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PERFORMANCE OF PARALLEL SCHWARZ ALGORITHM FOR A WAVE PROBLEM

ALEXANDROS KYRIAKIS*

*Department of Mathematics
Boston College, Skirbeck Road, Boston
PE21 6JF, United Kingdom

Abstract

In this paper, we aim to investigate the convergence behaviour of the Parallel Schwarz algorithm at the continuous level for the Helmholtz equation with absorption. More precisely, we examine the behaviour of the Parallel Schwarz method by taking into account different types of boundary conditions at the external boundaries of the subdomains and modifying the interface conditions. The boundary conditions on top and bottom of the rectangle are always Dirichlet. In this way, we see how the choice of the boundary conditions and interface conditions affect the overall performance of the iterative method. The global working domain is a rectangle with a Lipschitz boundary, decomposed into two sub-rectangles of equal size. To analyse the Performance of the Parallel Schwarz algorithm at the continuous level we use Fourier Analysis techniques and obtain the appropriate contraction factors which are revealing on how the errors contract. The Helmholtz equation frequently arises in the study and modelling of various physical phenomena. A few applications can be mentioned such as the scientific study of earthquakes, volcanic eruptions, Medical Imaging and Electromagnetism. There are various challenges to solving the Helmholtz equation numerically. One challenge stems from the fact that the problem is not symmetric and positive definite and the other one is from the fact that the wave number provides solutions with highly oscillatory behaviour. These issues that arise, make the Helmholtz equation substantially difficult to solve.

1. Introduction

During the last few decades, Domain decomposition methods have attracted a lot of attention from mathematical and engineering communities worldwide. The main reason is that this class of methods is very efficient for solving partial differential equations numerically, and presents computational and mathematical challenges. The spirit of Domain decomposition algorithms is to decompose the global boundary value problem on the whole domain into multiple local subproblems in multiple subdomains

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*Corresponding author; E-mail: alexanderkiriakis@hotmail.com

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[7], [8], [9], [10]. This approach enables us to reduce computational costs by solving smaller and cheaper local boundary value problems. By solving local subproblems, the computational effort is balanced in the different subdomains, and working with one global problem can be avoided. Domain decomposition methods can be used to solve partial differential equations sequentially or in parallel. Originally these schemes stem from the work of the famous German analyst Hermann Amandus Schwarz [4], [5], [6] back in 1869, who devised an iterative method to close a gap in Riemann's mapping theorem. This iterative method was used to solve the Laplace equation in an irregular domain (union of circle and rectangle) solving in an alternating approach, first in the circle and then in the rectangle, back and forth, passing the values at the interfaces. This is the so-called Alternating Schwarz algorithm. Later on, the French mathematician Pierre Luis Lions proposed a modification of the Classical Schwarz method [13], by changing one interface condition, leading to the Parallel Schwarz method. This strategy enables solving the two local boundary value problems in the two subdomains in parallel, passing the traces at the two interfaces. By appropriate finite difference or finite element discretization scheme, an algebraic solver is obtained that can be used by engineers and computational scientists. With regards to the behaviour of the Parallel Schwarz algorithm at the continuous level, there is one revealing analysis that predicts the overall behaviour. By performing a convergence analysis, we obtain a contraction factor which is a good measure of how the error behaves. This analysis can be conducted using Fourier transform if the domains are unbounded or the Fourier series if the domains are bounded. Usually, this error analysis is conducted for two subdomains and it is a common strategy in the literature [17]. For our analysis, we focus on the Helmholtz equation with absorption where we have two overlapping bounded domains, more precisely a union of two rectangles of equal size. We retrieve the reduction factors in each case with different boundary and interface conditions. The choices are either Dirichlet or Neumann. In addition, the Optimized Schwarz algorithm is analysed in similar fashion with [1], [2], [3], where impedance conditions are used on the left and right edge of the global domain and at the two interfaces. Lastly, we provide some illustrations of the methods, giving the figures of the reduction factors for each one of the different scenarios and the spectrum of the Schwarz iteration matrices in the complex plane for each of the algorithms.

2. The Model Problem

The equation of interest in the paper is the Helmholtz equation with absorption. This problem is ubiquitous in science and engineering with a variety of applications and one of the most important is the study of wave propagation phenomena. There are two aspects that make this scalar elliptic equation intriguing [11], [12]. One aspect is the fact that the problem is not symmetric positive definite. The other one is that due to the wave number, the solutions present highly oscillatory behaviour. We give the definition of the Helmholtz problem with damping where the domain $\Omega = (a_1, b_2) \times (0, L)$ is a rectangle with Lipschitz boundary $\partial\Omega$, where L is the height of the rectangle. The equation in strong form is

$$\begin{cases} \Delta u(x, y) + (k^2 - ik)u(x, y) = -f(x, y) \quad \text{in } \Omega, \\ \text{external b.c. on } \partial\Omega \end{cases} \tag{1}$$

where $k \in (0, +\infty)$ is the wave number, $u(x, y) \in H^1(\Omega)$ is the solution of the equation and $f(x, y) \in L_2(\Omega)$. The functional space $H^1(\Omega)$ is the classical Sobolev space and the functional space $L_2(\Omega)$ is the usual space of square-integrable functions. The $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator in the case of two dimensions.

3. Motivation for the Choice of the Boundary Conditions

The choice of the boundary conditions in the computational domain has an influence on the Schwarz algorithms and their behaviour. This will be demonstrated by obtaining the formulas for the reduction factors. There are three types of boundary conditions for the study of Schwarz algorithms:

- $u = u_D$ (Dirichlet boundary conditions),
- $\nabla u \cdot \vec{n} = g_N$ (Neumann boundary conditions),
- $\nabla u \cdot \vec{n} + cu = g_R$ (Robin boundary conditions).

4. The Formulation of the Schwarz algorithm-Case 1

In this section we formulate the Parallel Schwarz algorithm for the differential problem reported in (1). We decompose the domain $\Omega = (a_1, b_2) \times (0, L)$ into two bounded overlapping subdomains $\Omega_1 = (a_1, b_1) \times (0, L)$ and $\Omega_1 = (a_2, b_2) \times (0, L)$. The Parallel Schwarz algorithm in strong form reads

$$\begin{cases} \Delta u_1^{(m)} + (k^2 - ik) u_1^{(m)} = -f \text{ in } \Omega_1 \\ u_1^{(m)} = u_2^{(m-1)} \text{ at } \Gamma_1 \\ u_1^{(m)} = g \text{ on } \partial\Omega_1 \backslash \Gamma_1 \end{cases}, \begin{cases} \Delta u_2^{(m)} + (k^2 - ik) u_2^{(m)} = -f \text{ in } \Omega_2 \\ u_2^{(m)} = u_1^{(m-1)} \text{ at } \Gamma_2 \\ u_2^{(m)} = g \text{ on } \partial\Omega_2 \backslash \Gamma_2 \end{cases} \tag{2}$$

where the two interfaces are $\Gamma_1 = \partial\Omega_1 \backslash (\partial\Omega_1 \cap \partial\Omega)$ and $\Gamma_2 = \partial\Omega_2 \backslash (\partial\Omega_2 \cap \partial\Omega)$. The (m) denotes the number of iterations and it is evident that in order to start the iterative process the two initial guesses are required at the two interfaces, namely $u_1^{(0)}$ and $u_2^{(0)}$. At the external boundaries of the two subdomains, Dirichlet boundary conditions are imposed. It is observable that with this approach we break the boundary value problem in (1) into a collection of two subproblems that are solved in parallel using the traces at the interfaces, solving until convergence is reached.

Theorem 1. *The contraction factor of the Parallel Schwarz algorithm for case* 1 *is given by the formula*

$$rcs_1^2(\tilde{k}; k, a_1, a_2, b_1, b_2) = \left| \frac{\sinh(\lambda(\tilde{k})(a_2 - a_1))}{\sinh(\lambda(\tilde{k})(b_2 - a_2))} \frac{\sinh(\lambda(\tilde{k})(b_2 - b_1))}{\sinh(\lambda(\tilde{k})(b_1 - a_1))} \right| \tag{3}$$

where $\tilde{k}$ *is the Fourier number and* $\lambda(\tilde{k}) = \sqrt{\tilde{k}^2 - k^2 + ik}$.

Proof. The first step is to introduce the errors at each iteration (m) in each subdomain. The error is the first subdomain is given by $e_1^{(m)} = u|_{\Omega_1} - u_1^{(m)}$ and the error in the second subdomain is given by $e_2^{(m)} = u|_{\Omega_2} - u_2^{(m)}$, where the function $u|_{\Omega_1}$ is the restriction of the solution to the subdomain Ω_1 and $u|_{\Omega_2}$ is the restriction of the solution to the subdomain Ω_2. By the linearity of the two differential problems in (2), the two errors satisfy the homogeneous analogues of (2), and the algorithm takes the form

$$\begin{cases} \Delta e_1^{(m)} + (k^2 - ik)e_1^{(m)} = 0 \text{ in } \Omega_1 \\ e_1^{(m)} = e_2^{(m-1)} \text{ at } \Gamma_1 \\ e_1^{(m)} = 0 \text{ on } \partial\Omega_1 \setminus \Gamma_1 \end{cases}, \quad \begin{cases} \Delta e_2^{(m)} + (k^2 - ik)e_2^{(m)} = 0 \text{ in } \Omega_2 \\ e_2^{(m)} = e_1^{(m-1)} \text{ at } \Gamma_2 \\ e_2^{(m)} = 0 \text{ on } \partial\Omega_2 \setminus \Gamma_2 \end{cases} \tag{4}$$

Since we are working with bounded subdomains and more accurately rectangles, we can use the Fourier Series to expand the two solutions in the first subdomain and in the second subdomain respectively. The two solutions should satisfy the homogeneous Dirichlet boundary conditions on the top and bottom of the rectangles. In order to accommodate this requirement, we expand the functions $e_1^{(m)}$ and $e_2^{(m)}$ in Fourier sine series

$$e_1^{(m)} = \sum_{\tilde{k} \in K} \hat{e}_1^{(m)} \sin(\tilde{k}y), \tag{5}$$

$$e_2^{(m)} = \sum_{\tilde{k} \in K} \hat{e}_2^{(m)} \sin(\tilde{k}y), \tag{6}$$

where the set $K = \left\{ \frac{\pi}{L}, \frac{2\pi}{L}, \ldots, \frac{n\pi}{L} \right\}$ denotes the discrete Fourier frequencies. We plug in the formulas prescribed by (5), (6) back to the Parallel Schwarz method in (4) and it follows that the Fourier coefficients $\hat{e}_1^{(m)}$, $\hat{e}_2^{(m)}$ satisfy

$$\begin{cases} \dfrac{\partial^2 \hat{e}_1^{(m)}}{\partial x^2} - (\tilde{k}^2 - k^2 + ik)\hat{e}_1^{(m)} = 0 \text{ in } \Omega_1 \\ \hat{e}_1^{(m)} = \hat{e}_2^{(m-1)} \text{ at } x = b_1 \\ \hat{e}_1^{(m)} = 0 \text{ at } x = a_1 \end{cases}, \quad \begin{cases} \dfrac{\partial^2 \hat{e}_2^{(m)}}{\partial x^2} - (\tilde{k}^2 - k^2 + ik)\hat{e}_2^{(m)} = 0 \text{ in } \Omega_2 \\ \hat{e}_2^{(m)} = \hat{e}_1^{(m-1)} \text{ at } x = a_2 \\ \hat{e}_2^{(m)} = 0 \text{ at } x = b_2 \end{cases} \tag{7}$$

taking into account that

$$\Delta \hat{e}_j^{(m)} + (k^2 - ik)\hat{e}_j^{(m)} = \sum_{\tilde{k} \in K} \left(\frac{\partial^2 \hat{e}_j^{(m)}}{\partial x^2} - (\tilde{k}^2 - k^2 + ik)\hat{e}_j^{(m)} \right) \sin(\tilde{k}y)$$

for $j = 1, 2.$ There are two linear second order ordinary differential equations that arise in the two Schwarz problems in (7). These can be solved, and the general solutions are given by the formulas

$$\hat{e}_1^{(m)}(x, \tilde{k}) = A_1^{(m)}(\tilde{k})e^{\lambda(\tilde{k})x} + B_1^{(m)}(\tilde{k})e^{-\lambda(\tilde{k})x}, \tag{8}$$

$$\hat{e}_2^{(m)}(x, \tilde{k}) = A_2^{(m)}(\tilde{k})e^{\lambda(\tilde{k})x} + B_2^{(m)}(\tilde{k})e^{-\lambda(\tilde{k})x} \tag{9}$$

where $A_1^{(m)}(\tilde{k}), B_1^{(m)}(\tilde{k}), A_2^{(m)}(\tilde{k}), B_2^{(m)}(\tilde{k})$ are arbitrary coefficients, which can be specified by the boundary conditions. Also the $\lambda(\tilde{k})$ is provided by the relation $\lambda(\tilde{k}) = \sqrt{\tilde{k}^2 - k^2 + ik}.$ For the first Schwarz problem, the solution must satisfy the homogeneous Dirichlet boundary condition at $x = a_1$, therefore for this prescribed boundary condition the solution becomes

$$\hat{e}_1^{(m)}(x, \tilde{k}) = 2A_1^{(m)}(\tilde{k})e^{\lambda(\tilde{k})a_1} \sinh(\lambda(\tilde{k})(x - a_1)). \tag{10}$$

We move to the second Schwarz problem in (7), and by taking into account the boundary condition at $x = b_2$ the error becomes

$$\hat{e}_2^{(m)}(x, \tilde{k}) = -2A_2^{(m)}(\tilde{k})e^{\lambda(\tilde{k})b_2} \sinh(\lambda(\tilde{k})(b_2 - x)). \tag{11}$$

We substitute the two solutions in (10), (11) back to the transmission conditions in (7) to obtain

$$A_1^{(m)}(\tilde{k}) = -A_2^{(m-1)}(\tilde{k})e^{\lambda(\tilde{k})(b_2 - a_1)} \frac{\sinh(\lambda(\tilde{k})(b_2 - b_1))}{\sinh(\lambda(\tilde{k})(b_1 - a_1))},$$

$$A_2^{(m)}(\tilde{k}) = -A_1^{(m-1)}(\tilde{k})e^{-\lambda(\tilde{k})(b_2 - a_1)} \frac{\sinh(\lambda(\tilde{k})(a_2 - a_1))}{\sinh(\lambda(\tilde{k})(b_2 - a_2))}.$$

The expressions above can be written in a compact form

$$\begin{bmatrix} A_1^{(m)}(\tilde{k}) \\ A_2^{(m)}(\tilde{k}) \end{bmatrix} = \Psi_{cs1} \begin{bmatrix} A_1^{(m-1)}(\tilde{k}) \\ A_2^{(m-1)}(\tilde{k}) \end{bmatrix} \tag{12}$$

where

$$\Psi_{cs1} = \begin{bmatrix} 0 & -e^{\lambda(\tilde{k})(b_2 - a_1)} \dfrac{\sinh(\lambda(\tilde{k})(b_2 - b_1))}{\sinh(\lambda(\tilde{k})(b_1 - a_1))} \\ -e^{-\lambda(\tilde{k})(b_2 - a_1)} \dfrac{\sinh(\lambda(\tilde{k})(a_2 - a_1))}{\sinh(\lambda(\tilde{k})(b_2 - a_2))} & 0 \end{bmatrix} \tag{13}$$

is the Schwarz iteration matrix. Consequently, in (12) we have a stationary iteration, and the spectral properties of the Schwarz matrix determine the behavior of the algorithm. The eigenvalues of (13) are

$$\nu_+ = \sqrt{\frac{\sinh(\lambda(\tilde{k})(a_2 - a_1))}{\sinh(\lambda(\tilde{k})(b_2 - a_2))} \frac{\sinh(\lambda(\tilde{k})(b_2 - b_1))}{\sinh(\lambda(\tilde{k})(b_1 - a_1))}} = -\nu_-.$$

By definition, the spectral radius of the iteration matrix is

$$\rho(\Psi_{cs1}) = \max\{|\nu_+|, |\nu_-|\} = \left| \sqrt{\frac{\sinh(\lambda(\tilde{k})(a_2 - a_1))}{\sinh(\lambda(\tilde{k})(b_2 - a_2))} \frac{\sinh(\lambda(\tilde{k})(b_2 - b_1))}{\sinh(\lambda(\tilde{k})(b_1 - a_1))}} \right|.$$

As a consequence, we obtain

$$rcs_1^2(\tilde{k}; k, a_1, a_2, b_1, b_2) = \left| \frac{\sinh(\lambda(\tilde{k})(a_2 - a_1))}{\sinh(\lambda(\tilde{k})(b_2 - a_2))} \frac{\sinh(\lambda(\tilde{k})(b_2 - b_1))}{\sinh(\lambda(\tilde{k})(b_1 - a_1))} \right|. \qquad \square$$

5. The Formulation of the Parallel Schwarz algorithm - Case 2

In this section we modify the overlapping domain decomposition problem. We consider that the external boundary conditions are Dirichlet as before, but now the interface conditions are modified. Instead of using Dirichlet, we use the Neumann boundary conditions and now the formulation of the Schwarz method is

$$\begin{cases} \Delta u_1^{(m)} + (k^2 - ik)u_1^{(m)} = -f \text{ in } \Omega_1 \\ \nabla u_1^{(m)} \cdot \overrightarrow{n_1} = \nabla u_2^{(m-1)} \overrightarrow{n_1} \text{ at } \Gamma_1 \\ u_1^{(m)} = g \text{ on } \partial\Omega_1 \setminus \Gamma_1 \end{cases}, \begin{cases} \Delta u_2^{(m)} + (k^2 - ik)u_2^{(m)} = -f \text{ in } \Omega_2 \\ \nabla u_2^{(m)} \cdot \overrightarrow{n_2} = \nabla u_1^{(m-1)} \cdot \overrightarrow{n_1} \text{ at } \Gamma_2 \\ u_2^{(m)} = g \text{ on } \partial\Omega_2 \setminus \Gamma_2 \end{cases} \tag{14}$$

where $\overrightarrow{n_1} = (1, 0)$ and $\overrightarrow{n_2} = (-1, 0)$ are the outward normal vectors. Note that the fluxes $\nabla u_2^{(0)} \overrightarrow{n_1}$ and $\nabla u_1^{(0)} \overrightarrow{n_2}$ are needed to start the iterative procedure.

Theorem 2. *The reduction factor of the Parallel Schwarz algorithm for case 2 is given by the formula*

$$rcs_2^2(\tilde{k}; k, a_1, a_2, b_1, b_2) = \left| \frac{\cosh(\lambda(\tilde{k})(a_2 - a_1))}{\cosh(\lambda(\tilde{k})(b_2 - a_2))} \frac{\cosh(\lambda(\tilde{k})(b_2 - b_1))}{\cosh(\lambda(\tilde{k})(b_1 - a_1))} \right|. \quad (15)$$

Proof. We can write the iterative scheme prescribed by (14) in the form

$$\begin{cases} \Delta u_1^{(m)} + (k^2 - ik)u_1^{(m)} = -f \text{ in } \Omega_1 \\ \partial_x u_1^{(m)} = \partial_x u_2^{(m-1)} \text{ at } \Gamma_1 \\ u_1^{(m)} = g \text{ on } \partial\Omega_1 \setminus \Gamma_1 \end{cases}, \begin{cases} \Delta u_2^{(m)} + (k^2 - ik)u_2^{(m)} = -f \text{ in } \Omega_2 \\ \partial_x u_2^{(m)} = \partial_x u_1^{(m-1)} \text{ at } \Gamma_2 \\ u_2^{(m)} = g \text{ on } \partial\Omega_2 \setminus \Gamma_2 \end{cases} \quad (16)$$

taking into consideration that the minus signs that appear in $(16)_2$ (second Schwarz problem) are cancelled on both sides. For the convergence analysis, the errors are introduced at each iteration and due to the linearity of the two Schwarz problems in (16), the one level method reads

$$\begin{cases} \Delta e_1^{(m)} + (k^2 - ik)e_1^{(m)} = 0 \text{ in } \Omega_1 \\ \partial_x e_1^{(m)} = \partial_x e_2^{(m-1)} \text{ at } \Gamma_1 \\ e_1^{(m)} = 0 \text{ on } \partial\Omega_1 \setminus \Gamma_1 \end{cases}, \begin{cases} \Delta e_2^{(m)} + (k^2 - ik)e_2^{(m)} = 0 \text{ in } \Omega_2 \\ \partial_x e_2^{(m)} = \partial_x e_1^{(m-1)} \text{ at } \Gamma_2 \\ e_2^{(m)} = 0 \text{ on } \partial\Omega_2 \setminus \Gamma_2 \end{cases} \quad (17)$$

We use the Fourier series expansions for the two local errors prescribed by (5), (6) and replace the two expressions back in (17) giving

$$\begin{cases} \dfrac{\partial^2 \hat{e}_1^{(m)}}{\partial x^2} - (\tilde{k}^2 - k^2 + ik)\hat{e}_1^{(m)} = 0 \text{ in } \Omega_1 \\ \partial_x \hat{e}_1^{(m)} = \partial_x \hat{e}_2^{(m-1)} \text{ at } x = b_1 \\ \hat{e}_1^{(m)} = 0 \text{ at } x = a_1 \end{cases}, \begin{cases} \dfrac{\partial^2 \hat{e}_2^{(m)}}{\partial x^2} - (\tilde{k}^2 - k^2 + ik)\hat{e}_2^{(m)} = 0 \text{ in } \Omega_2 \\ \partial_x \hat{e}_2^{(m)} = \partial_x \hat{e}_1^{(m-1)} \text{ at } x = a_2 \\ \hat{e}_2^{(m)} = 0 \text{ at } x = b_2 \end{cases} \quad (18)$$

We can solve the two second order ordinary differential equations in (18) and by exploiting the boundary conditions, we write the solutions in the form

$$\hat{e}_1^{(m)}(x, \tilde{k}) = 2A_1^{(m)}(\tilde{k})e^{\lambda(\tilde{k})a_1} \sinh(\lambda(\tilde{k})(x - a_1)), \quad (19)$$

$$\hat{e}_2^{(m)}(x, \tilde{k}) = -2A_2^{(m)}(\tilde{k})e^{\lambda(\tilde{k})b_2} \sinh(\lambda(\tilde{k})(b_2 - x)). \quad (20)$$

We plug in the errors back to the interface conditions in (18), which yields

$$A_1^{(m)}(\tilde{k}) = -A_2^{(m-1)}(\tilde{k})e^{\lambda(\tilde{k})(b_2-a_1)}\frac{\cosh(\lambda(\tilde{k})(b_2-b_1))}{\cosh(\lambda(\tilde{k})(b_1-a_1))},$$

$$A_2^{(m)}(\tilde{k}) = A_1^{(m-1)}(\tilde{k})e^{-\lambda(\tilde{k})(b_2-a_1)}\frac{\cosh(\lambda(\tilde{k})(a_2-a_1))}{\cosh(\lambda(\tilde{k})(b_2-a_2))}.$$

We reformulate the above expressions in matrix form

$$\begin{bmatrix} A_1^{(m)}(\tilde{k}) \\ A_2^{(m)}(\tilde{k}) \end{bmatrix} = \Psi_{cs2}\begin{bmatrix} A_1^{(m-1)}(\tilde{k}) \\ A_2^{(m-1)}(\tilde{k}) \end{bmatrix}$$

where

$$\Psi_{cs2} = \begin{bmatrix} 0 & e^{\lambda(\tilde{k})(b_2-a_1)}\dfrac{\cosh(\lambda(\tilde{k})(b_2-b_1))}{\cosh(\lambda(\tilde{k})(b_1-a_1))} \\ e^{-\lambda(\tilde{k})(b_2-a_1)}\dfrac{\cosh(\lambda(\tilde{k})(a_2-a_1))}{\cosh(\lambda(\tilde{k})(b_2-a_2))} & 0 \end{bmatrix}$$

is the Schwarz iteration matrix. The spectral behaviour of the iteration matrix determines the performance of the Schwarz iterative method. The eigenvalues of Ψ_{cs2} are

$$\nu_+ = \sqrt{\frac{\cosh(\lambda(\tilde{k})(a_2-a_1))}{\cosh(\lambda(\tilde{k})(b_2-a_2))}\frac{\cosh(\lambda(\tilde{k})(b_2-b_1))}{\cosh(\lambda(\tilde{k})(b_1-a_1))}} = -\nu_-.$$

Finally, the reduction factor is

$$rcs_2^2(\tilde{k}; k, a_1, a_2, b_1, b_2) = \left|\frac{\cosh(\lambda(\tilde{k})(a_2-a_1))}{\cosh(\lambda(\tilde{k})(b_2-a_2))}\frac{\cosh(\lambda(\tilde{k})(b_2-b_1))}{\cosh(\lambda(\tilde{k})(b_1-a_1))}\right|. \qquad \square$$

6. The Formulation of the Parallel Schwarz algorithm - Case 3

In this scenario, we impose Dirichlet boundary conditions on the top and bottom of the rectangles, Dirichlet at the interfaces, and we alter the boundary conditions on the left and right edge of the global domain. More accurately, at the left edge of Ω_1 and the right edge of Ω_2. Neumann conditions are enforced. As a consequence, the Schwarz method reads

$$\begin{cases} \Delta u_1^{(m)} + (k^2 - ik)u_1^{(m)} = -f \text{ in } \Omega_1 \\ u_1^{(m)} = u_2^{(m-1)} \text{ at } \Gamma_1 \\ u_1^{(m)} = g \text{ on } \Gamma_t \cup \Gamma_b \\ -\nabla u_1^{(m)} \cdot \overrightarrow{n_1} = g_N \text{ at } \Gamma_L \end{cases}, \begin{cases} \Delta u_2^{(m)} + (k^2 - ik)u_2^{(m)} = -f \text{ in } \Omega_2 \\ u_2^{(m)} = u_1^{(m-1)} \text{ at } \Gamma_2 \\ u_2^{(m)} = g \text{ on } \Gamma_t \cup \Gamma_b \\ -\nabla u_2^{(m)} \cdot \overrightarrow{n_2} = g_N \text{ at } \Gamma_R \end{cases} \quad (21)$$

where Γ_t is the top part of the boundary, Γ_b is the bottom part, Γ_L is the left edge of the domain Ω and Γ_R is the right edge respectively.

Theorem 3. *The reduction factor of the Parallel Schwarz algorithm for case* 3 *is given by the formula*

$$rcs_3^2(\tilde{k}; k, a_1, a_2, b_1, b_2) = \left| \frac{\cosh(\lambda(\tilde{k})(a_2 - a_1))}{\cosh(\lambda(\tilde{k})(b_2 - a_2))} \frac{\cosh(\lambda(\tilde{k})(b_2 - b_1))}{\cosh(\lambda(\tilde{k})(b_1 - a_1))} \right|. \quad (22)$$

Proof. The result is proved using a similar approach to the two previous cases. Following the steps that were extensively explained in the two previous cases, the Schwarz algorithm can be written in a compact form

$$\begin{bmatrix} A_1^{(m)}(\tilde{k}) \\ A_2^{(m)}(\tilde{k}) \end{bmatrix} = \Psi_{cs3} \begin{bmatrix} A_1^{(m-1)}(\tilde{k}) \\ A_2^{(m-1)}(\tilde{k}) \end{bmatrix}$$

where

$$\Psi_{cs3} = \begin{bmatrix} 0 & e^{\lambda(\tilde{k})(b_2 - a_1)} \frac{\cosh(\lambda(\tilde{k})(b_2 - b_1))}{\cosh(\lambda(\tilde{k})(b_1 - a_1))} \\ e^{-\lambda(\tilde{k})(b_2 - a_1)} \frac{\cosh(\lambda(\tilde{k})(a_2 - a_1))}{\cosh(\lambda(\tilde{k})(b_2 - a_2))} & 0 \end{bmatrix}$$

is the iteration matrix. The spectrum of the iteration matrix is $\sigma(\Psi_{cs3}) = \{\nu_-, \nu_+\}$ where

$$\nu_+ = \sqrt{\frac{\cosh(\lambda(\tilde{k})(a_2 - a_1))}{\cosh(\lambda(\tilde{k})(b_2 - a_2))} \frac{\cosh(\lambda(\tilde{k})(b_2 - b_1))}{\cosh(\lambda(\tilde{k})(b_1 - a_1))}} = -\nu_-.$$

Consequently, we obtain

$$rcs_3^2(\tilde{k}; k, a_1, a_2, b_1, b_2) = \left| \frac{\cosh(\lambda(\tilde{k})(a_2 - a_1))}{\cosh(\lambda(\tilde{k})(b_2 - a_2))} \frac{\cosh(\lambda(\tilde{k})(b_2 - b_1))}{\cosh(\lambda(\tilde{k})(b_1 - a_1))} \right|. \quad \square$$

7. The Formulation of the Parallel Schwarz algorithm - Case 4

For this case, we consider Dirichlet boundary conditions on $\Gamma_t \cup \Gamma_b$ (top and bottom), Dirichlet interface conditions, Dirichlet at Γ_L (left edge of Ω) and Neumann at Γ_R (right edge of Ω). Therefore the Schwarz algorithm is

$$\begin{cases} \Delta u_1^{(m)} + (k^2 - ik)u_1^{(m)} = -f \text{ in } \Omega_1 \\ u_1^{(m)} = u_2^{(m-1)} \text{ at } \Gamma_1 \\ u_1^{(m)} = g \text{ on } \Gamma_t \cup \Gamma_b \\ u_1^{(m)} = q \text{ at } \Gamma_L \end{cases}, \quad \begin{cases} \Delta u_2^{(m)} + (k^2 - ik)u_2^{(m)} = -f \text{ in } \Omega_2 \\ u_2^{(m)} = u_1^{(m-1)} \text{ at } \Gamma_2 \\ u_2^{(m)} = g \text{ on } \Gamma_t \cup \Gamma_b \\ -\nabla u_2^{(m)} \cdot \overrightarrow{n_2} = h \text{ at } \Gamma_R. \end{cases}$$

Theorem 4. *The reduction factor of the Parallel Schwarz algorithm for case* 4 *is given by the formula*

$$rcs_4^2(\tilde{k}; k, a_1, a_2, b_1, b_2) = \left| \frac{\sinh(\lambda(\tilde{k})(a_2 - a_1))}{\cosh(\lambda(\tilde{k})(b_2 - a_2))} \frac{\cosh(\lambda(\tilde{k})(b_2 - b_1))}{\sinh(\lambda(\tilde{k})(b_1 - a_1))} \right|.$$

Proof. The result follows by implementing all the steps explained in detail in cases 1 and 2, performing all the necessary calculations.

8. The Formulation of the Parallel Schwarz algorithm - Case 5

For this problem configuration, we consider Dirichet boundary conditions on top and bottom of the rectangles, Dirichlet transmission conditions, Neumann at Γ_L (left edge of Ω) and Dirichlet at Γ_R (right edge of Ω). The Schwarz iterative scheme for this problem is

$$\begin{cases} \Delta u_1^{(m)} + (k^2 - ik)u_1^{(m)} = -f \text{ in } \Omega_1 \\ u_1^{(m)} = u_2^{(m-1)} \text{ at } \Gamma_1 \\ u_1^{(m)} = g \text{ on } \Gamma_t \cup \Gamma_b \\ -\nabla u_1^{(m)} \overrightarrow{n_1} = h \text{ at } \Gamma_L \end{cases}, \quad \begin{cases} \Delta u_2^{(m)} + (k^2 - ik)u_2^{(m)} = -f \text{ in } \Omega_2 \\ u_2^{(m)} = u_1^{(m-1)} \text{ at } \Gamma_2 \\ u_2^{(m)} = g \text{ on } \Gamma_t \cup \Gamma_b \\ u_2^{(m)} = q \text{ at } \Gamma_R. \end{cases}$$

Theorem 5. *The reduction factor of the Parallel Schwarz algorithm for case* 5 *is given by the formula*

$$rcs_5^2(\tilde{k}; k, a_1, a_2, b_1, b_2) = \left| \frac{\cosh(\lambda(\tilde{k})(a_2 - a_1))}{\sinh(\lambda(\tilde{k})(b_2 - a_2))} \frac{\sinh(\lambda(\tilde{k})(b_2 - b_1))}{\cosh(\lambda(\tilde{k})(b_1 - a_1))} \right|.$$

Proof. The result is obtained by finding the eigenvalues and the spectral radius of the iteration matrix.

9. The Formulation of the Parallel Schwarz algorithm - Case 6

For this particular problem, we specify the boundary conditions as follows: Neumann at the left and right edge of the global domain (Γ_L and Γ_R), Neumann conditions at the two interfaces, Dirichlet on top and bottom ($\Gamma_t \cup \Gamma_R$). The Schwarz method for this problem reads

$$\begin{cases} \Delta u_1^{(m)} + (k^2 - ik)u_1^{(m)} = -f \text{ in } \Omega_1 \\ \nabla u_1^{(m)}\overrightarrow{n_1} = \nabla u_2^{(m-1)}\overrightarrow{n_1} \text{ at } \Gamma_1 \\ u_1^{(m)} = g \text{ on } \Gamma_t \cup \Gamma_b \\ -\nabla u_1^{(m)}\overrightarrow{n_1} = h \text{ at } \Gamma_L \end{cases}, \begin{cases} \Delta u_2^{(m)} + (k^2 - ik)u_2^{(m)} = -f \text{ in } \Omega_2 \\ \nabla u_2^{(m)}\overrightarrow{n_2} = \nabla u_1^{(m-1)}\overrightarrow{n_2} \text{ at } \Gamma_2 \\ u_2^{(m)} = g \text{ on } \Gamma_t \cup \Gamma_b \\ -\nabla u_2^{(m)}\overrightarrow{n_2} = h \text{ at } \Gamma_R. \end{cases}$$

Theorem 6. *The reduction factor of the Parallel Schwarz algorithm for case* 6 *is given by the formula*

$$rcs_6^2(\widetilde{k}; k, a_1, a_2, b_1, b_2) = \left| \frac{\sinh(\lambda(\widetilde{k})(a_2 - a_1))}{\sinh(\lambda(\widetilde{k})(b_2 - a_2))} \frac{\sinh(\lambda(\widetilde{k})(b_2 - b_1))}{\sinh(\lambda(\widetilde{k})(b_1 - a_1))} \right|.$$

Proof. The result follows by explicitly finding the spectrum and the spectral radius of the iteration matrix.

10. The Formulation of the Parallel Schwarz algorithm - Case 7

In this scenario, we specify the boundary conditions as follows: Dirichlet conditions are imposed on top and bottom, Neumann at the interfaces, Dirichlet at the left edge of the global domain, Neumann at the right edge of the whole rectangle. For this configuration, we proceed giving the Schwarz iterative method

$$\begin{cases} \Delta u_1^{(m)} + (k^2 - ik)u_1^{(m)} = -f \text{ in } \Omega_1 \\ \nabla u_1^{(m)}\overrightarrow{n_1} = \nabla u_2^{(m-1)}\overrightarrow{n_1} \text{ at } \Gamma_1 \\ u_1^{(m)} = g \text{ on } \Gamma_t \cup \Gamma_b \\ u_1^{(m)} = g \text{ at } \Gamma_L \end{cases}, \begin{cases} \Delta u_2^{(m)} + (k^2 - ik)u_2^{(m)} = -f \text{ in } \Omega_2 \\ \nabla u_2^{(m)}\overrightarrow{n_2} = \nabla u_1^{(m-1)}\overrightarrow{n_2} \text{ at } \Gamma_2 \\ u_2^{(m)} = g \text{ on } \Gamma_t \cup \Gamma_b \\ -\nabla u_2^{(m)}\overrightarrow{n_2} = h \text{ at } \Gamma_R. \end{cases}$$

Theorem 7. *The reduction factor of the Parallel Schwarz algorithm for case* 7 *is given by the formula*

$$rcs_7^2(\tilde{k}; k, a_1, a_2, b_1, b_2) = \left| \frac{\cosh(\lambda(\tilde{k})(a_2 - a_1))}{\sinh(\lambda(\tilde{k})(b_2 - a_2))} \frac{\sinh(\lambda(\tilde{k})(b_2 - b_1))}{\cosh(\lambda(\tilde{k})(b_1 - a_1))} \right|.$$

Proof. The proof is obtained by investigating the spectral properties of the Schwarz iteration matrix. Computing explicitly the spectrum and the spectral radius of the Schwarz matrix leads to the desired formula for the convergence factor.

11. The Formulation of the Parallel Schwarz algorithm - Case 8

The boundary conditions are specified in the following way: Dirichlet on top and bottom, Neumann at the two interfaces, Neumann at the left edge of the global domain and Dirichlet at the right edge. As a result, the Schwarz algorithm reads

$$\begin{cases} \Delta u_1^{(m)} + (k^2 - ik)u_1^{(m)} = -f \text{ in } \Omega_1 \\ \nabla u_1^{(m)} \overrightarrow{n_1} = \nabla u_2^{(m-1)} \overrightarrow{n_1} \text{ at } \Gamma_1 \\ u_1^{(m)} = g \text{ on } \Gamma_t \cup \Gamma_b \\ -\nabla u_1^{(m)} \overrightarrow{n_1} = h \text{ at } \Gamma_L \end{cases}, \quad \begin{cases} \Delta u_2^{(m)} + (k^2 - ik)u_2^{(m)} = -f \text{ in } \Omega_2 \\ \nabla u_2^{(m)} \overrightarrow{n_2} = \nabla u_1^{(m-1)} \overrightarrow{n_2} \text{ at } \Gamma_2 \\ u_2^{(m)} = g \text{ on } \Gamma_t \cup \Gamma_b \\ u_2^{(m)} = g \text{ at } \Gamma_R. \end{cases}$$

Theorem 8. *The reduction factor of the Parallel Schwarz algorithm for case* 8 *is given by the formula*

$$rcs_8^2(\tilde{k}; k, a_1, a_2, b_1, b_2) = \left| \frac{\sinh(\lambda(\tilde{k})(a_2 - a_1))}{\cosh(\lambda(\tilde{k})(b_2 - a_2))} \frac{\cosh(\lambda(\tilde{k})(b_2 - b_1))}{\sinh(\lambda(\tilde{k})(b_1 - a_1))} \right|.$$

Proof. The result follows using the concepts and the steps from the previously analysed configurations.

12. The Optimized Schwarz Algorithm - Case 9

As a last scenario, we pay attention to the Optimized Schwarz method, where impedance conditions are employed on the left and right edge of the global domain and at the two interfaces. The Optimized Schwarz method in strong form reads

$$\begin{cases} \Delta u_1^{(m)} + (k^2 - ik)u_1^{(m)} = -f \text{ in } \Omega_1 \\ \nabla u_1^{(m)}\overrightarrow{n_1} + iku_1^{(m)} = \nabla u_2^{(m-1)}\overrightarrow{n_1} + iku_2^{(m-1)} \text{ at } \Gamma_1 \\ u_1^{(m)} = 0 \text{ on } \Gamma_t \cup \Gamma_b \\ \nabla u_1^{(m)}\overrightarrow{n_1} + iku_1^{(m)} = 0 \text{ at } \Gamma_L \end{cases},$$

$$\begin{cases} \Delta u_2^{(m)} + (k^2 - ik)u_2^{(m)} = -f \text{ in } \Omega_2 \\ \nabla u_2^{(m)}\overrightarrow{n_2} + iku_2^{(m)} = \nabla u_1^{(m-1)}\overrightarrow{n_2} + iku_1^{(m-1)} \text{ at } \Gamma_2 \\ u_2^{(m)} = 0 \text{ on } \Gamma_t \cup \Gamma_b \\ \nabla u_2^{(m)} + iku_2^{(m)} = 0 \text{ at } \Gamma_R. \end{cases},$$

Theorem 9. *The convergence factor of the Optimized Schwarz algorithm is given by the formula*

$$r_{os}^2(\tilde{k}; k, a_1, a_2, b_1, b_2) = \frac{|(1 - e^{2\lambda(\tilde{k})(b_2 - b_1)})(1 - e^{-2\lambda(\tilde{k})(a_2 - a_1)})|}{|(1 - \zeta^2 e^{-2\lambda(\tilde{k})(b_2 - a_1)})(1 - \zeta^{-2} e^{2\lambda(\tilde{k})(b_2 - a_2)})|}$$

where $\zeta = \dfrac{\lambda - ik}{\lambda + ik}$.

Proof. The proof is straightforward and it follows in the same fashion with the previous configurations. We can write the Optimized Schwarz algorithm in compact form

$$\begin{bmatrix} A_1^{(m)}(\tilde{k}) \\ A_2^{(m)}(\tilde{k}) \end{bmatrix} = \Psi_{os} \begin{bmatrix} A_1^{(m-1)}(\tilde{k}) \\ A_2^{(m-1)}(\tilde{k}) \end{bmatrix}$$

where $\Psi_{os} = \begin{bmatrix} 0 & \dfrac{(1 - e^{2\lambda(\tilde{k})(b_2 - b_1)})}{(1 - \zeta^2 e^{-2\lambda(\tilde{k})(b_2 - a_1)})} \\ \dfrac{(1 - e^{-2\lambda(\tilde{k})(a_2 - a_1)})}{(1 - \zeta^{-2} e^{2\lambda(\tilde{k})(b_2 - a_2)})} & 0 \end{bmatrix}$ is the Schwarz iteration matrix. By doing a little algebra, we obtain the reduction factor

$$r_{os}^2(\tilde{k}; k, a_1, a_2, b_1, b_2) = \frac{|(1 - e^{2\lambda(\tilde{k})(b_2 - b_1)})(1 - e^{-2\lambda(\tilde{k})(a_2 - a_1)})|}{|(1 - \zeta^2 e^{-2\lambda(\tilde{k})(b_2 - a_1)})(1 - \zeta^{-2} e^{2\lambda(\tilde{k})(b_2 - a_2)})|}.$$ □

13. Remarks

From the analysis conducted in the previous sections of the paper, there are some remarks to be made. In case 2 and 3, we observe that the reduction factors are the same, therefore the Schwarz algorithm has the same behaviour for these two configurations. Similarly case 1-6, case 4-8 and case 5-7 share the same formulas for the reduction factors. It is evident that for zero overlap the convergence factors become 1, so the Schwarz method stagnates, which is a very common experience in the literature. Also, for high Fourier frequencies, the contraction factors rapidly decay towards zero. For the evanescent modes, the Performance of the Schwarz method is very good and the contraction factors rapidly go to zero. However, the main concern for the Helmholtz equation is how the algorithm performs for the propagative modes. In cases 1-6 and 2-3 the performance of the algorithm is not so good as there are oscillations appearing for the propagative modes. These cases have in common the following: At the left and right edge and at the interfaces, the boundary conditions are purely Dirichlet or purely Neumann. Now, taking into account cases 4-8 and 5-7, we notice that there is something in common: One of the boundary conditions at the interfaces and the edges differs from the other boundary conditions. For instance, in case 4 we have Dirichlet interface conditions, Dirichlet at the left edge of Ω and Neumann at the right edge of Ω. Therefore the boundary condition at the right edge differs from the other ones. This enables us to have better performance of the Schwarz method for the propagative modes. In the table below there is comprehensive information for all the different configurations. The D denotes the Dirichlet, N denotes the Neumann, and R denotes the Robin boundary conditions. The

Table 1. All reported cases for varying the boundary conditions.

	Γ_t	Γ_b	Γ_L	Γ_R	Γ_1	Γ_2	prop. modes	ev. modes
Case 1	D	D	D	D	D	D	X	√
Case 2	D	D	D	D	N	N	X	√
Case 3	D	D	N	N	D	D	X	√
Case 4	D	D	D	N	D	D	√	√
Case 5	D	D	N	D	D	D	√	√
Case 6	D	D	N	N	N	N	X	√
Case 7	D	D	D	N	N	N	√	√
Case 8	D	D	N	D	N	N	√	√
Case 9	D	D	R	R	R	R	√	√

last columns represent the propagative modes and the evanescent modes, where the checkmark means that the Schwarz method performs well for these modes and the xmark means that the algorithm performs badly for this regime. We notice that for all the cases, the Schwarz method behaves well for the evanescent modes and the convergence factor rapidly decreases to zero. The problem occurs when we are in the region of the propagative modes. We clearly see that when the boundary conditions are the same on both edges (columns 4 and 5), then there is a problem with the propagative modes. On the contrary, using different boundary conditions on the two edges helps with the convergence of the Schwarz method. Lastly, in case 9 where we have Robin conditions at the interfaces and the right-left edges, the convergence is improved drastically for both evanescent and propagative modes.

14. Some Illustrations

In this section we provide the graphs of the convergence factors as a function of Fourier frequency. In addition to that the eigenvalues of Schwarz matrices for each algorithm is exhibited. These graphs are very revealing as they illustrate the behaviour of the Parallel Schwarz algorithm for the different Fourier modes. For the plots, we consider that $\Omega_1 = (0,\, 2 + h) \times (0,\, 1)$, $\Omega_2 = (2 - h,\, 4) \times (0,\, 1)$, where the overlap is $2h$ and h is the mesh size. The two subdomains have the same size and the height of the rectangle is 1.

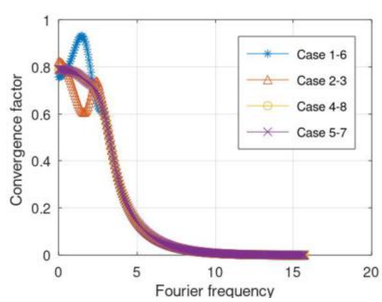

Figure 1. The convergence factor for different methods as a function of the Fourier frequency for wave number $k = \pi$. The mesh size is $h = 0.25$.

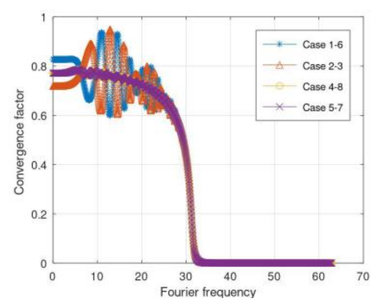

Figure 2. The convergence factor for different methods as a function of the Fourier frequency for wave number $k = 10\pi$. The mesh size is $h = 0.25$.

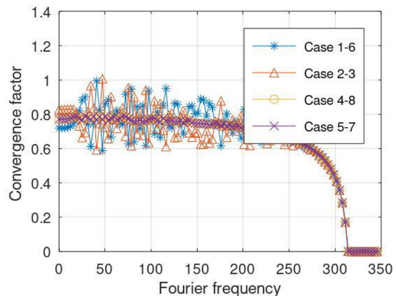

Figure 3. The convergence factor for different methods as a function of the Fourier frequency for wave number $k = 100\pi$. The mesh size is $h = 0.25$.

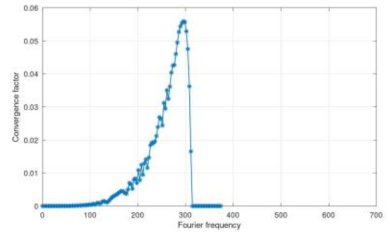

Figure 4. The convergence factor of the Optimized Schwarz method as a function of the Fourier frequency for wave number $k = 100\pi$. The mesh size is $h = 0.25$.

From the above convergence curves, it is evident that keeping different boundary conditions at the left and right edge of the global domain Ω helps with the convergence for the propagative modes. If the same boundary conditions are imposed at the two edges, then oscillations occur in the propagative regime. If Robin Conditions are imposed on the external edges and on the interfaces, the Schwarz method has good behaviour in propagative and evanescent regime.

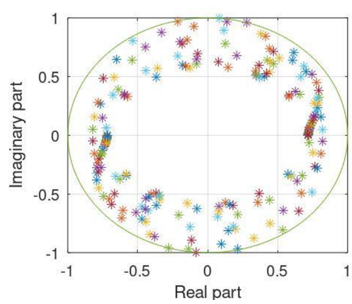

Figure 5. The spectrum of Schwarz iteration matrix (Cases 1-6) for a range of Fourier modes, for fixed wave number $k = 100\pi$. The mesh size is $h = 0.25$.

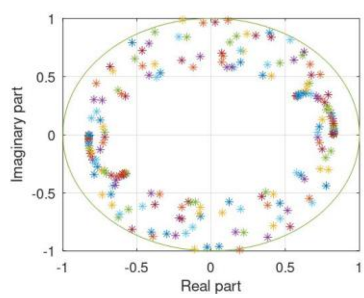

Figure 6. The spectrum of Schwarz iteration matrix (Cases 2-3) for a range of Fourier modes, for fixed wave number $k = 100\pi$. The mesh size is $h = 0.25$.

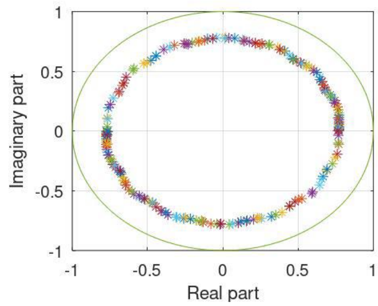

Figure 7. The spectrum of Schwarz iteration matrix (Cases 4-8) for a range of Fourier modes, for fixed wave number $k = 100\pi$. The mesh size is $h = 0.25$.

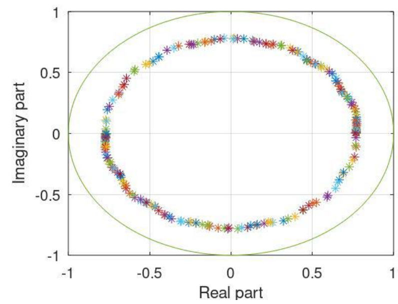

Figure 8. The spectrum of Schwarz iteration matrix (Cases 5-7) for a range of Fourier modes, for fixed wave number $k = 100\pi$. The mesh size is $h = 0.25$.

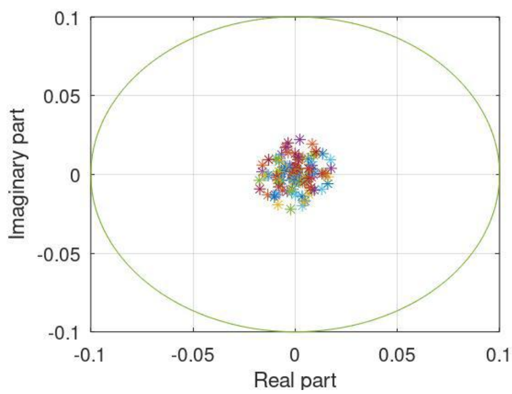

Figure 9. The spectrum of Schwarz iteration matrix (Case 9) for a range of Fourier modes, for fixed wave number $k = 100\pi$. The mesh size is $h = 0.25$.

15. Conclusions

In this paper, we have analysed the performance of the Parallel Schwarz algorithm for the perturbed Helmholtz problem for different configurations of the boundary conditions at the interfaces and exterior edges. We have obtained the appropriate convergence factors for all the different cases and we noticed that using the same boundary conditions at the external edges results in bad performance of the Schwarz method in the propagative regime. On the contrary, different boundary conditions at the external edges of the domain Ω result in better performance of the iterative method for the propagative modes. The choice of the boundary conditions is Dirichlet or Neumann. We did also consider the Optimized Schwarz algorithm using impedance conditions at the interfaces and at the left and right edge of the global domain. In addition to the curves of the reduction factors, the spectrum of the iteration matrices are provided for each method. Ultimately, Further analysis can be carried out for more than two subdomains [14], [15], [16] i.e. two, three etc., but this could be more complicated as we have more interface conditions. This analysis will appear in future work.

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