Advances and Applications in Mathematical Sciences
Volume 23, Issue 9, July 2024, Pages 769-794

ASYMPTOTE BEHAVIOR OF A SOLUTION TO NONLINEAR MIXED INTEGRAL EQUATION AND ITS NUMERICAL APPLICATIONS AT FRACTIONAL TIMES

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Abstract

In this study, we employ the weak noncompactness approach to analyze at least one solution of the nonlinear mixed integral equation (**NMIE**) in the space of position and time, $L_1[0, 1] \times C[0, T], 0 \leq t \leq T < 1$ using the De Blasi measure and Schauder fixed point. A nonlinear system of Hammerstein integral equations (**NSHIEs**) in $L_1[0, T]$-space can be generated using the quadrature approach. It is discussed whether there is at least one solution for (**NSHIEs**). By using the Collocation method and the Galerkin method, the (**NSHIEs**), in each case, reduced to an algebraic system of nonlinear type (**NAS**). Matlab 2023 software is used to compute and plot the solutions at specific fractional times.

1. Introduction

Many integral equations (**IEs**) of various types contain mathematical models that describe the general form of the problem that appears in many different fields of applied sciences. For some of them, see Jan et al. [1], Mikula [2], Nasr et al. [3], Basim [4] and Alharbi and Alhindi [5]. That is why previous and current authors were interested in developing some methods to solve various types of **IEs**. We refer to some methods, such as the Collocation method in Mirzaee, and Samadyar [6] and, the Degenerate kernel method in Nasr and Abdel-Aty [3]. Toeplitz matrix method in Basseem and Alalyani [7].

2020 Mathematics Subject Classification: 65R20, 45B05, 65H10.
Keywords: De Blasi measure, weak noncompactness method, mixed integral equation, Hammerstein integral equations, algebraic system of nonlinear type.
Received July 17, 2024; Accepted July 28, 2024

The collocation technique was applied by Diego and Lima in [8] to discuss numerically the solution of IE with a weakly singular kernel. In [9], Baksheesh used the Galerkin method to solve an IE of Volterra type with a convolution kernel. Mirzaee and Hoseini [10] used the Fibonacci Collocation method to solve an **IE** of the Volterra-Fredholm $(V - F)$ second type with a continuous kernel. In [11] Al-Bugami applied the Collocation and Galerkin method for solving contact problems in the elastic material. In [12], He et al. developed block-pulse functions to solve an **IE** of **VF** type. In [13], Abd-Elhameed used Chebyshev polynomials of the sixth type to obtain numerically the solution of Burgers equation in one-dimension. In [14], Matoog et al., used the orthogonal polynomials method in of Chebyshev polynomials form to solve **MIE** in time and position. In [15], Brezinski, et al. discussed the numerical solution of **HIEs** using Extrapolation methods. In [16], Al-Bugame et al. used Chebyshev polynomials and Bernoulli polynomials for solving numerically nonlinear mixed partial integro differential equations. Using the modified least square method, Majouti et al. [17] obtained the solution of **N FIE** in numerical form. In [18], Jebreen used the Multi-wavelets Galerkin method to compute a numerical solution for **VFIE**. More different methods and its solution can be found in [19-22]. On the other side, many authors in the filled of (IEs) focused on understanding the properties of the solutions of (IEs). This kinds of study provides a deep understanding of behavior for modifying systems, see [23, 24, 25].

Assume the NMIE of second type,

$$\mu\chi(u, t) = q(u, t) + \lambda\int_0^t\int_0^1 \xi(t, \tau)\,\omega(u, v)\,\varphi(v, \tau, \chi(v, \tau))\,dv d\tau \tag{1}$$

Here, the two known functions $q(u, t), \varphi(u, t, \chi(u, t))$ belong to the space $L_1[0, T] \times C[0, T], 0 \leq t \leq T < 1$. The know functions $\omega(u, v)$ and $\xi(t, \tau)$ represent the continuous kernel of position and time, respectively. While, $\chi(u, t)$ is unknown function represents the solution of Eq.(1) that will be obtained. λ is a constant that has many physical meaning and μ is a constant that defines the type of IE. The goal here is to prove the existence of at least one solution of MIE (1) by using the method of weak noncompactness and Shauder theorem. Then, a suitable quadrature method is used to reduce the

MIE to a set of Hammerstein integral equations SHIEs with continuous kernels. Therefore, the existence of at least one solution for SHIEs is considered. By using the Collocation method and the Galerkin method, the SHIEs reduced to the nonlinear algebraic system NAS which is solved numerically. Finally, numerical results are calculated and the error estimate in each case is computed.

2. The Strategy of Finding at Least a Solution

The Schauder theorem is considered a source to prove that there exists at least one solution to Eq.(1). Therefore, we state the following basic definitions, [25]

Definition 1. (measure of weak noncompactness)

Consider $\gamma : m_E \to R^+, \gamma$ is a measurable function of weak noncompactness with kernel $\rho(\ker \gamma = \rho, \rho C n^w)$, if it satisfies the following conditions:

1. $\gamma(X) = 0 \Leftrightarrow X \in \rho.$

2. $\gamma(\overline{X}) = \gamma(X).$

3. $X \subset Y \Rightarrow \gamma(X) \leq \gamma(Y).$

4. $\gamma(conX) = \gamma(X).$

5. $\gamma(\lambda X + (1 - \lambda)Y) \leq \lambda\gamma(X) + (1 - \lambda)\gamma(Y).$

6. If $X_n \in m, X_n = \overline{X_n^w}$ and $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if $\lim\limits_{n \to \infty} \gamma(X_n) = 0$, then, $X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \phi.$

Definition 2. (De Blas measure). The weak noncompactness $\beta(X)$,

$\beta(X) = \inf\{r > 0 :$ there exists a weakly comact subset W of E such that $X \subset W + B\_r\}.$

Definition 3. The convenient of the function $\beta(X)$ in the space $L_1[0, 1]$ was given by,

$$\beta(X) = \lim_{\varepsilon \to 0} \left\{ \sup_{x \in X} \left\{ \sup \left[\int_D | X(t) | dt : D \subset [0, 1], m(D) \in \varepsilon \right] \right\} \right\}$$

$m(D) -$ the Lebesgues measure of subset D.

Before proving the principle theorem, we consider the integral operator form,

$$\mu L(\chi(u, t)) = q(u, t) + \lambda Z \Omega \Phi(\chi(u, t)) \tag{2}$$

where,

$$\Phi(\chi(u, t)) = \varphi(u, t, \chi(u, t)).$$

$$\Omega \Phi(\chi(u, t)) = \int_0^1 \omega(u, v) \varphi(v, \tau, \chi(u, \tau)) dv$$

$$Z \Omega \Phi(\chi(u, t)) = \int_0^1 \xi(t, \tau) \omega(u, v) \varphi(v, \tau, \chi(v, \tau, \chi(v, \tau))) dv d\tau. \tag{3}$$

Here, Φ is a superposition operator generated by the function φ, while Ω and Z generated by ω and ξ respectively. Then assume the assumptions:

i) The functions $\varphi(u, t, \chi(u, t)) \in L_1[0, 1] \times C[0, T], T < 1$ satisfies Caratheodory and growth assumptions. i.e. we have a function $A(u, t) \in L_1[0, 1] \times C[0, T]$ and constant $b > 0$ such that,

$$| \varphi(u, t, \chi(u, t)) | \leq A(u, t) + b | A(u, t) | + b | \chi(u, t) |, \| A \| = \overline{A}, (\overline{A} \text{ is constant})$$

Moreover, for every positive $\varepsilon_1 < \varepsilon$, we can find $\delta(\varepsilon)$ in which,

$$\max_{0 \leq t \leq T} \int_0^t \int_0^1 | \varphi(u, \tau, \chi_1(u, \tau) - \varphi(u, \tau, \chi_2(u, \tau)) | du d\tau \leq \varepsilon_1, \text{ such that}$$

$$\| \chi_1 - \chi_2 \| \leq \delta(\varepsilon)$$

ii) The kernel ω satisfies $| \omega(u, v) | = \eta$, η is constant.

iii) The kernel term ξ for all value of τ satisfies, $| \xi(t, \tau) | \leq S^-$ (S^- is constant).

iv) $q(u, t) \in L_1[0, 1] \times C[0, T]$ and satisfies,

$$\| q \| = \max_{0 \leq t \leq T} \int_0^t \int_0^1 | q(u, \tau) | du d\tau = N^-.$$

Theorem 1. *By considering the pervious conditions, Eq.*(1) *has at least one solution in* $L_1[0, 1] \times C[0, T]$ *under one solution in* $L_1[0, 1] \times C[0, T]$ *under the assumption*,

$$| \lambda | b \eta S^- T \leq | \mu |. \tag{4}$$

Lemma 1. *Under the assumptions* (i)-(iv), *the integral* operator L *of Eq.*(2) *maps* $L_1[0, 1] \times C[0, T],\ T < 1$ *into itself continuously*.

Proof. First, let $\chi(u, t) \in B_r$ and write Eq.(2) in the normal form,

$$| \mu | \| L(u, t) \| \leq \| q(u, t) \| + | \lambda | \left\| \int_0^t \int_0^1 | \xi(t, \tau) | | \omega(u, v) | | \varphi(v, \tau, (v, \tau)) | dv d\tau \right\| \tag{5}$$

applying Hölder inequality and using the conditions (i)-(iv), we get

$$\| L\chi \| \leq \frac{\gamma}{| \mu |} + \frac{| \lambda | b n S^- T}{| \mu |} \| \chi \|,\ \gamma = N + | \lambda | \bar{A} S^- T,\ T = \max_{0 \leq t \leq T} \int_0^t dt \tag{6}$$

the last inequality shows that, the operator q maps the ball $B_{r_1} \subset L_1[0, 1] \times C[0, T]$ into itself, where $r_1 = \dfrac{\gamma}{\mu - | \lambda | b \eta S^- T}$.

Since $r_1 > 0$, therefore, $| \lambda | b \eta S^- T < | \mu |$.

Also, Eq. (6) leads to the boundedness of $L\chi$ where

$$\| L\chi \| \leq \frac{| \lambda | b \eta S^- T}{| \mu |} \| \chi \|. \tag{7}$$

Lemma 2. *The integral operator* L *of Eq.*(2) *is continuous in* $L_1[0, 1] \times C[0, T],\ T < 1$.

Proof. Let $\chi_1(u, t)$ and $\chi_2(u, t)$ be two functions in B_{r_1}, such that

$\| \chi_1 - \chi_2 \|_{L_1[0,1]\times C[0,T]} \le \delta(\varepsilon)$ for every $\varepsilon > 0$, then form Eq.(2) we find

$$|\mu| \| L\chi_1(u,t) - L\chi_2(u,t) \| \le |\lambda| \int_0^t \int_0^1 |\xi(t,\tau)| |\omega(u,v)\varphi(v,\tau,\chi(u,\tau))| dv d\tau$$

the above inequality after applying Hölder inequality and then the conditions (i)-(iii) takes the form

$$\| L\chi_1(u,t) - L\chi_2(u,t) \|_{L_1[0,1]\times C[0,T]} \le \varepsilon,\ \varepsilon = |\lambda| \eta S^{-} T\varepsilon_1 / |\mu| \tag{8}$$

hence the operator L is a continuous operator in B_{r_1}.

Lemma 3. *Under the weak noncompactness measure* β, *the operator L is a contraction.*

Proof. Let $X \subset B_{r_1}$, $\sigma > 0$ and $Y \times \Theta \subset L_1[0,1] \times C[0,T]$, such that $m(\Upsilon \times \Theta) \le \sigma$ then for any $\chi(u,t) \in X$ we have,

$$|\mu| \max_{0\le t\le T} \int_\Upsilon \int_\Theta |L\chi(u,t)| du dt$$

$$\le \max_{0\le t\le T} \int_\Upsilon \int_\Theta |q(u,t)| du dt$$

$$+ \max_{0\le t\le T} \int_\Upsilon \int_\Theta \left| \int_0^t \int_0^1 |\xi(t,\tau)| |\omega(u,v)\varphi(v,\tau,\chi(v,\tau))| dv d\tau \right| du dt$$

after using Lebesgue integral [ ]

$$\lim_{\sigma\to 0} \max_{0\le t\le T} \int_\Upsilon \int_\Theta \int_0^t \int_0^1 a(u,\tau) dv d\tau du dt = 0,$$

$$\lim_{\sigma\to 0} \max_{0\le t\le T} \int_\Upsilon \int_\Theta \int_0^t \int_0^1 |f(u,t)| du dt = 0$$

and the definition of De Blasi measure at weak noncompactness, β can be written by:

$$\beta(L\chi) \leq \alpha\beta(X),\ \alpha = |\lambda| b\eta S^{-}T / |\mu| < 1 \tag{9}$$

$$\beta(X) = \| \chi \|_{L_1(\Upsilon)\times C(\Theta)}$$

then L is contraction operator with respect to weak noncompactness β.

Lemma 4. $Y = \bigcap_{m\in\mathbb{N}} B_{r_1}$ *is non empty, closed, bounded convex and relatively weak compact.*

Proof. Let where $Conv(LB_{r_1})$ denoted the smallest closed convex set containing LB_{r_1}, i.e. $LB_{r_1} \subset Conv(LB_{r_1})$ and since $LB_{r_1} \subset B_{r_1}$, $L : B_{r_1} \to B_{r_1}$, then we obtain $Conv(LB_{r_1}) \subset B_{r_1}$, then we have $B^1_{r_1} \subset B_{r_1}$. Similarly, we have $B^2_{r_1} = Conv(LB^1_{r_1})$ and $Conv(LB^1_{r_1}) \subset B^1_{r_1} \to B^2_{r_1} \subset B^1_{r_1}$, then $B^3_{r_1} = Conv(LB^2_{r_1})$ and $Conv(LB^2_{r_1}) \subset B^2_{r_1} \to B^3_{r_1} \subset B^2_{r_1}$ and so on to have a sequence which is decreasing, bounded, convex, closed subset $B^m_{r_1}$ of B_{r_1} such that $LB^m_{r_1} \subset B^m_{r_1}$, $m \in \mathbb{N}$. Using the properties of the De Blasi measure of weak noncompactness β, we get $B^{m+1}_{r_1} = conv(LB^m_{r_1})$ $\to \beta(B^{m+1}_{r_1}) = \beta(conv(LB^m_{r_1})) \to \beta(B^{m+1}_{r_1}) = \beta(LB^m_{r_1})$. Hence, $\beta(B^{m+1}_{r_1})$ $\leq s\beta(B^m_{r_1})$, $m \in \mathbb{N}$. Repeat the process m times, we get $\beta(B^{m+1}_{r_1}) \leq s^{m+1}\beta(B_{r_1})$, $m \in \mathbb{N}$. Since $s < 1$ then $\lim_{m\to\infty} \beta(B^m_{r_1}) = 0$.

This implies that $Y = \bigcap_{m\in\mathbb{N}} B_{r_1}$ is a set nonempty, closed, bounded, convex and relatively weakly compact subset of B_{r_1}. It is also clear that $LY \subset Y$, as $LB^m_{r_1} \subset B^m_{r_1} \Rightarrow L(\bigcap_{m\in\mathbb{N}} B_{r_1}) \subset \bigcap_{m\in\mathbb{N}} B_{r_1} \Rightarrow LY \subset Y$. The lemma is proved.

Lemma 5. *For a subset* Υ *of* B_{r_1}, *we have* LY *is relatively compact.*

Proof. Let $\{\chi_n(u, t)\}$ be a sequences in Y and $\varsigma > 0$, then by using

Dragoni theorem, there exist a closed measurable subset $\Upsilon_\varsigma \times \Theta_\varsigma \subset L_1[0, 1] \times C[0, T]$, such that $m((\Upsilon_\varsigma \times \Theta_\varsigma)^c) < \varsigma$, $\gamma|_{[0,1]\times[0,T]\times\mathbb{R}}$, $v|_{\Theta_\varsigma\times[0,T]}$ and $\omega|_{\Upsilon_\varsigma\times[0,1]}$ are uniformly continuous.

Let

$$Q_n(u, t) = \int_0^t \int_0^1 \xi(t, \tau)\omega(u, v)\varphi(v, \tau, \chi_n(v, \tau))dvd\tau \tag{10}$$

then if $t_1, t_2 \in \Theta_\varsigma$, $\forall\, u \in \Upsilon_\varsigma$, we have

$$| \theta_n(u, t_1) - Q_n(u, t_2) |$$

$$\leq \int_0^{t_1}\int_0^1 | \xi(t_1, \tau) - \xi(t_2, \tau) || \omega(u, v) || \varphi(v, \tau, \chi_n(v, \tau)) | dvd\tau$$

$$+ \int_0^{t_2}\int_0^1 | \xi(t_2, \tau) || \omega(u, v) || \varphi(v, \tau, \chi_n(v, \tau)) | dvd\tau$$

since $\{\chi_n(u, t)\} \subset Y$ is bounded, then $\{\chi_n(u, t)\}$ is in the space $L_1[0, 1] \times C[0, T]$. Therefore, $\{Q_n\}$ is a sequence of an equicontinuous and uniformly bounded function in $C^0(\Upsilon_\varsigma \times \Theta_\varsigma)$. Hence, for $u \in \Upsilon_\varsigma$, $t \in \Theta_\varsigma$ we get $\max_{n\in\mathbb{N}} | Q_n(u, t) | = E$. Then $Q_n|_{\Upsilon_\varsigma\times\Theta_\varsigma}$ are uniformly continuous, so $\{L\chi_n(u, t)\}$ is a sequence of an equicontinuous and uniformly bounded function in $C^0(\Upsilon_\varsigma \times \Theta_\varsigma)$. By using Ascoli-Arzela theorem [26], we deduce that LY is relatively compact subset of $C^0(\Upsilon_\varsigma \times \Theta_\varsigma)$ then $\{L\chi_n(u, t)\}$ is Cauchy sequence in $C^0(\Upsilon_\varsigma \times \Theta_\varsigma)$.

Since LY is relatively compact subset of $C^0(\Upsilon_\varsigma \times \Theta_\varsigma)$, then $\{L\chi\}$ is uniformly integrable i.e. for a given $\varepsilon > 0$, there exist $\delta > 0$, such that,

$$\sup_{\chi} \max_{0\leq t\leq T} \int_{\Upsilon_\varepsilon}\int_{\Theta_\varepsilon} | L\chi(u, t) | dudt < \frac{\varepsilon}{4},\ m(\Upsilon_\varepsilon \times \Theta_\varepsilon) < \delta. \tag{11}$$

Choosing $r \in \mathbb{N}$ with $m(\Upsilon_\varepsilon \times \Theta_\varepsilon)^c < \delta,$ we have

$$\max_{0\le t\le T}\int_0^t\int_0^1 | L\chi_{n_1}(u, t) - L\chi_{n_2}(u, t) |dudt$$

$$\le \max_{0\le t\le T}\int_{\Upsilon^C}\int_{\Theta^C} | L\chi_{n_1}(u, t) - L\chi_{n_2}(u, t) |dudt$$

$$+ \max_{0\le t\le T}\int_{\Upsilon^C}\int_{\Theta^C} | L\chi_{n_1}(u, t) - L\chi_{n_2}(u, t) |dudt$$

since $\{L\chi_n\}$ is Cauchy sequence in $C^0(\Upsilon_\varsigma \times \Theta_\varsigma),$ we get

$$\max_{0\le t\le T}\int_0^t\int_0^1 | L\chi_{n_1}(u, t) - L\chi_{n_2}(u, t) |dudt \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \tag{12}$$

hence for large n_1, n_2 we deduce that $\{L\chi_n\}$ is Cauchy sequence in $L_1[0, 1]\times C[0, T],$ therefore $\{L\chi_n\}$ is relatively compact. In addition, LY is relatively compact in the space $L_1[0, 1]\times C[0, T].$ Finally, form the previous lemma and after using Schauder fixed point theorem we deduce that L has at least one fixed point $\chi(u, t) \in L_1[0, 1]\times C[0, T],$ which is the solution of Eq.(1).

3. System of Hammerstein Integral Equations (SHIEs)

To find SHIEs of the second type, a quadratic numerical method is utilized in this section.

For this, dividing the interval $[0, T]$ into s sub-intervals, $0 = t_0 < t_1 < \ldots < t_s = T$ where $t = t_i, \tau = t_j, i, j = 0, 1, 2, \ldots, S.$ Hence, the integral term of Eq.(1) becomes

$$\int_0^t\int_0^1 \xi(t, \tau)\omega(u, v)\varphi(v, \tau, \chi(v, \tau))dvd\tau$$

$$= \sum_{j=0}^{t} W_j\xi(t_i, t_j)\int_0^1 \omega(u, v)\varphi(v, t_j, \chi(v, t_j))dv + O(h_i^{p+1}),\ h \to 0,\ p > 0. \tag{13}$$

where $h_i = \max_{0 \le j \le m} a_j$, $a_j = t_{j+1} - t_j$, h_i the step size of integration and W_j are the weights such that $W_j = a/2,\ j = 0, i$ and $W_j = a,\ 0 < j < i,$ the value of i and p depend on the number of derivative of $\xi(t, \tau)$ with respect to t for all $\tau \in [0, T]$. Hence, $O(h_i^{p+1})$ is the order of sum of errors of the method.

After using Eq.(13), Eq.(1) yields a value that doesn't include $O(h_i^{p+1})$,

$$\mu\chi_i(u) = q_i(u) + \lambda \sum_{j=0}^{i} W_j \xi_{ij} \int_0^1 \omega(u, v)\varphi_j(v, \chi_j(v))dv,\ i = 0, 1, 2, \ldots, S \tag{14}$$

where,

$$\chi(u, t_i) = \chi_i(u),\ q(u, t_i) = q_i(u),\ \xi(t_i, t_j) = \xi_{ij},\ \varphi(v, t_j, \chi(v, t_j)) = \varphi_j(v, \chi_j(v)).$$

The formula (14) represents system of $(s+1)$ HIEs in position and its solution depends on the function $q_i(u)$, the kernel $\omega(u, v)$ and the degree of $\varphi_j(v, \chi_j(v))$.

Lemma 6. *The estimate local error* $R_{s,j}$ *of the quadrature numerical method is determined by the relation*,

$$R_{s,j} = \left| \int_0^t \int_0^1 \xi(t, \tau)\omega(u, v)\varphi(v, \tau, \chi(v, \tau, \chi(v, \tau)))dvd\tau \right.$$

$$\left. - \sum_{j=0}^{i} W_j \xi_{ij} \int_0^1 \omega(u, v)\varphi_j(v, \chi_j(v))dv \right|,\ i = 0, 1, 2, \ldots, S \tag{15}$$

4. The study of exist at least a solution of SHIEs

The existence of at least a solution of SHIEs (14) of the second kind, will be proved according to the Schauder fixed point theorem in the Banach space $L_1[0, 1]$.

For this aim write Eq.(14) in the operator form

$\mathcal{H}\chi_i(u) = \frac{1}{\mu}[q_i(u) + \lambda\mathcal{H}_1\chi_i(u)],$

$$\mathcal{H}_1\chi_i(u) = \sum_{j=0}^{i} W_j \xi_{ij} \int_0^1 \omega(u, v)\varphi_j(v, \chi_j(v))dv, \ \forall\, i \tag{16}$$

then assume the assumptions:

1. The given function $\varphi_j(v, \chi_j(v))$ satisfies the two assumptions:

1.1. For all integers i, $|\varphi_i(u, \chi_i(u))| \le \mathcal{X}_i(u) + b|\chi_i(u)|$, $A^* = |\mathcal{X}_i|$.

1.2. For every positive $\mathcal{E}_1^* < \mathcal{E}_2^*$ we can find $\delta(\mathcal{E}^*)$ in which,

1.3. $\int_0^1 |\varphi_i(u, \chi_{i1}(u)) - \varphi_i(u, \chi_{i2}(u))|du \le \mathcal{E}_1^*$, such that $\|\chi_{i1} - \chi_{i1}\| \le \delta(\mathcal{E}^*)$

2. $\sum_{j=0}^{i} |W_j \xi_{ij}| \le S^*$.

3. $\int_0^1 |q_i(u)|du \le N^*$.

Theorem 2. *In view of condition* (i) *of theorem* 1 *and conditions* (1)-(3) *the SHIEs* (14) *have at least a solution in* $L_1[0, 1]$ *under the relation,*

$$|\lambda| b\eta S^* < |\mu|$$

Lemma 7. *Under the conditions* (ii) *of theorem* 1 *and conditions* (1)-(2), *the operator* $\mathcal{H}$ *maps the ball* B_{r_2} *into itself.*

Proof. Let $\chi_i(u) \in B_{r_2}$, $B_{r_2} = \{\chi_i(u) \in R : \|\chi_i(u)\|_{L_1[0, 1]}\}$, r_2 is constant be nonempty, closed, bounded and convex subset.

Define the norm of the operator $\mathcal{H}\chi_i(u)$ in $L_1[0, 1]$ by

$$\|\mathcal{H}\chi_i(u)\|_{L_1[0, 1]} = \int_0^1 |\mathcal{H}\chi_i(u)|du, \ \forall\, i \tag{17}$$

In the light of Eq.(16), after applying Hölder inequality and integrating, we obtain

$$\int_0^1 |\mathcal{H}\chi_i(u)|du \le \frac{1}{|\mu|}\left\{\int_0^1 |q_i(u)|du\right.$$

$$\left. + |\lambda|\sum_{j=0}^{i} W_j\xi_{ij}\int_0^1\int_0^1 |\omega(u, v)|dudv\int_0^1 |\varphi_j(v, \chi_j(v))|dv\right\}$$

with the aid of above conditions, we get

$$\|\mathcal{H}\chi_i(u)\|_{L_1[0, 1]} \le \mathcal{F}/|\mu| + g\|\chi_i(u)\|_E,\ \mathcal{F} = N^* + |\lambda|A^*b\eta,\ g = \frac{|\lambda|b\eta S^*}{|\mu|}. \tag{18}$$

hence from inequality (18), the operator $\mathcal{H}$ maps B_{r_2} into itself, where

$$r_2 = \frac{\mathcal{F}}{|\mu| - |\lambda|b\eta S^*}. \tag{19}$$

since $r_2 > 0$ then $|\lambda|b\eta S^* < |\mu|$, therefore we have $g < 1$. Moreover, the inequality (18) leads to the boundedness of $\mathcal{H}$.

Lemma 8. *If the assumption* (ii) *of theorem* 1, *and assumptions* (1), (2) *are satisfied then* $\mathcal{H}$ *is continuous in* B_{r_2}.

Proof. Let $\chi_{i1}(u)$ and $\chi_{i2}(u)$ be two functions in B_{r_2}, and from Eq.(16) we find

$$\|\mathcal{H}\chi_{i1}(u) - \mathcal{H}\chi_{i2}(u)\|$$

$$\le \frac{|\lambda|}{|\mu|}\left|\sum_{j=0}^{i} W_j\xi_{ij}\int_0^1 \omega(u, v)(\varphi_j(v, \chi_{j1}(v)) - \varphi_j(v, \chi_{j2}(v))dv\right|$$

applying Hölder inequality and then using the conditions (1) and (ii) of theorem 1 to get

$$\|\mathcal{H}\chi_{i1}(u) - \mathcal{H}\chi_{i2}(u)\|_{L_1[0, 1]} < \mathcal{E}^*,\ \mathcal{E}^* = |\lambda|E_1^*\eta S^*/|\mu| \tag{20}$$

the last inequality shows that the operator $\mathcal{H}$ is continuous operator in B_{r_2}.

Lemma 9. *The operator* $\mathcal{H}$ *is compact in the ball* B_{r_2}.

Proof. Since, $B_{r_2} \subseteq L_1[0, 1]$, B_{r_2} is bounded in $L1[0,1]$. Therefore $\mathcal{H}B_{r_2}$ in $L_1[0, 1]$ is bounded. Then we will show that $(\mathcal{H}\chi_i(u))_n \to (\mathcal{H}\chi_i(u))$ in $L_1[0, 1]$. Let $\chi_i(u) \in B_{r_2}$, then

$$\| (\mathcal{H}\chi_i(u))_n - (\mathcal{H}\chi_i(u)) \|_{L_1[0,1]} = \int_0^1 | (\mathcal{H}\chi_i(u))_n - (\mathcal{H}\chi_i(u)) | du$$

$$= \int_0^1 \left| \frac{1}{n} \int_u^{u+n} (\mathcal{H}\chi_i(v))_n - (\mathcal{H}\chi_i(u)) \right| dv du \qquad (21)$$

since,

$$\lim_{n\to 0} \frac{1}{n} \int_u^{u+n} | (\mathcal{H}\chi_i(v))_n - (\mathcal{H}\chi_i(u)) | dv = 0 \to \lim_{n\to 0} \| (\mathcal{H}\chi_i)_n - (\mathcal{H}\chi_i) \|_{L_1[0,1]} = 0.$$

This implies that, $(\mathcal{H}\chi_i)_n \to (\mathcal{H}\chi_i)$ uniformly as $n \to 0$. Then $\mathcal{H}B_{r_2}$ is relatively compact. Since B_{r_2} is bounded, hence $\mathcal{H}$ is compact operator.

According to previous lemmas 8 and 9, the operator defined by Eq.(18) is continuous and compact, and has the ability to map a closed convex set B_{r_2} into itself. Hence by theorem fixed point of Schauder, the operator $\mathcal{H}$ has at least one fixed point in B_{r_2}, therefore Eq.(14) has at least one solution $\chi_i(u) \in L_1[0, 1]$. Hence the proof of theorem 2 is obtained. Also, for $s \to \infty$, then

$$\sum_{j=0}^{s} W_j \xi_{ij} \int_0^1 \omega(u, v) \varphi_j(v, \chi_j(v)) dv \to \int_0^t \int_0^1 \xi(t, \tau) \omega(u, v) \varphi(v, \tau, \chi(v, \tau)) dv d\tau$$

Thus, the solution of SHIEs (14) becomes the solution of Eq.(1). 14

Theorem 3. *Under the conditions of theorem* 2, *and if, in the space* $L_1[0, 1]$ *the sequence* $\{P_s\} = \{(q_i(u))_s\}$ *converges uniformly to the function* $P = \{q_i(u)\}$, *then the sequence of functions* $\{X_s\} = \{(X_i(u))_s\}$ *of Eq.*(14) *converges uniformly to the solution* $X = \{X_i(u)\}$ *of Eq.*(14) *in the same space.*

Proof. From Eq.(14), we write

$$\| \chi_i(u) - (\chi_i(u))_s \| \leq \frac{1}{|\mu|} \{\| q_i(u) - (q_i(u))_s |$$

$$+ |\lambda|| \sum_{j=0}^{i-1} W_j \xi_{ij} \int_0^1 W(u, v)(\varphi_i(v, \chi_i(u)) - \varphi_i(v, (\chi_i(v))_s dv |\}$$

After applying Hölder inequality, integrating both sides and using the conditions of theorem 2, we get

$$\| \chi - \chi_s \|_{L_1[0, 1]} \leq \frac{1}{|\mu|} \| P - P_s \|_{L_1[0, 1]} + \mathcal{E}, \mathcal{E} = |\lambda| \eta S^* \varepsilon_1 /|\mu| \tag{22}$$

Since, $\| P - P_s \|_{L_1[0, 1]} \to 0$ as $s \to \infty$, then $\chi \to \chi_s$ uniformly when $s \to \infty$ in the space $L_1[0, 1]$.

5. Numerical Methods

The Collocation method and the Galerkin method are used to solve SHIEs with continuous kernel in this section.

5.1 The Collocation method

Here, we use the Collocation method to obtain the numerical solution of SHIEs of the second kind with continuous kernel. The idea of this method is to transform it to SHIEs in terms of the linear combination coefficients appearing in the representation of the solution $\chi(u, t_i)$ in Eq. (14) by a partial sum,

$$G(u, t_i) = \sum_{k=0}^{m} c_k(t_i) \chi_k(u) \tag{23}$$

of $m + 1$ linearly independent functions $\chi_0(u), \chi_1(u), \ldots, \chi_m(u)$ on the interval $[0, 1]$, therefore, we have

$$\mu G_i(u) = q_i + \lambda \sum_{j=0}^{t} W_j \xi_{ij} \int_0^1 \omega(u, v) \varphi_j(v, G_i(v, \chi_j(v)) dv$$

$$+ E(u, c_0(t_i), c_1(t_i), \ldots, c_m(t_i)), i = 0, 1, 2, \ldots, s \tag{24}$$

Since, the error in Eq.(25) vanishes at $m+1$ point $u_0, u_1, \ldots, u_m$ then, we get

$$\mu \sum_{k=0}^{m} c_k(t_i)\chi_k(u) - \lambda \sum_{j=0}^{i} W_j \xi_{ij} \int_0^1 \omega(u, v)\varphi_j\left(v, \sum_{k=0}^{m} c_k(t_j)\chi_k(v)\right) dv$$

$$0 \le i \le s, \; 0 \le n \le m. \tag{25}$$

5.2. Galerkin method

In this method the error in Eq.(14) is orthogonal to $m+1$ given linear independent functions on $G_0(u), G_1(u), \ldots, G_m(u)$ the interval $(0, 1)$. Then From the definition of orthogonally on $E(u, c_0(t_i), c_1(t_i), \ldots, c_m(t_i))$ in Eq.(14), we get

$$\int_0^1 G_n(u)\left[\mu \sum_{k=0}^{m} c_k(t_i)\chi_k(u) - \lambda \sum_{j=0}^{i} W_j \xi_{ij} \int_0^1 \omega(u, v)\varphi_j\left(v, \sum_{k=0}^{m} c_k(t_j)\chi_k(v)\right) dv\right] du$$

$$= \int_0^1 G_n(u) q_i(u) du, \; 0 \le i \le s, \; 0 \le n \le m.$$

6. Numerical Applications

The Collocation method and Galerkin method are used to discuss the numerical solution for Eq.(1) with continuous kernels. using Mable 22 program. Studying the solution included two aspects regarding the difference in the value of time.

Application 1. Consider the following NMIE,

$$\chi(u, t) = q(u, t) + 0.01 \int_0^t \int_0^1 \tau^2 \cos v \chi^2(v, \tau) dv d\tau, \; 0 \le t \le T < 1,$$

the exact solution is $\chi(u, t) = t^2 \exp(x)$.

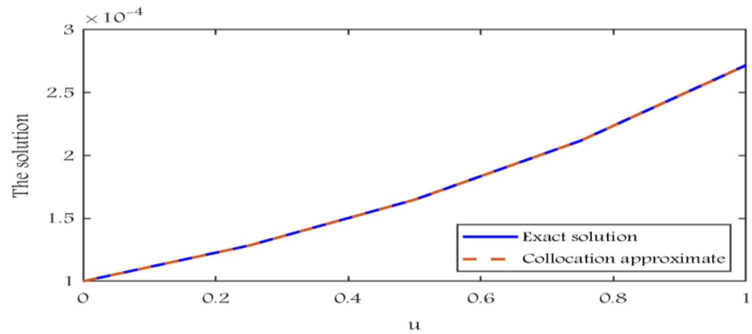

Figure 1. The relation between the collocation approximate at $T = 0.01$ and the exact solution.

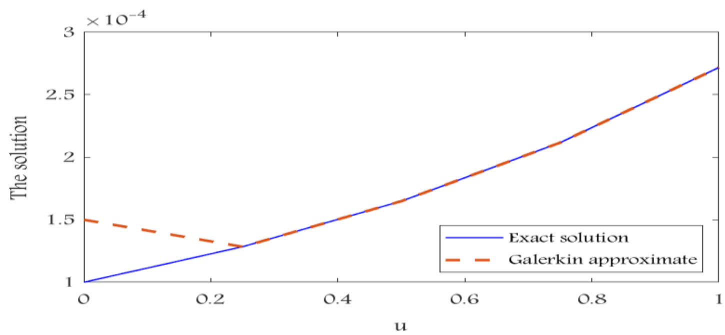

Figure 2. The relation between the Galerkin approximate at $T = 0.01$ and the exact solution.

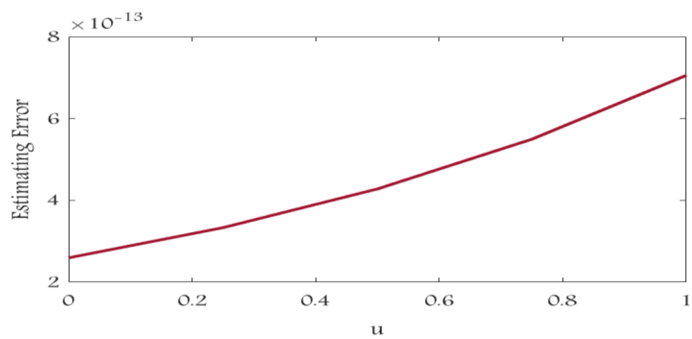

Figure 3. The Collocation estimating error at $T = 0.01$

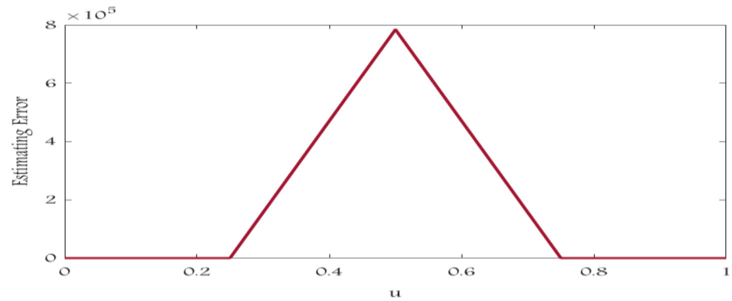

Figure 4. The Galerkin estimating error at $T = 0.01$

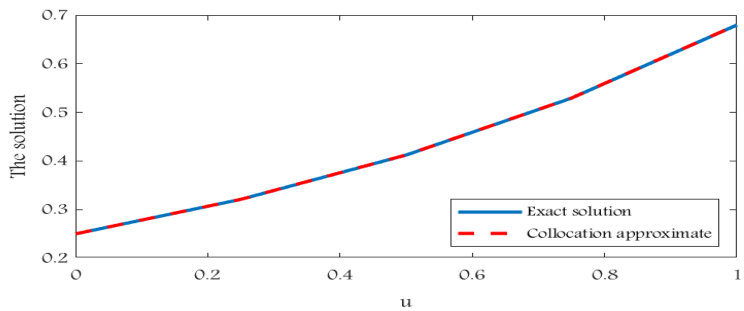

Figure 5. The relation between the collocation approximate at $T = 0.5$ ant the exact solution.

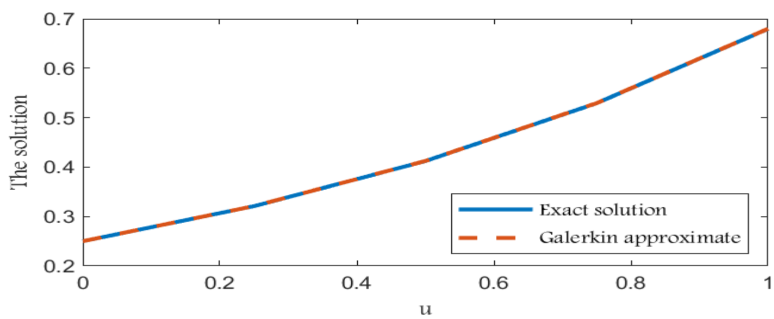

Figure 6. The relation between the Galerkin approximate at $T = 0.5$ and the exact solution.

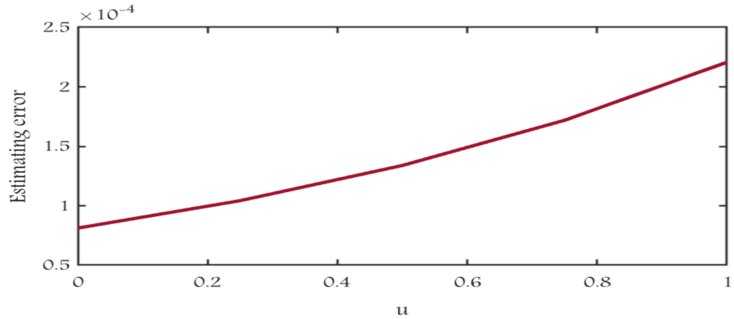

Figure 7. The Collocation estimating error at $T = 0.5$

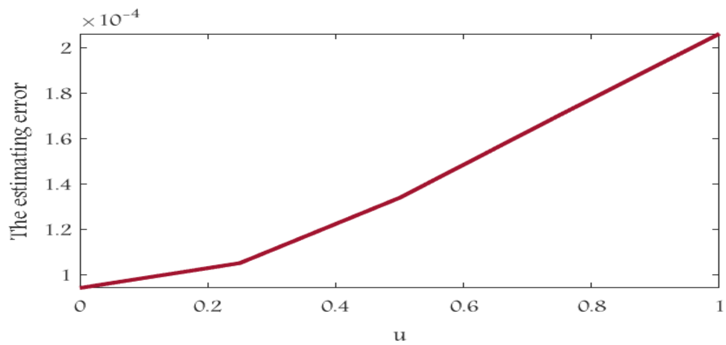

Figure 8. The Galerkin estimating error at $T = 0.5$

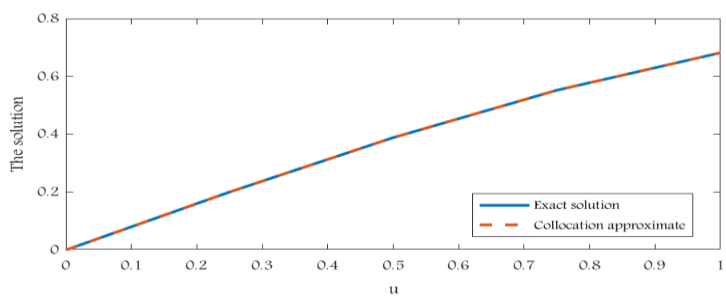

Figure 9. The relation between the collocation approximate at $T = 0.9$ and the exact solution.

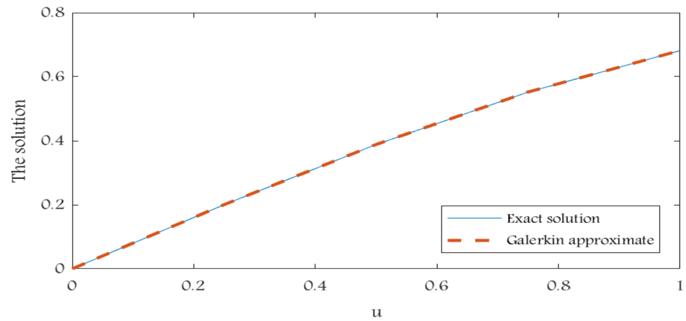

Figure 10. The relation between the Galerkin approximate at $T = 0.9$ and the exact solution.

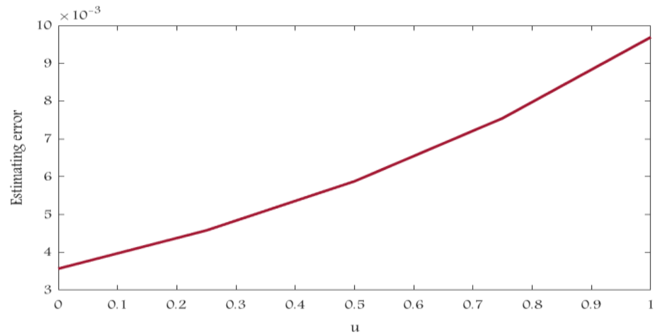

Figure 11. The Collocation estimating error at $T = 0.9$

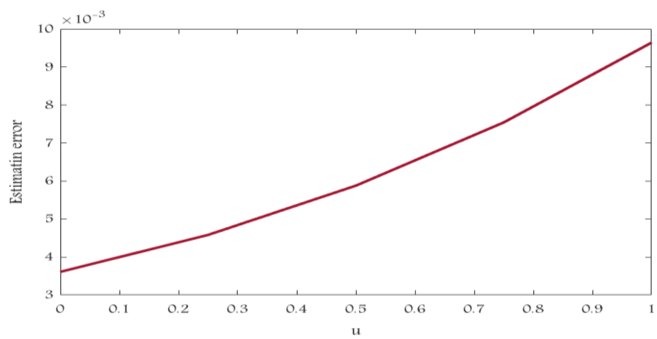

Figure 12. The Galerkin estimating error at $T = 0.9$

Application 2. Consider the NMIE,

$$\chi(u, t) = q(u, t) + 0.01\int_0^t \int_0^1 \tau^2 \cos v \chi^2(v, \tau) dv d\tau, \; 0 \le t \le T < 1,$$

exact solution is, $\chi(u, t) = t^2 \sin u.$

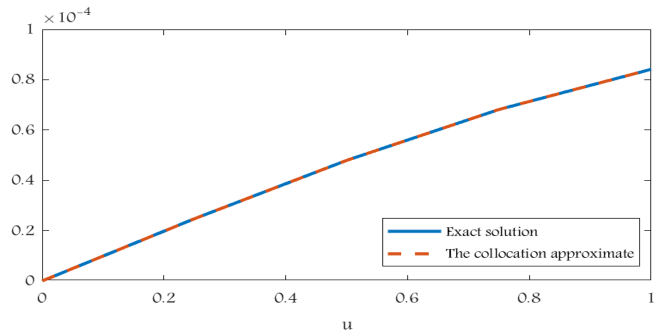

Figure 13. The relation between the collocation approximate at $T = 0.01$ and the exact solution.

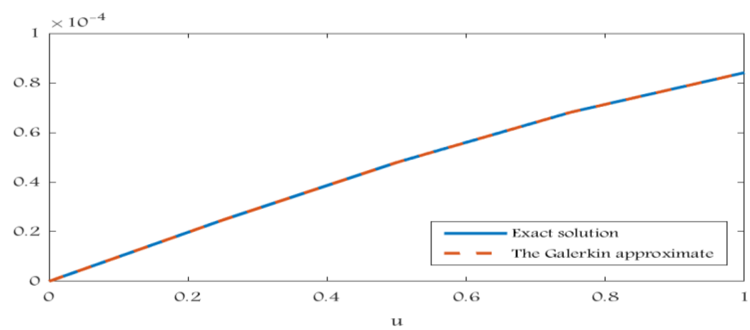

Figure 14. The relation between the Galerkin approximate at $T = 0.01$ and the exact solution.

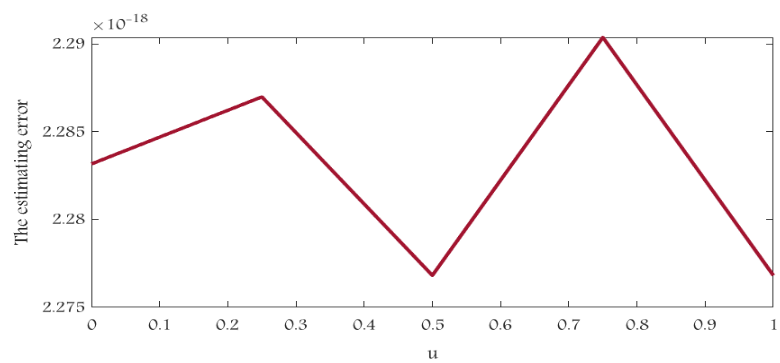

Figure 15. The Collocation estimating error at $T = 0.01$

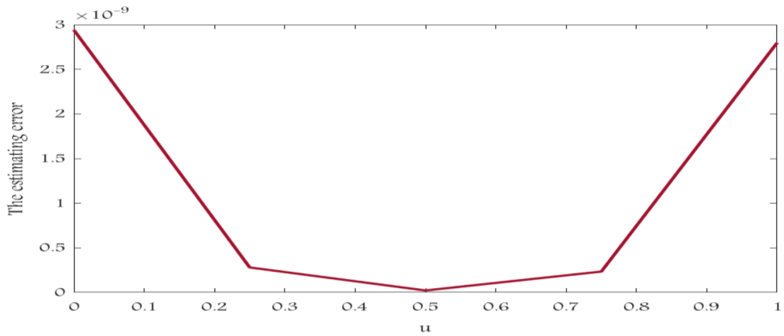

Figure 16. The Galerkin estimating error at $T = 0.01$

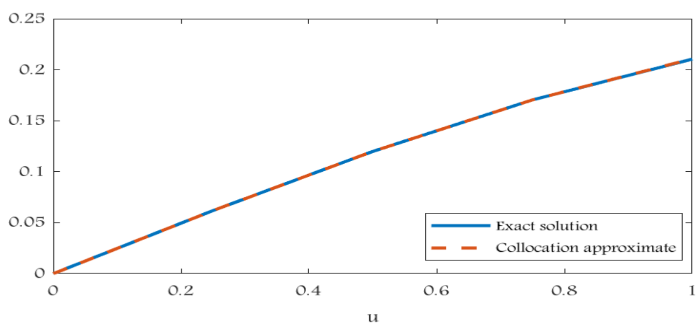

Figure 17. The relation between the collocation approximate at $T = 0.5$ and the exact solution.

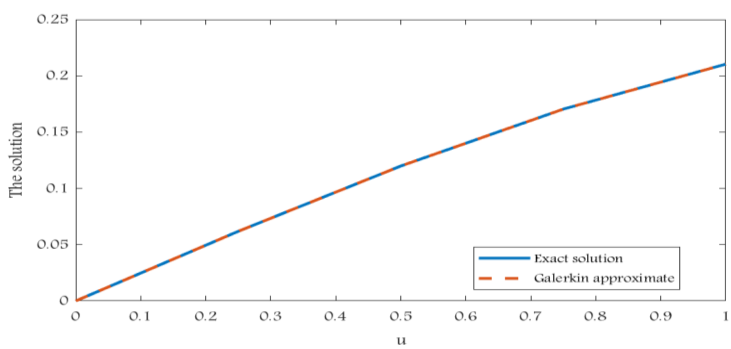

Figure 18. The relation between the Galerkin approximate at $T = 0.5$ and the exact solution.

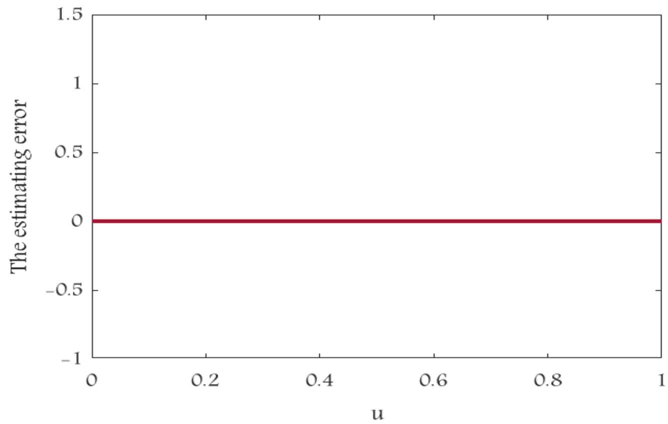

Figure 19. The Collocation estimating error at $T = 0.5$

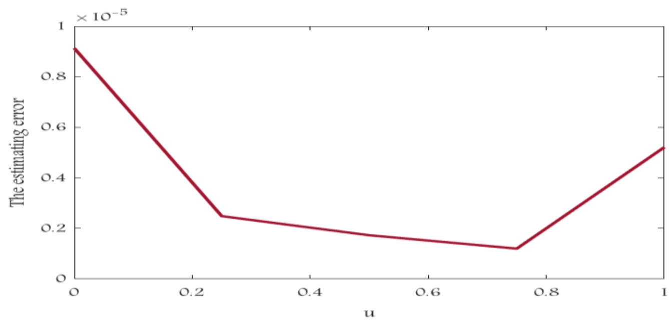

Figure 20. The Galerkin estimating error at $T = 0.5$

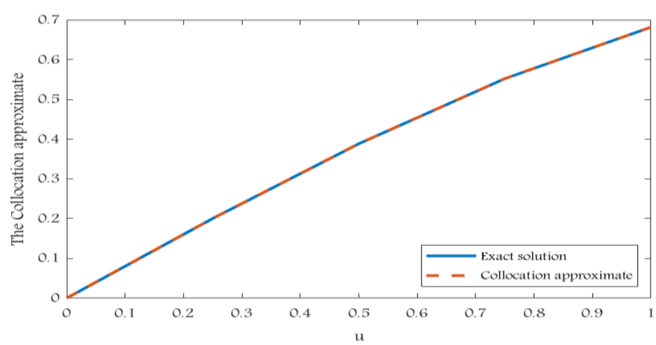

Figure 21. The relation between the collocation approximate at $T = 0.9$ and the exact solution.

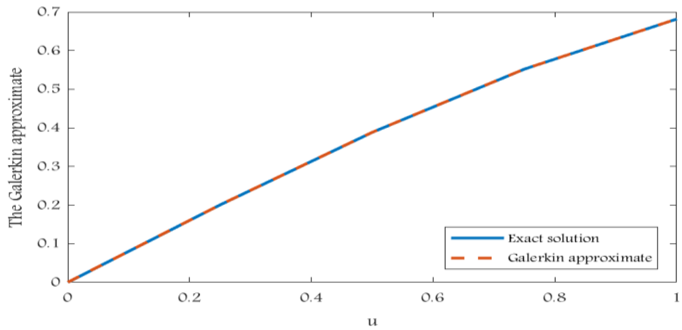

Figure 22. The relation between the Galerkin approximate at $T = 0.9$ and the exact solution.

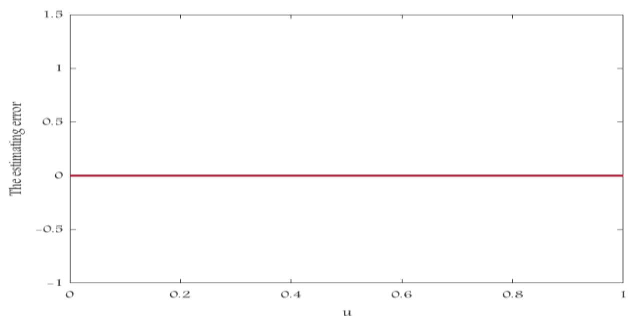

Figure 23. The Collocation estimating error at $T = 0.9$

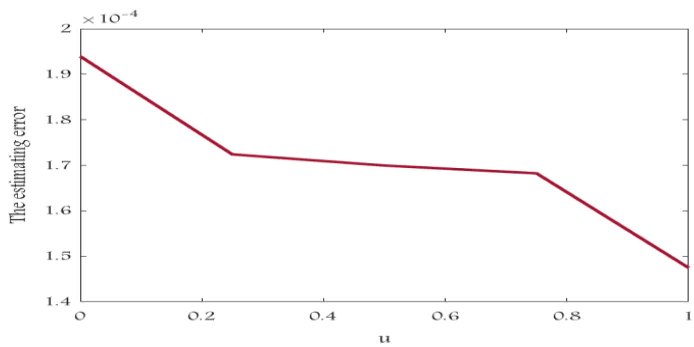

Figure 24. The Galerkin estimating error at $T = 0.9$

Conclusion

The research is a completed study that includes a theoretical side and is supported by mathematical applications. This work gives a sophisticated study that focuses on the properties and stability of the solution of the nonlinear mixed integral equation so as to ensure that there is at least a single solution to it and a deep understanding of the way to behave with modified systems from a wide range of scientific and engineering disciplines. We focus on the study of nonlinear mixed-integral equations. The quadratic numerical method is a powerful mathematical tool that allows us to transform nonlinear mixed integral equations in position and time into a system of nonlinear integral equations in position only. This technique enables us to handle the integral equations more efficiently. The collocation method and the Galerkin method are numerical methods used to solve the system of nonlinear integral equations with high accuracy and computational efficiency. Using the two methods, we compute and plot the error estimates associated with the proposed integral equation. We can establish the following:

- The absolute value of the error increases as the time value T, $0 < T < 1$ increases in both study cases for the two methods used in the two applications.
- The error of the Galerkin method is close to the error of the collocation method.

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