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MORE COMPARISON RESULTS FOR PROPER WEAK SPLITTINGS OF RECTANGULAR MATRICES

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Abstract

The iterative solution of a rectangular linear system of equations of the form $Ax = b$ may be found using the theory of matrix splittings. In order to improve the rate of convergence of such an iterative method, several comparison results for different classes of proper splitting have been proved in the literature. In this article, we also establish a few comparison results for different proper weak splittings for rectangular matrices by extending the work of Cao et al. [Cao, Z. H.; Wu, H. B.; Liu, Z., A note on weak splitting of matrices, Appl. Math. Comput. 112 (2000), 265-275].

1. Introduction

Consider a linear system

$$Ax = b,\ A \in \mathbb{R}^{m \times n},\ x \in \mathbb{R}^n \text{ and } b \in \mathbb{R}^m, \tag{1.1}$$

where $A \in \mathbb{R}^{m \times n}$ is a given matrix, $x \in \mathbb{R}^n$ is the unknown vector and $b \in \mathbb{R}^m$ is a given vector. When the coefficient matrix A is very large and sparse, iterative methods become more efficient. In order to find an iterative solution of (1.1), in [3], by considering $A = M - N$ is a proper splitting, the

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authors introduced the following iterative method:

$$x^{k+1} = M^{\dagger}Nx^{k} + M^{\dagger}b,\ k = 0, 1, 2, \ldots, \tag{1.2}$$

where $M^{\dagger}N$ is the iteration matrix and $M^{\dagger}$ is the Moore-Penrose inverse of M [20]. The same authors also proved that the sequence defined in (1.2) converges to $A^{\dagger}b$ for any initial guess x^0, if and only if $\rho(M^{\dagger}N)$, the spectral radius of the iteration matrix $M^{\dagger}N$ is less than one. (see Corollary 1, [3], for instance). Therefore, the rate of convergence of the iterative method (1.2) depends on $\rho(M^{\dagger}N)$ and so, the spectral radius of the iteration matrix is crucial in comparing the rate of convergence of various iterative methods for the same system. In this context, many comparison theorems are proved in the literature; see, e.g., [1, 3, 5, 6, 7, 8, 10, 11, 15, 16, 18, 19, 22].

The major goal of this article is to provide more comparison results for the proper weak splitting of type I and type II [6, 11, 17]. To this end, the article is organized as follows. Section 2 introduces notations, definitions and a few preliminary results that are commonly utilized in deriving the main results. In Section 3, we derive several comparison results for proper weak splittings.

2. Prerequisites

Throughout the article, $\mathbb{R}^{m\times n}$, A^T, $R(A)$, and $N(A)$ denote, the set of all real matrices of order $m \times n$, the transpose, the range space, and the null space of $A \in \mathbb{R}^{m\times n}$, respectively. If $T \oplus S = \mathbb{R}^n$, then $P_{T,S}$ is a projection onto T along S. Then $P_{T,S}A = A$ if and only if $R(A) \subseteq T$ and $AP_{T,S} = A$ if and only if $N(A) \supseteq S$. If $T \perp S$, then $P_{T,S}$ will be denoted by P_T. The spectral radius of $A \in \mathbb{R}^{n\times n}$, denoted by $\rho(A)$ is defined by $\rho(A) = \max_{1\le i\le n} |\lambda_i|$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A. It is well known that $\rho(AB) = \rho(BA)$, where A and B are two matrices such that AB and BA are defined. A matrix $A \in \mathbb{R}^{n\times n}$, is said to be convergent if $\lim_{n\to\infty} A^k = O$, where

O is the null matrix. It is well known that $A \in \mathbb{R}^{n \times n}$ is convergent if and only if $\rho(A) < 1$. $A \in \mathbb{R}^{m \times n}$ is called non-negative if $A \geq 0$, where $A \geq 0$ means each entry of A is non-negative For $A, B \in \mathbb{R}^{m \times n}$, $A \leq B$ means $B - A \geq 0$. Similarly, $B > 0$ means all the entries of B are positive. By e_i, we denote the i-th column of an identity matrix I of appropriate order. The Moore-Penrose inverse of $A \in \mathbb{R}^{m \times n}$ is the unique matrix $X \in \mathbb{R}^{n \times m}$ which satisfies the following matrix equations: $AXA = A$, $XAX = X$, $(AX)^T = AX$ and $(XA)^T = XA$. It always exists, and is denoted by $A^{\dagger}$. The Moore-Penrose inverse of a nonsingular matrix is same as the ordinary inverse. The following properties of $A^{\dagger}$ are well-known [9]: $R(A^{\dagger}) = R(A^T)$; $N(A^{\dagger}) = N(A^T)$; $AA^{\dagger} = P_{R(A)}$; $A^{\dagger}A = P_{R(A^T)}$. In particular, if $x \in R(A^T)$, then $x = A^{\dagger}Ax$. These will be used frequently in our main results. For more details one may refer to [2].

2.1. Proper splittings. Let $A \in \mathbb{R}^{m \times n}$. Then the splitting $A = M - N$ is called a proper splitting if $R(M) = R(A)$ and $N(M) = N(A)$ (see [3] for instance).

Following that, we will gather a few properties of a proper splitting.

Theorem 2.1. (Theorem 1, [3]). *Let* $A = M - N$ *be a proper splitting of* $A \in \mathbb{R}^{n \times m}$. *Then*

(a) $A = M(I - M^{\dagger}N)$;

(b) $I - M^{\dagger}N$ *is nonsingular*;

(c) $A^{\dagger} = (I - M^{\dagger}N)^{-1}M^{\dagger}$.

Theorem 2.2. (Theorem 1, [6]). *Let* $A = M - N$ *be a proper splitting of* $A \in \mathbb{R}^{n \times m}$. *Then*

(a) $A = (I - M^{\dagger}N)M$;

(b) $I - NM^{\dagger}$ *is nonsingular*;

(c) $A^{\dagger} = M^{\dagger}(I - NM^{\dagger})^{-1}$.

Different subclasses of a proper splitting are recalled next.

Definition 2.3. A proper splitting $A = M - N$ of $A \in \mathbb{R}^{m \times n}$ is called:

(a) convergent if and only if $\rho(M^{\dagger}N) < 1$;

(b) a proper weak splitting of type I, if $M^{\dagger}N \geq 0$ [11];

(c) a proper weak splitting of type II, if $NM^{\dagger} \geq 0$ [6].

3. Main Results

In this section, we will prove some comparison results. Before that we will begin with the following result.

Theorem 3.1. *Let $A = M_1 - N_1$ be a convergent proper weak splitting of type* II *and $A = M_2 - N_2$ be a convergent proper weak splitting of type* I *of $A \in \mathbb{R}^{m \times n}$.*

Then

(i) *If $M_1^{\dagger} \geq M_2^{\dagger}$, then $A^{\dagger}N_2A^{\dagger} \geq A^{\dagger}N_1A^{\dagger}$. In particular, if $M_1^{\dagger} > M_2^{\dagger}$, then $A^{\dagger}N_2A^{\dagger} > A^{\dagger}N_2A^{\dagger}$.*

(ii) *If $A^{\dagger}N_2A^{\dagger} \geq A^{\dagger}N_1A^{\dagger}$, then $(A^{\dagger}N_2)^p A^{\dagger} \geq (A^{\dagger}N_1)^p A^{\dagger}$ for all positive integers $p > 1$.*

In particular, if $A^{\dagger}N_2A^{\dagger} > A^{\dagger}N_1A^{\dagger}$, then for all positive integers $p > 1$, $(A^{\dagger}N_2)^p A^{\dagger} > (A^{\dagger}N_1)^p A^{\dagger}$.

Proof. (i) Since $\rho(M_1^{\dagger}N_1) < 1$ and $\rho(M_2^{\dagger}N_2) < 1$, it follows from [21, Theorem 3.15] that $(I - N_1M_1^{\dagger})^{-1} \geq 0$ and $(I - N_2M_2^{\dagger})^{-1} \geq 0$. Therefore, we have

$$A^{\dagger}N_2A^{\dagger} - A^{\dagger}N_1A^{\dagger} = A^{\dagger}(N_2 - N_1)A^{\dagger}$$

$$= A^{\dagger}(M_2 - M_1)A^{\dagger}$$

$$= A^{\dagger}(M_2 M_1^{\dagger} M_1 - M_2 M_2^{\dagger} M_1)A^{\dagger}$$

$$= A^{\dagger} M_2 (M_1^{\dagger} - M_2^{\dagger}) M_1 A^{\dagger}$$

$$= (I - M_2^{\dagger} N_2^{\dagger})^{-1} M_2^{\dagger} M_2 (M_1^{\dagger} - M_2^{\dagger}) M_1 M_1^{\dagger} (I - N_1 M_1^{\dagger})^{-1}$$

$$= (I - M_2^{\dagger} N_2^{\dagger})^{-1} (M_1^{\dagger} - M_2^{\dagger}) (I - N_1 M_1^{\dagger})^{-1} \geq 0.$$

Moreover, if $M_2^{\dagger} > M_2^{\dagger}$, then $A^{\dagger} N_2 A^{\dagger} - A^{\dagger} N_1 A^{\dagger} > 0$.

(ii) It suffices to prove that it holds for $p = 2$. So,

$$(A^{\dagger} N_2)^2 A^{\dagger} = A^{\dagger} N_2 A^{\dagger} N_2 A^{\dagger}$$

$$\geq A^{\dagger} N_2 A^{\dagger} N_1 A^{\dagger}$$

$$\geq A^{\dagger} N_1 A^{\dagger} N_1 A^{\dagger}$$

$$= (A^{\dagger} N_1)^2 A^{\dagger}.$$

Now assume that $A^{\dagger} N_2 A^{\dagger} > A^{\dagger} N_1 A^{\dagger}$. We will prove that $(A^{\dagger} N_2)^2 A^{\dagger} > A^{\dagger} N_2 A^{\dagger} N_1 A^{\dagger}$. The proof is by contradiction. If $(A^{\dagger} N_2)^2 A^{\dagger} > A^{\dagger} N_2 A^{\dagger} N_1 A^{\dagger}$ does not hold, then there exists e_i and e_j such that $e_i^T (A^{\dagger} N_2)^2 A^{\dagger} e_j = e_i^T A^{\dagger} N_2 A^{\dagger} N_1 A^{\dagger} e_j$, i.e., $e_i^T A^{\dagger} N_2 (A^{\dagger} N_2 A^{\dagger} - A^{\dagger} N_1 A^{\dagger}) e_j = 0$. But since $A^{\dagger} N_2 A^{\dagger} - A^{\dagger} N_1 A^{\dagger} > 0$, this leads to $e_i^T A^{\dagger} N_2 = 0$. Thus, we have $A^{\dagger} N_2 A^{\dagger} > A^{\dagger} N_1 A^{\dagger}$, which implies $e_i^T A^{\dagger} N_2 > e_i^T A^{\dagger} N_1 A^{\dagger}$. Since $e_i^T A^{\dagger} N_2 A^{\dagger} = 0$ we have $e_i^T A^{\dagger} N_1 A^{\dagger} < 0$, which is a contradiction. Thus $(A^{\dagger} N_2)^2 A^{\dagger} > (A^{\dagger} N_1)^2 A^{\dagger}$. □

The aforementioned theorem is illustrated in the following example.

Example 3.2. Let $A = \begin{pmatrix} 1 & -3/2 & 1 \\ -4/3 & 7/3 & -4/3 \end{pmatrix} = M_1 - N_1 = M_2 - N_2,$

where $M_1 = \begin{pmatrix} 2 & -3 & 2 \\ -4 & 7 & -4 \end{pmatrix}$, $N_1 = \begin{pmatrix} 1 & -3/2 & 1 \\ -8/3 & 14/3 & -8/3 \end{pmatrix}$,

$M_2 = \begin{pmatrix} 3 & -2 & 2 \\ -3 & 7 & -3 \end{pmatrix}$, $N_2 = \begin{pmatrix} 2 & -2 & 3 \\ -5/3 & 14/3 & -5/3 \end{pmatrix}$. Then, $A = M_1 - N_1$ is a convergent proper weak splitting of type I and $A = M_2 - N_2$ is a convergent proper weak splitting of type II, respectively. We also have

$$M_1^{\dagger} = \begin{pmatrix} 7/4 & 3/4 \\ 2 & 1 \\ 7/4 & 3/4 \end{pmatrix} \geq \begin{pmatrix} 7/30 & 1/15 \\ 1/5 & 1 \\ 7/30 & 1/15 \end{pmatrix} = M_2^{\dagger},\ A^{\dagger}N_2A^{\dagger}$$

$$= \begin{pmatrix} 231/4 & 327/8 \\ 2 & 99/8 \\ 231/4 & 327/8 \end{pmatrix} \geq \begin{pmatrix} 7/2 & 9/2 \\ 4 & 6 \\ 7/2 & 9/2 \end{pmatrix} = A^{\dagger}N_1A^{\dagger} \text{ and } (A^{\dagger}N_2)^2A^{\dagger}$$

$$= \begin{pmatrix} 7833/8 & 11181/16 \\ 2349/2 & 3357/4 \\ 7833/8 & 11181/16 \end{pmatrix} \geq \begin{pmatrix} 7/2 & 9 \\ 4 & 12 \\ 7/2 & 9 \end{pmatrix} = (A^{\dagger}N_1)^2A^{\dagger}.$$

Now we have the following comparison results for proper weak splittings.

Theorem 3.3. *Let* $A = M_1 - N_1 = M_2 - N_2$ *be convergent weak splittings of both types such that* $A^{\dagger}N_2A^{\dagger} > A^{\dagger}N_1A^{\dagger}$. *Let* u *and* v *be nonnegative vectors such that* $A^{\dagger}N_1A^{\dagger}N_2u = \rho(A^{\dagger}N_1A^{\dagger}N_2)u$ *and* $A^{\dagger}N_2A^{\dagger}N_1v = \rho(A^{\dagger}N_2A^{\dagger}N_1)v$. *If* $N_2u \geq 0$, $N_1v \geq 0$ *with* $v > 0$, *and* $N_2u \neq 0$ *or* $N_1v \neq 0$, *then* $\rho(M_1^{\dagger}N_1) < \rho(M_2^{\dagger}N_2) < 1$.

Proof. Assume that $N_2u \geq 0$, $N_2u \neq 0$. Then $(A^{\dagger}N_2)^2u = A^{\dagger}N_2A^{\dagger}N_2u > A^{\dagger}N_1A^{\dagger}N_2u = \rho(A^{\dagger}N_1A^{\dagger}N_2)u$. Hence, by [4, Theorem 2.1.11], we have $(\rho(A^{\dagger}N_2))^2 > \rho(A^{\dagger}N_1A^{\dagger}N_2)$, i.e., $\rho(A^{\dagger}N_1A^{\dagger}N_2) < (\rho(A^{\dagger}N_2))^2$ which implies that $(\rho(A^{\dagger}N_1))^2 \leq \rho(A^{\dagger}N_1A^{\dagger}N_2) < (\rho(A^{\dagger}N_2))^2$. Hence, $\rho(A^{\dagger}N_1) < \rho(A^{\dagger}N_2)$. By [11, Lemma 3.5], we have $\rho(M_1^{\dagger}N_1) < \rho(M_2^{\dagger}N_2) < 1$. Similarly, when $N_1v \geq 0$ with $v > 0$ holds, we can prove the required inequality. □

Theorem 3.4. *Let* $A = M_1 - N_1 = M_2 - N_2$ *be convergent weak splittings of both types and* $(A^{\dagger}N_2)^p A^{\dagger} \geq (A^{\dagger}N_1)^p A^{\dagger}$ *for some positive integer p. Let u and u be nonnegative vectors such that* $(A^{\dagger}N_1)^p (A^{\dagger}N_2)^p u = \rho((A^{\dagger}N_2)^p (A^{\dagger}N_2)^p)u$ *and* $(A^{\dagger}N_2)^p (A^{\dagger}N_1)^p v = \rho((A^{\dagger}N_2)^p (A^{\dagger}N_2)^p)v$, *respectively. If* $N_2 u \geq 0$, $N_1 v \geq 0$ *with* $v > 0$, *then* $\rho(M_1 N_1) \leq \rho(M_2 N_2) < 1$.

Proof. Since $N_2 M_2^{\dagger} \geq 0$ and $\rho(N_2 M_2^{\dagger}) < 1$, by [21, Theorem 3.15] we have $N_2 A^{\dagger} = N_2 M_2^{\dagger}(I - N_2 M_2^{\dagger})^{-1} \geq 0$ and so $(N_2 A^{\dagger})^{p-1} N_2 u \geq 0$. The given hypothesis $(A^{\dagger}N_2)^p A^{\dagger} \geq (A^{\dagger}N_1)^p A^{\dagger}$, implies

$$\begin{aligned}(A^{\dagger}N_2)^p A^{\dagger}(N_2 A^{\dagger})^{p-1} N_2 u &\geq (A^{\dagger}N_1)^p A^{\dagger}(N_2 A^{\dagger})^{p-1} N_2 u \\ &= (A^{\dagger}N_1)^p (A^{\dagger}N_2)^p u \\ &= \rho((A^{\dagger}N_1)^p (A^{\dagger}N_2)^p)u.\end{aligned}$$

So, $(A^{\dagger}N_2)^{2p} u \geq \rho((A^{\dagger}N_1)^p (A^{\dagger}N_1)^p)u$. Again, $\rho(M_2^{\dagger}N_2) < 1$ implies that $A^{\dagger}N_2 = (I - M_2^{\dagger}N_2)^{-1} M_2^{\dagger}N_2 \geq 0$. Hence, $(\rho(A^{\dagger}N_1))^{2p} \geq \rho((A^{\dagger}N_1)^p (A^{\dagger}N_2)^p)$, by [4, Theorem 2.1.11]. Similarly, $(\rho(A^{\dagger}V_1))^{2p} \leq \rho((A^{\dagger}N_2)^p (A^{\dagger}N_1)^p)$. Finally, we have $(\rho(A^{\dagger}N_1))^{2p} \geq \rho((A^{\dagger}N_2)^p (A^{\dagger}N_1)^p) \leq (\rho(A^{\dagger}N_2))^{2p}$, which implies $\rho(A^{\dagger}N_1) \leq \rho(A^{\dagger}N_2)$. By [11, Lemma 3.5] we get $\rho(M_1^{\dagger}N_1) \leq \rho(M_2^{\dagger}N_2) < 1$. □

The following example illustrates the aforementioned theorems.

Example 3.5. Let $A = \begin{pmatrix} 3 & -1 & 3 \\ -1 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -0.5 & 3 \\ -0.5 & 3 & -0.5 \end{pmatrix} - \begin{pmatrix} 0 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = M_1 - N_1 = M_2 - N_2$.

Then, the splittings are convergent proper weak splittings of both types.

Here, for $p = 1$, we have $A^{\dagger}N_2A^{\dagger} > A^{\dagger}N_1A^{\dagger}$. Here, $u = v = \begin{pmatrix} 229/561 \\ 574/703 \\ 229/561 \end{pmatrix} \geq 0$

with $N_2u = \begin{pmatrix} 574/703 \\ 458/561 \end{pmatrix} \geq 0$, $N_1v = \begin{pmatrix} 287/703 \\ 229/561 \end{pmatrix} > 0$. Hence,

$$\rho(M_1^{\dagger}N_1) = \frac{1}{5} < \frac{1}{3} = (M_2^{\dagger}N_2) < 1.$$

Theorem 3.6. *Let* $A = M_1 - N_1 = M_2 - N_2$ *be convergent proper weak splitting of* $A \in \mathbb{R}^{m\times n}$ *with* $A^{\dagger} > 0$. *If* $\rho(M_1^{\dagger}N_1) < \rho(M_2^{\dagger}N_2)$ *and either of the following cases holds*:

(i) $A^{\dagger}N_2 > 0$;

(ii) *There exists a permutation matrix* P *such that* $P^TA^{\dagger}N_2P = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}$, *where* $B_{11}, B_{12}, B_{22} > 0$ *and* $\rho(B_{11}) = \rho(B_{22})$;

(iii) *There exists a permutation matrix P such that*

$$P^TA^{\dagger}N_2P = \begin{pmatrix} 0 & B_{12} \\ 0 & B_{22} \end{pmatrix}, \tag{3.1}$$

where $B_{12}, B_{22} > 0$.

Then there must be a positive integer p_0 *such that* $(A^{\dagger}N_2)^pA^{\dagger} > (A^{\dagger}N_1)^pA^{\dagger}$ *for all positive integer* $p \geq p_0$.

Proof. We will only prove for the case (b), the other cases are analogous. Clearly, for any positive integer $p > 1$, we have

$$P^T(A^{\dagger}N_2)^pP \geq \begin{pmatrix} B_{11}^p & B_{11}^{p-1}B_{12} + B_{12}B_{22}^{p-1} \\ 0 & B_{22}^p \end{pmatrix}.$$

Let $\rho(A^{\dagger}N_2) = \rho(> 0)$. Then $\rho(B_{ii}) = \rho(A^{\dagger}N_2)$, $i = 1, 2$ by the assumption. As B_{11}, B_{12}, B_{22}, by [4, Theorem 2.4.1], we have

$$\lim_{p\to\infty} \frac{B_{ii}^p}{\rho^p} := \hat{B}_{ii} > 0,\ i = 1, 2.$$

Hence,

$$\lim_{p\to\infty} \frac{P^T(A^\dagger N_2)^p P}{\rho^p} \geq \begin{pmatrix} \hat{B}_{11} & (\hat{B}_{11}B_{12} + B_{12}\hat{B}_{22})/\rho \\ 0 & \hat{B}_{22} \end{pmatrix}.$$

Since $A^\dagger > 0$, partitioning $A^\dagger$ conformally with respect to the partition in (3.1):

$$A^\dagger = \begin{pmatrix} A_{11}^p & A_{12} \\ A_{21} & A_{22}^p \end{pmatrix} > 0.$$

and we can conclude that

$$\lim_{p\to\infty} \frac{P^T(A^\dagger N_2)^p PA^\dagger}{\rho^p}$$

$$\geq \begin{pmatrix} \hat{B}_{11}A_{12} + (\hat{B}_{11}B_{12} + B_{12}\hat{B}_{22})A_{21}/\rho & \hat{B}_{11}A_{12} + (\hat{B}_{11}B_{12} + B_{12} + B_{12}\hat{B}_{22})A_{22}/\rho \\ \hat{B}_{21}A_{22} & \hat{B}_{22}A_{22} \end{pmatrix}.$$

So, all the entries in the above equation are positive. On the other hand, $\rho(M_1^\dagger N_1) < \rho(M_2^\dagger N_2)$, implies that

$$\lim_{p\to\infty} \frac{P^T(A^\dagger N_1)^p P}{\rho^p} = 0.$$

Thus

$$\lim_{p\to\infty} \frac{P^T(A^\dagger N_1)^p PA^\dagger}{\rho^p} = 0.$$

So, there must be a positive integer p_0 such that

$$\frac{(A^\dagger N_2)^p A^\dagger}{\rho^p} > \frac{(A^\dagger N_1)^p A^\dagger}{\rho^p},$$

for all $p \geq p_0$, i.e.,

$$(A^\dagger N_2)^p A^\dagger > (A^\dagger N_2)^p A^\dagger,$$

for all $p \geq p_0$.

References

[1] A. K. Baliarsingh and D. Mishra, Comparison results for proper nonnegative splittings of matrices, Results. Math. 71 (2017), 93-109.

[2] A. Ben-Israel and T. N. E. Greville, Generalized Inverses. Theory and Applications, Springer-Verlag, New York, 2003.

[3] A. Berman and R. J. Plemmons, Cones and iterative methods for best least squares solutions of linear systems, SIAM J. Numer. Anal. 11 (1974), 145-154.

[4] A. Berman and R. J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, SIAM, Philadelphia, 1994.

[5] C. K. Giri and D. Mishra, Additional results on convergence of alternating iterations involving rectangular matrices, Numer. Funct. Anal. Optimiz. 38 (2017), 160-180.

[6] C. K. Giri and D. Mishra, Some comparison theorems for proper weak splittings of type II, J. Anal. 25 (2017), 267-279.

[7] C. K. Giri and D. Mishra, Comparison results for proper multisplittings of rectangular matrices, Adv. Oper. Theory 2 (2017), 334-352.

[8] C. K. Giri and D. Mishra, More on convergence theory of proper multisplittings, Khayyam J. Math. 4 (2018), 144-154.

[9] C. W. Groetsch, Generalized Inverses of Linear Operators: Representation and Approximation, Mercel Dekker, New York, 1974.

[10] D. Mishra, Proper weak regular splitting and its applications to convergence of alternating methods, Filomat 32 (2018), 6563-6573.

[11] D. Mishra, Nonnegative splittings for rectangular matrices, Comput. Math. Appl. 67 (2014), 136-144.

[12] D. Mishra and K. C. Sivakumar, Comparison theorems for a subclass of proper splittings of matrices, Appl. Math. Lett. 25 (2012), 2339-2343.

[13] D. Mishra and K. C. Sivakumar, On splittings of matrices and nonnegative generalized inverses, Oper. Matrices 6 (2012), 85-95.

[14] I. Marek and D. B. Szyld, Comparison theorems for weak splittings of bounded operators, Numer. Math. 58 (1990), 387-397.

[15] J. J. Climent, A. Devesa and C. Perea, Convergence results for proper splittings, Recent Advances in Applied and Theoretical Mathematics, World Scientific and Engineering Society Press, Singapore (2000), 39-44.

[16] J. J. Climent and C. Perea, Iterative methods for least square problems based on proper splittings, J. Comput. Appl. Math. 158 (2003), 43-48.

[17] J. J. Climent and C. Perea, Some comparison theorems for weak nonnegative splittings of bounded operators, Linear Algebra Appl. 275/276 (1998), 77-106.

[18] L. Jena, D. Mishra and S. Pani, Convergence and comparisons of single and double decompositions of rectangular matrices, Calcolo 51 (2014), 141-149.

[19] N. Mishra and D. Mishra, Two-stage iterations based on composite splittings for rectangular linear systems, Comput. Math. Appl. 75 (2018), 2746-2756.

[20] R. Penrose, A generalized inverse for matrices, Proc. Cambridge Philos. Soc. 51 (1955), 406-413.

[21] R. S. Varga, Matrix Iterative Analysis, Springer-Verlag, Berlin, 2000.

[22] V. Shekhar, C. K. Giri and D. Mishra, Convergence theory of iterative methods based on proper splittings and proper multisplittings for rectangular linear systems, Filomat 34 (2020), 1835-1851.

[23] Z. H. Cao, H. B. Wu and Z. Liu, A note on weak splittings of matrices, Appl. Math. Comput. 112 (2000), 265-275.