $u^* = q$ which completes the proof.

Theorem 3. *Under the assumptions of theorem 1, if, additionally, $\lim_{n \rightarrow \infty} \alpha_n = 0$, then Mann's iteration is weakly T -stable.*

Proof. By virtue of Theorem 1, we assume that u^* is a unique fixed point of T in X . Assuming that $\{v_n\}$ is a sequence in X which satisfies $\limsup_{l \rightarrow \infty} G(v_{n+1}W, (v_n, Tv_n; \alpha_n), \alpha) = 0$ and $\{G(v_n, Tv_n, \alpha)\}$ is bounded. We obtain

$$\begin{aligned} G(v_{n+1}, u^*, \alpha) &\leq G(v_{n+1}W(v_n, Tv_n; \alpha_n), W(v_n, Tv_n; \alpha_n)) \\ &+ G(W(v_n, Tv_n; \alpha_n), u^*, \alpha) \\ &\leq G(v_{n+1}, W(v_n, Tv_n; \alpha_n), W(v_n, Tv_n; \alpha_n)) + G(W(v_n, Tv_n; \alpha_n), Tv_n, Tv_n) \\ &+ G(Tv_n, u^*, \alpha) \\ &\leq G(v_{n+1}, W(v_n, Tv_n; \alpha_n), W(v_n, Tv_n; \alpha_n)) + \alpha_n G(v_n, Tv_n, Tv_n) \\ &+ BG(v_n, u^*, \alpha). \end{aligned}$$

Noticing that $B < 1$, $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\lim_{n \rightarrow \infty} G(v_{n+1}W(v_n, Tv_n; \alpha_n), W(v_n, Tv_n; \alpha_n)) = 0$ and $\{G(v_n, Tv_n, Tv_n)\}$ is bounded, and taking into account Lemma 1, we get that

$$\lim_{n \rightarrow \infty} G(v_n, u^*, \alpha) = 0,$$

which completes the proof.

Theorem 4. *Under all the assumptions of Theorem 2, if, $\lim_{n \rightarrow \infty} \alpha_n = 0$ and if the positive real number k from Theorem 2 satisfies, as well as the condition*

$\frac{k}{1-k} < 1$, then Mann's iteration is weakly T -stable.

Proof. From Theorem 2, it follows that T has a unique fixed point of u^* in X . Assume that $\{v_n\}$ is a sequence in X which satisfies

$$\lim_{l \rightarrow \infty} G(v_{n+1}W(v_n, Tv_n; \alpha_n), \alpha) = 0.$$

and $\{G(v_n, Tv_n, \alpha)\}$ is bounded. We obtain

$$\begin{aligned} G(v_n, u^*, \alpha) &\leq G(v_{n+1}W(v_n, Tv_n; \alpha_n), W(v_n, Tv_n; \alpha_n)) \\ &+ G(W(v_n, Tv_n; \alpha_n), u^*, \alpha) \\ &\leq G(v_{n+1}W(v_n, Tv_n; \alpha_n), W(v_n, Tv_n; \alpha_n)) + G(W(v_n, Tv_n; \alpha_n), Tv_n, Tv_n) \\ &+ G(Tv_n, u^*, \alpha) \\ &\leq G(v_{n+1}W(v_n, Tv_n; \alpha_n), W(v_n, Tv_n; \alpha_n)) + \alpha_n G(v_n, Tv_n, Tv_n) + G(Tv_n, u^*, \alpha) \end{aligned}$$

Noticing that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\lim_{n \rightarrow \infty} G(v_{n+1}W(v_n, Tv_n; \alpha_n), W(v_n, Tv_n; \alpha_n)) = 0$ and $\{G(v_n, Tv_n, Tv_n)\}$ is bounded, and taking into account Lemma 1, we get that

$$\lim_{n \rightarrow \infty} G(v_n, u^*, \alpha) = 0,$$

which completes the proof.

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