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NEW INFINITE NUMBERS TO QUANTIFY INFINITY AND THEIR APPLICATIONS

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Abstract

In the present study, the new infinite numbers and functions briefly mentioned in the author's previous short conference paper are here introduced, formulated and demonstrated. In particular, the fundamental definitions, lemmas, theorems, properties, and illustrative mathematical and engineering applications are presented and proved. Using the well-founded theory of limits of functions, the developed new infinite numbers quantify infinity (in a way different than past efforts) and are a useful tool for solving problems where infinity appears. The reason is that they allow arithmetic operations and calculus in mathematical expressions where infinity occurs. The set of infinite numbers is a superset of the complex numbers set. The extended in the infinite numbers bilateral Laplace transform, also proposed here, makes it possible to solve specific differential equations defined piecewise over the entire domain of real numbers $(-\infty, +\infty)$. Their solutions in general belong to the set of infinite functions. However, they also include the solutions belonging to the well-known real-complex functions which are a subset of the previous set. Solving these problems is not possible using the normal Laplace transform, since it is only defined for positive real values. It is interesting to note, that by using the infinite number-functions, long series of infinite terms can be nicely transformed into short elegant infinite functions whose computation is an easy task. In this way a simple, efficient criterion for series convergence is also developed. Furthermore, by calculating and using the derivatives/integrals of these infinite functions, complicated limits of infinite series of numbers, as well as ratios of the form ∞/∞ (involving series and improper integrals), can be easily calculated in cases where L'Hopital's rule cannot be applied. These complex limits and ratios are difficult to calculate in the conventional way, as there is no general method for their calculation. In addition, a new numerical method is developed for the easier and more accurate computation of a series of numbers where the sum is not known analytically. Furthermore, the abstract structure of these new infinite numbers is investigated and presented. Finally, certain

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technological applications are presented, such as complex, infinite, electrical networks (proposed by H. Zemanian, H. Flanders, C. Thomassen, etc) and specific kinematic problems in which infinity appears. In conclusion, infinite numbers easily solve problems that are quite difficult or impossible to solve by conventional methods.

I. Introduction

INFINITY represents something that is boundless or endless or something that is larger than any real or natural number [1]. Three main types of infinity may be distinguished: the mathematical, the physical, and the metaphysical. Since the age of the ancient Greeks, philosophers have engaged in many debates over the philosophical essence of infinity. The earliest recorded concept of infinity would probably be that of Anaximander (c. 610-c. 546 BC), a pre-Socratic philosopher.

In the seventeenth century, mathematicians started working with infinite series and also what mathematicians (notably l'Hopital and Bernoulli) saw as infinitely small quantities, but infinity remained linked to unending processes [2]. The infinity symbol, ∞, was introduced in 1655 by John Wallis [3, 4].

The topic of infinitely small numbers prompted the discovery of calculus in the late 1600s by the English mathematician Isaac Newton and the German mathematician Gottfried Wilhelm Leibniz. Newton presented his own theory of infinitely small numbers, or infinitesimals, to justify the calculation of derivatives or slopes.

A more direct use of infinity in mathematics emerges with attempts to compare the sizes of infinite sets, such as the set of points on a line (real numbers) or the set of counting numbers. In a demonstration by Galileo, he showed that the set of counting numbers could be put into one-to-one correspondence with the obviously much smaller set of their squares and, similarly, illustrated that the set of counting numbers and their doubles (i.e., the set of even numbers) could be paired together. Galileo concluded that "we cannot speak of infinite quantities as one greater than or less than or equal to another".

Richard Dedekind, a German mathematician, proposed a definition of an infinite set as one that may be related to some proper subset by one-to-one correspondence. At the turn of the nineteenth century, the German mathematician Georg Cantor extended the mathematical study of infinity by classifying infinite sets and infinite (transfinite) numbers, demonstrating that they may have different sizes [2, 5]. In this sense, infinity is a mathematical notion, and infinite mathematical objects are exactly like any other mathematical item in that they may be examined, handled, and utilized in the same way as any other mathematical object. Cantor defined two kinds of infinite numbers: ordinal numbers and cardinal numbers. The smallest ordinal of infinity is that of the positive integers. In 1873, Cantor clearly demonstrated that the set of rational numbers is the same size as the counting numbers; therefore, they are called countable or denumerable. Cantor also proved that not all infinities are equal by using a so-called "diagonal argument". He was able to show that the size of the counting numbers is strictly less than the size of the real numbers. This outcome is known as Cantor's theorem [6].

Cantor distinguished between a particular set and the abstract concept of its size or cardinality when comparing sets. Cantor called the sizes of his infinite sets "transfinite cardinals". His arguments showed that there are transfinite cardinals of endlessly numerous different sizes. The transfinite cardinals include aleph-null (the size of the set of natural numbers), aleph-one (the next larger infinity), and the continuum (the size of real numbers). These three numbers are also written as $\aleph_0$, $\aleph_1$, and c, respectively. One of Cantor's most important results was that the cardinality of the continuum c is greater than that of the natural numbers $\aleph_0$; that is, there are more real numbers, $\mathbb{R}$, than natural numbers, $\mathbb{N}$. Namely, Cantor showed that $c = 2^{\aleph_0} > \aleph_0$.

So, the modern mathematical conception of quantitative infinity developed in the late 19th century from works by Cantor, Gottlob Frege, Richard Dedekind, and others using the idea of collections or sets [2, 7]. Cantor's ideas mainly prevailed, and contemporary mathematics now recognizes actual infinity as a necessary component of a consistent and coherent theory. Certain extended number systems, such as hyperreal numbers, incorporate ordinary (finite) numbers and infinite numbers of different sizes. Surreal numbers are the most natural collection of numbers that includes both the real numbers and the infinite ordinal numbers of Georg Cantor. They were invented by John H. Conway in 1969. Every real number is surrounded by surreals, which are closer to it than any real number.

Nonstandard analysis is used to reformulate calculus by using a logically rigorous notion of infinitesimal numbers [8, 12]. Nonstandard analysis arose in the early 1960s with the mathematician Abraham Robinson. According to him, the theory of infinitesimals was eventually replaced by the classical theory of limits [9, 10, 13]. In nonstandard analysis, infinitesimals are invertible, and their inverses are infinite numbers. These types of infinities are part of a hyperreal field; there is no equivalence between them, as there is with the Cantorian transfinites. For example, if H is an infinite number in this sense, then $\mathrm{H} + \mathrm{H} = 2\mathrm{H}$ and $\mathrm{H} + 1$ are distinct infinite numbers. This method of nonstandard calculus was fully developed by Keisler (1986) [11].

In addition to the above, based on the works of Flanders, Zemanian, and Thomassen, the theory and applications of infinite-transinfinite graphs and the corresponding electrical networks, the element values of which can be considered operators in a Hilbert-space H, were developed [14-22], [35-37]. Such infinite electrical networks are of practical importance since they arise naturally from the discretization of physical phenomena [21, 23]. According to Zemanian, there are very few classes of infinite electrical networks for which computational methods of their solution exist [23].

In the author's previous 4-page conference paper [38], the new infinite numbers and functions were briefly described. These new infinite numbers retain most of the properties of the real-complex numbers (arithmetic operations, powers, roots, etc.). In [39], the mirror infinite numbers and the infinite geometrical shapes with their properties were also described and used.

In the present study, the fundamental definitions, lemmas, theorems, properties concerning the new infinite numbers and functions, and illustrative mathematical and engineering applications are presented and proved. Unlike previous attempts to quantify infinity the new infinite numbers are defined as limits of complex functions tending to infinity. Using these numbers and functions, the extended (in infinite numbers) bilateral

Laplace transform, also proposed here, makes it possible to solve specific differential equations defined piecewise over the entire domain of real numbers $(-\infty, +\infty)$. Their solutions (verified to be true) in general belong to the set of infinite functions. However, they also include the solutions belonging to the well-known real-complex functions set. Solving these problems is not possible using the normal Laplace transform, since it is only defined for positive real values. By using the infinite numbers, long series of infinite terms can be nicely transformed into simple infinite functions whose computation is an easy task. Moreover, lemmas and theorems about the derivatives of infinite number functions are proved, and unusual limits of infinite series of numbers, as well as the ratios of the form ∞/∞ are calculated in cases where L'Hopital's rule cannot be applied. These complex limits and ratios are difficult to calculate with conventional methods. Also, a simple and efficient criterion for the convergence of series of numbers is developed. In addition, it is shown that the infinite numbers constitute an algebraic structure that is "non-Archimedean", and, moreover, in contrast to Hardy fields, the set of infinite numbers is not an ordered field. Furthermore, by using these infinite numbers and functions, a simple-to-apply numerical method is developed for the easier and accurate calculation of a series of numbers where the sum is not known analytically. Finally, these new numbers are used to model, analyze, and solve technological problems where infinity occurs, such as complex infinite networks and specific kinematic problems. In conclusion, infinite numbers can easily solve problems that are quite difficult or impossible to solve by conventional methods. Such problems are those below, described by eqs. (100), (102), (110), (113), (118), (119), (120), (123), ((124)-(126)), also infinite electrical networks of Figures 1-3 as well as the problem of Figure 4.

The rest of the paper is organized as follows: in Section II, the fundamental definitions and lemmas concerning the infinite numbers and functions are presented. Section III deals with the issues of the arithmetic operations and calculations on the infinite numbers. Section IV deals with the proposed extension of the Laplace transform in the set of infinite functions and the bilateral Laplace transform, as well as their applications in solving specific differential equations. Section V deals with the infinite series calculation (analytical-numerical) by using the new infinite functions, their derivatives/integrals, and their properties. Section VI deals with the abstract structure of the new infinite numbers. In Section VII, specific number series, ratios of the form ∞/∞ where L'Hopital's rule cannot be applied, and specific differential equations defined piecewise over the entire domain of real numbers are solved in the broader set of infinite functions. Furthermore, complex, difficult to solve, infinite electrical networks (proposed by H. Zemanian, H. Flanders, C. Thomassen, etc.) as well as specific kinematic problems are modeled, analyzed and solved based on the above. In Section VIII, the conclusions of the paper are presented.

II. Initial Definitions and Lemmas

If one proposes new numbers, one must establish all the necessary definitions, indicatively, the unit number, the inverse number, the opposite number, the arithmetic operations, etc. The key point in the new infinite numbers presented is the fact that the usual arithmetic operations and calculations apply and there is no contradiction. As seen in [38], the infinity unit, (ξ), the negative infinity unit, $(-\xi)$, the imaginary infinity unit, $(i\xi)$, and the inverse of the infinity unit, $(1/\xi)$ are defined by the following formulas (1)-(4). Moreover, the multiples of the infinity unit, $(\alpha\xi)$, where $\alpha \in \mathbb{C}$, the negative imaginary infinity unit, $(-i\xi)$, and all the natural powers of ξ were also defined [38].

$$\xi \equiv \lim_{x \to +\infty} (x) \tag{1}$$

$$-\xi = -\lim_{x \to +\infty} (x) = \lim_{x \to +\infty} (-x) \tag{2}$$

$$i\xi = i \lim_{x \to +\infty} (x) = \lim_{x \to +\infty} (ix) \tag{3}$$

$$\frac{1}{\xi} = \frac{1}{\lim\limits_{x \to +\infty} (x)} = \lim_{x \to +\infty} \left(\frac{1}{x}\right) = 0 \tag{4}$$

where $x \in \mathbb{R}$.

Lemma 1. *The square of the infinity unit* (ξ^2) *is the same as the square of the negative infinity unit* $(-\xi)^2$ *and is opposite to the square of the imaginary infinity unit* $(i\xi)^2$.

Proof. Based on eq. (1), the following applies:

$$\xi^2 = \left(\lim_{x \to +\infty} x\right)^2 \tag{5}$$

$$(-\xi)^2 = \left(-\lim_{x \to +\infty} x\right)^2 = \left(\lim_{x \to +\infty} x\right)^2 \tag{6}$$

$$(i\xi)^2 = \left(i \lim_{x \to +\infty} x\right)^2 = i^2\left(\lim_{x \to +\infty} x\right)^2 = -\left(\lim_{x \to +\infty} x\right)^2 \tag{7}$$

where $x \in \mathbb{R}$.

$$(5), (6), (7) \Rightarrow \xi^2 = (-\xi)^2 = -(i\xi)^2 \qquad \square$$

Lemma 2. *The following relation is true, i.e., the negative integer powers of the infinity unit*, ξ, *are all zero.*

$$\xi^{-n} = 0^n = 0 \qquad n \in \mathbb{N} \tag{8}$$

Proof. Based on eq. (1): $\xi^{-n} = \left(\lim_{x \to +\infty} x\right)^{-n} = \frac{1}{\xi^n} = \left(\frac{1}{\xi}\right)^n$

where $x \in \mathbb{R}$ and by using eq. (4), the following applies: $\xi^{-n} = 0^n = 0$ $\square$

The following (Definition 1) completes and specifies the definition of infinite numbers and infinite functions briefly mentioned in [38].

Definition 1. By definition, each single-value, continuous, differentiable, and non-oscillating complex function, ϕ, of the infinity unit ξ, that is, the function $\phi(\xi)$, is an "infinite number function" which is also an "infinite number". The set of all these numbers is the set of infinite numbers and is denoted by the capital letter **A** (in bold), which is the first letter of the Greek word for infinity: "**Α**πειρο-**A**piro".

For instance:

$$\phi(\xi) = \phi_1(\xi) + i\phi_2(\xi) = (4\xi^2 - 3\xi + \ln \xi) + i(\sqrt{\xi})$$

$$= 4\left(\lim_{x \to +\infty} x\right)^2 - 3\left(\lim_{x \to +\infty} x\right) + \ln\left(\lim_{x \to +\infty} x\right) + i\sqrt{\lim_{x \to +\infty} x}$$

$$\lim_{x \to +\infty} (4x^2 - 3x + \ln x) + i \lim_{x \to +\infty} \sqrt{x}$$

where $x \in \mathbb{R}$.

Remark. It should be noticed that the limit of a single-value, continuous, differentiable, and non-oscillating complex function $\phi(x)$, when $x \in \mathbb{R}$ tends to infinity (ξ), that is $\phi(\xi)$, it can be either infinity or a complex/real number. Therefore, infinite numbers also include complex numbers and constitute a wider set of numbers. For example,

- $\phi_1(\xi) = 3\xi + 2\xi = 3 \lim_{x \to +\infty} (x) + 2 \lim_{x \to +\infty} (x) = 5 \lim_{x \to +\infty} (x) = 5\xi$ (infinity)
- $\phi_2(\xi) = \dfrac{7\xi + 5.5\xi}{5\xi} = \dfrac{7 \lim_{x \to +\infty} (x) + 5.5 \lim_{x \to +\infty} (x)}{5 \lim_{x \to +\infty} (x)} = 2.5$ (rational real number)
- $\phi_3(\xi) = \dfrac{e\xi}{3\xi - \xi} = \dfrac{e \lim_{x \to +\infty} (x)}{3 \lim_{x \to +\infty} (x) - \lim_{x \to +\infty} (x)} = \dfrac{e}{2}$ (irrational real number)
- $\phi_4(\xi) = \dfrac{9\xi}{3\xi} + i \cdot \dfrac{2\xi}{\xi} = \dfrac{9 \lim_{x \to +\infty} (x)}{3 \lim_{x \to +\infty} (x)} + i \cdot \dfrac{2 \lim_{x \to +\infty} (x)}{\lim_{x \to +\infty} (x)} = 3 + 2i$ (complex number)
- $\phi_5(\xi) = \sin(\xi) = \sin\left(\lim_{x \to +\infty} (x)\right)$ (it oscillates)

where $x \in \mathbb{R}$.

In the last case above, we notice that $\varphi_5(\xi)$ oscillates, meaning that it does not converge to a finite number nor does it diverge to infinity, but it receives all the real values from -1 to +1 repeatedly. As stated in Definition 1, this oscillating function does not represent a single infinite number, as defined here. It represents a set of infinite numbers that will be presented in a later paper.

Thus, according to the above, the set of infinite numbers is a superset of the set of complex numbers.

Definition 2. For each different number $\varphi(\xi)$, there is a corresponding infinite number of the form $[1/\varphi(\xi)]$, which is its “inverse number”.

For instance, for the infinite number $(\xi^2 + 2)$, we can obtain its inverse number $1/(\xi^2 + 2)$, which is equal to zero.

III. Arithmetic Operations and Calculations on the Infinite Numbers

Lemma 3. *The sum/subtraction/product/ratio of two infinite numbers, $\phi_1(\xi)$ and $\phi_2(\xi)$, is obtained if one obtains the sum/subtraction/product/ratio of the respective functions $\phi_1(x)$ and $\phi_2(x)$, where $x \in \mathbb{R}$, and then takes the limit for x tending to infinity*.

Proof. According to formula (1), the infinity unit (ξ) can be replaced by the limit: $\lim_{x \to +\infty} (x)$. By also taking Definition 1 into account, $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$ are single-valued, continuous, differentiable, and non-oscillating complex functions and therefore, it holds that: $\phi_1(\xi) = \phi_1\left(\lim_{x \to +\infty} (x)\right) = \lim_{x \to +\infty} \phi_1(x)$ and $\phi_2(\xi) = \phi_2\left(\lim_{x \to +\infty} (x)\right) = \lim_{x \to +\infty} \phi_2(x)$.

Thus, relations (9), (10), and (11) apply:

$$\phi_1(\xi) \pm \phi_2(\xi) = \phi_1\left(\lim_{x \to +\infty} x\right) \pm \phi_2\left(\lim_{x \to +\infty} x\right) \tag{9}$$

$$= \lim_{x \to +\infty} (\phi_1(x) \pm \phi_2(x))$$

$$\phi_1(\xi) \cdot \phi_2(\xi) = \phi_1\left(\lim_{x \to +\infty} x\right) \cdot \phi_2\left(\lim_{x \to +\infty} x\right) \tag{10}$$

$$= \lim_{x \to +\infty} (\phi_1(x) \cdot \phi_2(x))$$

$$\frac{\phi_1(\xi)}{\phi_2(\xi)} = \frac{\phi_1\left(\lim_{x \to +\infty} x\right)}{\phi_2\left(\lim_{x \to +\infty} x\right)} = \lim_{x \to +\infty} \frac{\phi_1(x)}{\phi_2(x)} \qquad \square \tag{11}$$

Example:

$$\frac{(3\xi^2 + 2\xi) \cdot (2\xi - 1)}{\xi} = \lim_{x \to +\infty} \frac{(3x^2 + 2x) \cdot (2x - 1)}{x}$$

$$= \lim_{x\to+\infty} \frac{6x^3 - 3x^2 + 4x^2 - 2x}{x} = \lim_{x\to+\infty} (6x^2 + x - 2)$$

$$= 6\left(\lim_{x\to+\infty} x\right)^2 + \lim_{x\to+\infty} x - 2 = 6\xi^2 + \xi - 2.$$

Remark. Unlike infinity in its general consideration (∞), where arithmetic operations do not apply (e.g.: $\infty - \infty =$ undefined, $\infty/\infty =$ undefined, etc), arithmetic operations are possible on infinite numbers (e.g.: $\xi - \xi = 0$ also $\xi/\xi = 1$ also $4\xi/\xi = 4$, etc), given that (ξ) no longer represents infinity in its general determination, but its specific univocal form that is the infinity unit (ξ) defined by eq. (1).

The infinite numbers can be distinguished into infinite numbers of a polynomial form and infinite numbers of a non-polynomial form.

A. *Infinite Numbers of a Polynomial Form*

The infinite numbers of a polynomial form are defined as shown in eq. (12) below [38].

$$A = \phi(\xi) = \alpha_n\xi^n + a_{n-1}\xi^{n-1} + \ldots + a_1\xi + a_0, \tag{12}$$

where $\alpha_n, \alpha_{n-1}, \ldots, a_0 \in \mathbb{C}$ and $n \in \mathbb{N}$.

Lemma 4. *It applies that* $A = \alpha_n\xi^n \pm a_{n-1}\xi^n \pm \ldots \pm a_0\xi^n$

$$= (\alpha_n \pm a_{n-1} \pm \ldots \pm a_0)\xi^n \tag{13}$$

where $\alpha_n, \alpha_{n-1}, \ldots, a_0 \in \mathbb{C}$ *and* $n \in \mathbb{N}$.

Proof. By using eq. (1): $A = \alpha_n\xi^n \pm a_{n-1}\xi^n \pm \ldots \pm a_0\xi^n$

$$= a_n \cdot \left(\lim_{x\to+\infty} x\right)^n \pm a_{n-1} \cdot \left(\lim_{x\to+\infty} x\right)^n \pm \ldots \pm a_0\left(\lim_{x\to+\infty} x\right)^n$$

$$= a_n \lim_{x\to+\infty} x^n \pm a_{n-1} \lim_{x\to+\infty} x^n \pm \ldots \pm a_0 \lim_{x\to+\infty} x^n,\ x \in \mathbb{R}$$

$$= (\alpha_n \pm a_{n-1} \pm \ldots \pm a_0) \cdot \lim_{x\to+\infty} x^n = (\alpha_n \pm a_{n-1} \pm \ldots \pm a_0)\xi^n \qquad \square$$

Example.

$$3\xi^4 - 5\xi^4 = 3\left(\lim_{x\to+\infty} x\right)^4 - 5\left(\lim_{x\to+\infty} x\right)^4 = -2\left(\lim_{x\to+\infty} x\right)^4 = -2\xi^4.$$

Remark. Therefore, if we add or subtract infinite numbers of a polynomial form of the same power of the infinity unit (ξ), this operation is precisely executed as it is in the polynomials.

Lemma 5. *The following relation applies where the right-hand side of this equation represents the term with the highest power of the infinity unit* (ξ).

$$A = \alpha_n\xi^n + a_{n-1}\xi^{n-1} + \dots + a_1\xi + a_0 = \alpha_n\xi^n \tag{14}$$

where $\alpha_n, \alpha_{n-1}, \dots, a_0 \in \mathbb{C}$ *and* $n \in \mathbb{N}$.

Proof.

$$A = \alpha_n\xi^n + a_{n-1}\xi^{n-1} + \dots + a_1\xi + a_0$$

$$= \alpha_n\left(\lim_{x\to+\infty} x\right)^n + a_{n-1}\left(\lim_{x\to+\infty} x\right)^{n-1} + \dots + a_1\left(\lim_{x\to+\infty} x\right) + a_0$$

$$= \lim_{x\to+\infty}\left[\alpha_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0\right]$$

$$= \lim_{x\to+\infty}[\alpha_n x^n] \cdot \lim_{x\to+\infty}\left[1 + \frac{a_{n-1}}{\alpha_n x} + \dots + \frac{a_1}{\alpha_n x^{n-1}} + \frac{a_0}{\alpha_n x^n}\right]$$

$$= \lim_{x\to+\infty}[\alpha_n x^n] \cdot 1 = \alpha_n\xi^n \qquad \text{where } x \in \mathbb{R}.$$ □

B. *Infinite Numbers of a Non-polynomial Form*

Apart from the previous infinite numbers of a polynomial form, there are other continuous, differentiable, and non-oscillating functions of the infinity unit (ξ), e.g.,

$$e^{\xi+4} = \lim_{x\to+\infty} e^{x+4}, \quad \frac{\xi^2 + e^{2/\xi}}{\sqrt{\xi}} = \lim_{x\to+\infty} \frac{x^2 + e^{2/x}}{\sqrt{x}},$$

$$2e^{\sqrt{\xi}} - 3i\xi = 2 \lim_{x \to +\infty} e^{\sqrt{x}} - 3i \lim_{x \to +\infty} x,$$

$$\ln\left(\frac{\xi^2}{2}\right) = \lim_{x \to +\infty}\left(\ln \frac{x^2}{2}\right) \quad \text{etc.}$$

where $x \in \mathbb{R}$.

The following (Definition 3) introduce and formulates the terms "Simplified form" and "Advancement" about the infinite numbers, initially mentioned in paper [38].

Definition 3. By definition, for the "Full form" of an infinite number, $\phi(\xi)$, if $\varphi(\cdot)$ can be written as a finite sum of other functions $(\varphi = \varphi_1 + \varphi_2 + \ldots + \varphi_n)$, then the fastest growing one (φ_j) (where $1 \leq j \leq n$) is called its "Simplified form", i.e., $\varphi_j(\xi)$. Additionally, by definition, the remaining part of $\varphi(\xi)$, that is, $\varphi(\xi) - \varphi_j(\xi)$, is called the "Advancement" of the infinite number $\varphi(\xi)$. Moreover, by definition, the symbol $(\approx)$ will be used to distinguish the "Simplified form" from the "Full form" of any infinite number $\varphi(\xi)$.

Remark. Based on this definition, for infinite numbers of a polynomial form, the "Advancement" of the full infinite number is the part of it that includes all the terms with the lower powers of ξ, whereas the term with the highest power of ξ is its "Simplified form".

Therefore, the elimination of the "Advancement" leads to the "Simplified form" of an infinite number. For the sake of brevity, the "Simplified form" of an infinite number is abbreviated to (SF), the full form to (FF), and the advancement to (AD). Thus, eq. (15) applies. Hereafter, the symbol $(\approx)$ will be used to differentiate between FF and SF after simplifying an infinite number. Since AD contains the terms with the lower powers of ξ (and AD and SF are continuous functions), eq. (16) also applies and, therefore, eqs. (17) and (18) are valid.

$$FF = SF + AD \tag{15}$$

$$\frac{AD}{SF} = \frac{AD(\xi)}{SF(\xi)} = \frac{AD\left(\lim\limits_{x \to +\infty} x\right)}{SF\left(\lim\limits_{x \to +\infty} x\right)} = \lim_{x \to +\infty} \frac{AD(x)}{SF(x)} = 0 \tag{16}$$

$$\frac{FF}{SF} = \frac{SF + AD}{SF} = 1 + \frac{AD}{SF} = 1 \tag{17}$$

which is equivalently written as

$$\lim_{x \to +\infty} \frac{FF(x)}{SF(x)} = 1 \tag{18}$$

For instance, for the infinite number $\phi(\xi) = 5\xi^4 - 3\xi^3 + 1$, the following applies:

$$FF = 5\xi^4 - 3\xi^3 + 1,\ SF = 5\xi^4 \text{ and } AD = -3\xi^3 + 1.$$

Lemma 6. *Suppose that the infinite number A of the form of the following function has the powers* d_n, *which are real but not natural numbers. Prove that the simplified form SF of A is given by eq.* (19) *below* (*where the symbol* $(\approx)$ *is used as per Definition* 3).

$$A = \alpha_n \xi^{d_n} + a_{n-1}\xi^{d_{n-1}} + \ldots + a_1 \xi^{d_1}$$

where $d_j (j = 1, \ldots, n) \in \mathbb{R}$ *and* $d_n > d_{n-1} > \ldots > d_1$ *and* $\alpha_n, a_{n-1}, \ldots, a_1 \in \mathbb{C}$

$$A \approx \alpha_n \xi^{d_n}. \tag{19}$$

Proof. According to the above, respectively as in a polynomial form numbers (Lemma 5), it is similarly proved that the simplified form of A is given by the relation (19). Namely, by multiplying and dividing by $\alpha_n \xi^{d_n}$. □

Examples:

- Using the above lemmas and definitions, the following calculations hold about the infinite numbers $A_1 = \alpha_1 \xi^{d_1}$ and $A_2 = \alpha_2 \xi^{d_2}$, where $\alpha_1, \alpha_2 \in \mathbb{C}$ and $d_1, d_2 \in \mathbb{R}$.

if $d_1 > d_2 \Rightarrow A_1 \pm A_2 \approx A_1 = \alpha_1 \xi^{d_1}$ $(AD = +/- A_2$ and $SF = A_1)$

if $d_1 < d_2 \Rightarrow A_1 \pm A_2 \approx \pm A_2 = \pm\alpha_2 \xi^{d_2}$ $(AD = A_1$ and $SF = +/- A_2)$

if $d_1 = d_2 = d \Rightarrow A_1 \pm A_2 = (\alpha_1 \pm \alpha_2)\xi^d$

For instance: $0.1 \cdot \xi^{0.4} + 10^{100} \cdot \xi^{0.3} \approx 0.1 \cdot \xi^{0.4}$

Also, $A_1 \cdot A_2 = (\alpha_1 \cdot \alpha_2)\xi^{(d_1 + d_2)}$ (20)

and $A_1 / A_2 = (\alpha_1 / \alpha_2)\xi^{(d_1 - d_2)}$ (21)

- Find the absolute values of λ that satisfy the infinite numbers equation: $\xi^{(3+i)} = \lambda \cdot e^{3 \ln \xi} + \ln \xi$

Using formula (1), the previous equation becomes:

$$\lim_{x \to +\infty} x^{(3+i)} = \lim_{x \to +\infty} \lambda e^{3 \ln x} + \lim_{x \to +\infty} \ln x \tag{22}$$

where:

$$\lim_{x \to +\infty} x^{(3+i)} = \lim_{x \to +\infty} x^3 e^{i \ln x} = \lim_{x \to +\infty} x^3 (\cos \ln x + i \sin \ln x) \tag{23}$$

From eqs. (22) and (23) it follows:

$$\lim_{x \to +\infty} x^3 (\cos \ln x + i \sin \ln x) = \lim_{x \to +\infty} \lambda x^3 + \lim_{x \to +\infty} \ln x \Rightarrow$$

$$\lim_{x \to +\infty} (\cos \ln x + i \sin \ln x) = \lambda + \lim_{x \to +\infty} \frac{\ln x}{x^3} \Rightarrow$$

$$|\lambda| = \left| \lim_{x \to +\infty} (\cos \ln x + i \sin \ln x) \right| \Rightarrow |\lambda| = 1$$

Obviously, we reach exactly the same result if we work directly with ξ, without using the limits. It is noted that, in general, the solutions of the infinite numbers equations are infinite numbers.

Lemma 7. *The commutative and associative laws for addition and multiplication also apply to infinite numbers.*

Proof. By taking into account Definition 1, the functions $\varphi_1(\cdot)$, $\varphi_2(\cdot)$, and $\varphi_3(\cdot)$ (below) are single-valued, continuous, differentiable, and non-oscillating complex functions, and thus, the following applies, where $x \in \mathbb{R}$.

Commutative law for the addition:

$$\phi_1(\xi) + \phi_2(\xi) = \phi_1\left(\lim_{x \to +\infty} x\right) + \phi_2\left(\lim_{x \to +\infty} x\right) = \lim_{x \to +\infty} \phi_1(x) + \lim_{x \to +\infty} \phi_2(x)$$

$$= \lim_{x \to +\infty} \phi_2(x) + \lim_{x \to +\infty} \phi_1(x) = \phi_2\left(\lim_{x \to +\infty} x\right) + \phi_1\left(\lim_{x \to +\infty} x\right) = \phi_2(\xi) + \phi_1(\xi)$$

Associative law for the addition:

$$(\phi_1(\xi) + \phi_2(\xi)) + \phi_3(\xi) = \left(\phi_1\left(\lim_{x \to +\infty} x\right) + \phi_2\left(\lim_{x \to +\infty} x\right)\right) + \phi_3\left(\lim_{x \to +\infty} x\right)$$

$$= \lim_{x \to +\infty} (\phi_1(x) + \phi_2(x)) + \lim_{x \to +\infty} \phi_3(x) = \lim_{x \to +\infty} ((\phi_1(x) + \phi_2(x)) + \phi_3(x))$$

$$= \lim_{x \to +\infty} (\phi_1(x) + (\phi_2(x)) + \phi_3(x)) = \phi_1(\xi) + (\phi_2(\xi) + \phi_3(\xi))$$

Similarly, the rest of the above laws are proven. □

IV. Extension of Laplace Transform and Bilateral Laplace Transform to the Set of Infinite Numbers

Since we have quantified infinity using infinite numbers, it is possible and would be a good idea to calculate the Laplace transform over the entire frequency spectrum (also for values of $s \in \mathbb{C}$ $(s = \sigma + i\omega)$, for which the corresponding generalized integral does not converge). In ref. [38], the Laplace transform extension in the wider set of infinite functions is introduced by using the formula (24) below:

$$F(s, \xi) = \int_{t=0}^{t=+\infty} f(t)e^{-st}dt = \int_{t=1/\xi}^{t=\xi} f(t)e^{-st}dt \tag{24}$$

where $t >= 0$, $s \in \mathbb{C}$, and $f(t)$ is a single-value, continuous, differentiable, and non-oscillating real function and is, therefore, integrable on $[0, +\infty)$.

Additionally, the bilateral Laplace transform (or two-sided Laplace transform) in the wider set of infinite functions is introduced by using the formula (25) below:

$$F(s, \xi) = \int_{t=-\infty}^{t=+\infty} f(t)e^{-st}dt = \int_{t=-\xi}^{t=\xi} f(t)e^{-st}dt \tag{25}$$

where $-\infty < t < \infty$, $s \in \mathbb{C}$, and $f(t)$ is a single-value, continuous, differentiable, and non-oscillating real function and is, therefore, an integrable on $(-\infty, +\infty)$.

Based on the above, the Laplace transform extension and bilateral Laplace transform of the unit function $(f(t) = 1)$ and rump function $(f(t) = t)$ for $s < 0$ and $s = 0$ (in infinite numbers) were estimated [38].

Lemma 8. *The Laplace transform of* $f(t) = e^{\alpha t}$ *where* $\alpha \in \mathbb{R}$ *for* $s < \alpha$ *and* $s = \alpha$ (*in infinite numbers and functions*) *is given by the following formulas, respectively*:

$$F(s) = \frac{1 - e^{-(s-a)\xi}}{s - a} \approx -\frac{e^{-(s-a)\xi}}{s - a} \quad \text{for } s < \alpha \tag{26}$$

$$\text{and } F(s) = \xi - \frac{1}{\xi} \approx \xi \quad \text{for } s = \alpha \tag{27}$$

Proof. For $s \neq \alpha$, according to eq. (24), the following applies:

$$F(s) = \int_{t=1/\xi}^{t=\xi} e^{-(s-a)t} dt \approx -\frac{e^{-(s-a)\xi} - 1}{s - a} \tag{28}$$

Thus,

$$\text{for } s > \alpha : F(s) \approx \frac{1}{s - \alpha} \tag{29}$$

$$\text{for } s < \alpha : F(s) = \frac{1 - e^{-(s-a)\xi}}{s - a} \approx -\frac{e^{-(s-a)\xi}}{s - a} \tag{30}$$

Finally, for $s = \alpha$, eq. (24) becomes

$$F(s) = \int_{t=1/\xi}^{t=\xi} e^{0} dt = \int_{t=1/\xi}^{t=\xi} 1 dt = t\Big|_{1/\xi}^{\xi} = \xi - \frac{1}{\xi} \approx \xi \qquad \square \tag{31}$$

Moreover, by using relation (25), the bilateral Laplace transform for $s > \alpha$, is calculated:

$$F(s) = \int_{t=-\xi}^{t=\xi} e^{at} \cdot e^{-st} dt \approx \frac{e^{(s-a)\xi}}{s - a} \tag{32}$$

Theorem 1. *The Laplace transform of the sinusoidal functions* $f(t) = \cos(\omega t)$ *and* $g(t) = \sin(\omega t)$, *where* ω *is a real constant*, *for* $s < 0$ *and* $s = 0$ (*in infinite numbers*) *are given by the following formulas*, *respectively*:

$$F(s) \approx -\frac{e^{-s\xi}}{\sqrt{s^2 + \omega^2}} \cdot \cos[\omega\xi + \arctan(\omega/s)] \quad \textit{for } s < 0,$$

$$F(s) = \frac{\sin(\omega\xi)}{\omega} \quad \textit{for } s = 0$$

$$G(s) \approx -\frac{e^{-s\xi}}{\sqrt{s^2 + \omega^2}} \cdot \sin[\omega\xi + \arctan(\omega/s)] \quad \textit{for } s < 0,$$

$$G(s) = \frac{1 - \cos(\omega\xi)}{\omega} \quad \textit{for } s = 0.$$

Proof. When using relation (24), it holds that

$$F(s) = \int_{t=0}^{t=+\infty} \cos(\omega t) \cdot e^{-st} dt \tag{33}$$

$$G(s) = \int_{t=0}^{t=+\infty} \sin(\omega t) \cdot e^{-st} dt \tag{34}$$

Now, let us take expression A:

$$A = F(s) + iG(s) = \int_{t=0}^{t=+\infty} \cos(\omega t) \cdot e^{-st} dt + i \int_{t=0}^{t=+\infty} \sin(\omega t) \cdot e^{-st} dt$$

$$= \int_{t=0}^{t=+\infty} e^{i\omega t} e^{-st} dt = \lim_{T \to +\infty} \frac{e^{(i\omega - s)t}}{i\omega - s} \Bigg|_{t=0}^{t=T} = \lim_{T \to +\infty} \frac{e^{(i\omega - s)T} - 1}{i\omega - s}$$

$$\Rightarrow A = \lim_{T \to +\infty} \frac{e^{(iw - s)T} - 1}{i\omega - s} = \frac{e^{(i\omega - s)\xi} - 1}{i\omega - s} \tag{35}$$

From eq. (35), it follows that

- on the one hand, for $s > 0$,

$$A \approx \frac{s}{s^2 + \omega^2} + i \frac{\omega}{s^2 + \omega^2} \tag{36}$$

- on the other hand, for $s < 0$, after the calculations,

$$A \approx -\frac{e^{-s\xi} \cdot e^{i[\omega\xi + \arctan(\omega/s)]}}{\sqrt{s^2 + \omega^2}} \Rightarrow$$

$$A \approx -\frac{e^{-s\xi}}{\sqrt{s^2 + \omega^2}} \cdot \cos[\omega\xi + \arctan(\omega/s)]$$

$$+ \left(-i\frac{e^{-s\xi}}{\sqrt{s^2 + \omega^2}} \sin[\omega\xi + \arctan(\omega/s)]\right) \tag{37}$$

Hence, as expected, for $s > 0$, from (33), (34), (35), and (36), it follows that

$$F(s) \approx \frac{s}{s^2 + \omega^2} \tag{38}$$

$$G(s) \approx \frac{\omega}{s^2 + \omega^2} \tag{39}$$

Additionally, for $s < 0$, from (33), (34), (35), and (37), it follows that

$$F(s) \approx -\frac{e^{-s\xi}}{\sqrt{s^2 + \omega^2}} \cdot \cos[\omega\xi + \arctan(\omega/s)] \tag{40}$$

$$G(s) \approx -\frac{e^{-s\xi}}{\sqrt{s^2 + \omega^2}} \cdot \sin[\omega\xi + \arctan(\omega/s)] \tag{41}$$

Finally, for $s = 0$, the relations (33) and (34) become

$$F(s) = \int_{t=0}^{t=+\infty} \cos(\omega t)dt = \frac{\sin(\omega t)}{\omega}\Big|_0^{\xi} = \frac{\sin(\omega\xi)}{\omega}$$

$$G(s) = \int_{t=0}^{t=+\infty} \sin(\omega t)dt = -\frac{\cos(\omega t)}{\omega}\Big|_0^{\xi} = -\frac{\cos(\omega\xi) - 1}{\omega} = \frac{1 - \cos(\omega\xi)}{\omega}$$

Therefore, for $s = 0$,

$$F(s) = \frac{\sin(\omega\xi)}{\omega} \tag{42}$$

$$G(s) = \frac{1 - \cos(\omega\xi)}{\omega} \qquad \square \tag{43}$$

Solving specific differential equations by using infinite numbers and functions

As is known, the Laplace transform is often used as a mathematical tool to solve ordinary differential equations. The method, presented here, for solving specific differential equations using the infinite numbers and functions will be best demonstrated and illustrated by means of a typical example. Let us look at the following specific differential equation that is defined piecewise over the entire domain of real numbers, $t \in \mathbb{R}$, (and not only on positive real numbers), where, as is known, for $t < 0$, we cannot obtain the Laplace transform (without using infinite numbers).

$$y'(t) + y(t) = f(t) \quad \text{where } \lim_{t \to -\infty} y(t) = 1 \tag{44}$$

and where

$$f(t) = e^{3t} \text{ for } t \in (-\infty, +5) \tag{45}$$

$$f(t) = 2 \text{ for } t \in [+5, +\infty) \tag{46}$$

Proof. By using infinite numbers/functions, we can certainly obtain the bilateral Laplace transform of eq. (44); that is, over the entire domain of the real numbers $(-\infty, +\infty)$. The problem cannot be solved by using the one-sided Laplace transform. It is noted that, for the bilateral Laplace transform, the derivative property is as follows:

$$L(y'(t)) = sY(s) + y(t)e^{-st} \Big|_{-\infty}^{+\infty} \tag{47}$$

where $L(\cdot)$ is the Laplace transformation symbol.

Additionally, when using infinite numbers, the limit, $\lim_{t \to -\infty} y(t) = 1$, is equivalently written as $y(-\xi) = 1$.

Based on eq. (52), eq. (51) is transformed:

$$sY(s) + y(\xi)e^{-s\xi} - y(-\xi)e^{s\xi} + Y(s) = \int_{t=-\xi}^{t=\xi} f(t)e^{-st}dt$$

where, for $s > 3$, we have

$$sY(s) - y(-\xi)e^{s\xi} + Y(s) = \int_{t=-\xi}^{t=\xi} f(t)e^{-st}dt$$

and since $y(-\xi) = 1$, we have

$$sY(s) - e^{s\xi} + Y(s) = \int_{t=-\xi}^{t=\xi} f(t)e^{-st}dt$$

$$= \int_{t=-\xi}^{t=0} e^{3t}e^{-st}dt + \int_{t=0}^{t=5} e^{3t}e^{-st}dt + \int_{t=5}^{t=\xi} 2 \cdot e^{-st}dt$$

$$= \int_{t=-\xi}^{t=0} e^{-(s-3)t}dt + \int_{t=0}^{t=5} e^{-(s-3)t}dt + 2\int_{t=5}^{t=\xi} e^{-st}dt$$

$$= -\frac{1}{s-3} e^{-(s-3)t} \Big|_{t=-\xi}^{t=0} - \frac{1}{s-3} e^{-(s-3)t} \Big|_{t=0}^{t=5} - \frac{2}{s} e^{-st} \Big|_{t=5}^{t=\xi}$$

$$= -\frac{1 - e^{(s-3)\xi}}{s-3} - \frac{e^{-(s-3)5} - 1}{s-3} - \frac{0 - 2e^{-5s}}{s}$$

$$= \frac{e^{(s-3)\xi}}{s-3} - \frac{e^{-(s-3)5}}{s-3} + \frac{2e^{-5s}}{s} \tag{48}$$

Therefore,

$$(s+1)Y(s) = \frac{e^{(s-3)\xi}}{s-3} - \frac{e^{-(s-3)5}}{s-3} + 2\frac{e^{-5s}}{s} + e^{s\xi} \Rightarrow$$

$$Y(s) = \frac{e^{(s-3)\xi}}{(s+1)(s-3)} - \frac{e^{-(s-3)5}}{(s+1)(s-3)} + 2\frac{e^{-5s}}{s(s+1)} + \frac{e^{s\xi}}{s+1}$$

Thus,

$$Y(s) = -\frac{1}{4}\frac{e^{(s-3)\xi}}{(s+1)} + \frac{1}{4}\frac{e^{(s-3)5}}{(s-3)} + \frac{1}{4}\frac{e^{-(s-3)5}}{(s+1)} - \frac{1}{4}\frac{e^{-(s-3)5}}{(s-3)}$$

$$+ 2\frac{e^{-5s}}{s} + 2\frac{e^{-5s}}{s+1} + \frac{e^{s\xi}}{s+1}$$

Furthermore,

$$Y(s) = -\frac{1}{4}e^{-4\xi}\frac{e^{(s+1)\xi}}{(s+1)} + \frac{1}{4}\frac{e^{(s-3)5}}{(s-3)} + \frac{1}{4}e^{20}\frac{e^{-5(s+1)}}{(s+1)}$$
$$-\frac{1}{4}\frac{e^{-5(s-3)}}{(s-3)} + 2\frac{e^{-5s}}{s} + 2e^{5}\frac{e^{-5(s+1)}}{s+1} + e^{-\xi}\frac{e^{(s+1)\xi}}{s+1}$$

It is noted that the inverse bilateral Laplace transform of the expression $\frac{e^{(s-a)\xi}}{(s-\alpha)}$, where $s > \alpha$, is

$$L^{-1}\left(\frac{e^{(s-\alpha)\xi}}{(s-\alpha)}\right) = e^{\alpha t}u(t-(-\xi)) = e^{\alpha t}$$

Indeed,

$$\int_{-\xi}^{\xi} e^{\alpha t}e^{-st}dt = \int_{-\xi}^{+\xi} e^{-(s-\alpha)t}dt = -\frac{1}{s-a}e^{-(s-\alpha)}\Big|_{-\xi}^{+\xi}$$
$$-\frac{e^{-(s-\alpha)\xi} - e^{(s-\alpha)\xi}}{s-\alpha} = \frac{e^{(s-\alpha)\xi}}{s-\alpha}$$

Hence, by taking the inverse Laplace transform of $Y(s)$, we have

$$y(t) = -\frac{1}{4}e^{-4\xi}e^{-t}u(t-(-\xi)) + \frac{1}{4}e^{3t}u(t-(-\xi))$$
$$+\frac{1}{4}e^{20}e^{-t}u(t-5) - \frac{1}{4}e^{3t}u(t-5) + 2u(t-5) + 2e^{5}e^{-t}u(t-5)$$
$$+ e^{-\xi}e^{-t}u(t-(-\xi))$$

Since $u(t-(-\xi)) = 1$, we obtain the solution:

$$y(t) = -\frac{1}{4}e^{-4\xi}e^{-t} + \frac{1}{4}e^{3t} + \frac{1}{4}e^{20}e^{-t}u(t-5) - \frac{1}{4}e^{3t}u(t-5)$$
$$+ 2u(t-5) + 2e^{5}e^{-t}u(t-5) + e^{-\xi}e^{-t} \tag{49}$$

Relation (49) can be written equivalently:

For $t < 5$,

$$y(t) = -\frac{1}{4}e^{-4\xi}e^{-t} + \frac{1}{4}e^{3t} + e^{-\xi}e^{-t} \tag{50}$$

For $t >= 5$,

$$y(t) = -\frac{1}{4}e^{-4\xi}e^{-t} + \frac{1}{4}e^{3t} + \frac{1}{4}e^{20}e^{-t} - \frac{1}{4}e^{3t} + 2 + 2e^{5}e^{-t} + e^{-\xi}e^{-t}$$

$$= -\frac{1}{4}e^{-4\xi}e^{-t} + \frac{1}{4}e^{20}e^{-t} + 2 + 2e^{5}e^{-t} + e^{-\xi}e^{-t} \quad (51)$$

Let us now proceed to verify the above solution generally described by eq. (49). First, we will examine whether eq. (49) and, consequently, eq. (50) satisfies the specific initial condition, i.e., if $y(-\xi) = 1$ holds. For $t = -\xi$, eq. (50) is transformed into eq. (52):

$$y(-\xi) = -\frac{1}{4}e^{-4\xi}e^{\xi} + \frac{1}{4}e^{-3\xi} + e^{-\xi}e^{\xi} \Rightarrow$$

$$y(-\xi) = -\frac{1}{4}e^{-3\xi} + \frac{1}{4}e^{-3\xi} + e^{-\xi}e^{\xi} = e^{0} = 1 \quad (52)$$

Therefore, the specific initial condition (which is a limit on minus infinity) is satisfied.

Let us now see if the original differential equation (44) is also verified by the solution in question. To this end, let us form the derivative $y'(t)$.

For $t < 5$,

$$y'(t) = \frac{1}{4}e^{-4\xi}e^{-t} + \frac{3}{4}e^{3t} - e^{-\xi}e^{-t} \quad (53)$$

So, based on eqs. (50) and (53), it holds that $y'(t) + y(t) = e^{3t}$ when $t < 5$, as it was expected.

For $t >= 5$,

$$y'(t) = \frac{1}{4}e^{-4\xi}e^{-t} - \frac{1}{4}e^{20}e^{-t} - 2e^{5}e^{-t} - e^{-\xi}e^{-t} \quad (54)$$

So, based on eqs. (51) and (54), it holds that $y'(t) + y(t) = 2$ when $t >= 5$, as it was also expected.

Hence, it is confirmed that eq. (49), which is written in infinite numbers, is indeed the solution to the problem, in the broader set of infinite functions. Certainly, the first and last terms of eq. (49) are infinite numbers, which, however, are zero. Therefore, they can be omitted, and consequently, our solution is finally described by the function (55), which is a real function.

$$y(t) = \frac{1}{4}e^{3t} + \frac{1}{4}e^{20}e^{-t}u(t-5) - \frac{1}{4}e^{3t}u(t-5) + 2u(t-5) + 2e^{5}e^{-t}u(t-5) \quad \square \tag{55}$$

Remark. Furthermore, if we solve the above problem (following the same procedure) but for $f(t) = e^{-3t}$ instead of $f(t) = e^{3t}$, we arrive at the solution below:

$$y(t) = \frac{1}{2}e^{2\xi}e^{-t} - \frac{1}{2}e^{-3t} - \frac{1}{2}e^{-10}e^{-t}u(t-5) + \frac{1}{2}e^{-3t}u(t-5)$$
$$+ 2u(t-5) + 2e^{5}e^{-t}u(t-5) + e^{-\xi}e^{-t} \tag{56}$$

However, in this solution, the first term is an infinite number, which is infinity and not zero, and therefore it cannot be omitted. So, this new differential equation has a solution in the broader set of infinite functions, but it does not have a solution in the set of real functions. The reason is that, in this differential equation, the condition $\lim_{t \to -\infty} y(t) = 1$ cannot be satisfied (in real numbers) since the exponential terms with the type Ce^{-3t}, where $C \in \mathbb{R}$, for $t \to -\infty$ all become $+\infty$ and do not cancel each other.

Consequently, as is shown above, the Laplace transform is extended, and now it is defined in the larger set of infinite functions for the values of s where it does not converge (and hence, it was not defined in the narrower set of complex functions). Additionally, the bilateral Laplace transform can be calculated using infinite numbers, even in the cases where the corresponding integral tends to infinity. As the Laplace transform is often used in applied mathematics and physics, it could be useful for applications where a conclusion cannot be drawn due to the divergence of calculations.

V. Derivatives/Integrals of Infinite Number Functions and Infinite Series Calculations

Lemma 9. *The derivative/integral of an infinite number function*, $\varphi(\xi)$, is *calculated if one obtains the derivative/integral of* $\varphi(x)$, *where* $x \in \mathbb{R}$, *and then one takes the limit for x tending to infinity*.

Proof. According to Definition 1, an infinite number function is a function $\varphi(\xi)$, where $\xi = \lim_{x \to +\infty} (x)$ and $\varphi(\cdot)$ is any single-value, continuous, differentiable, and non-oscillating complex function. Since each continuous differentiable function $\varphi(x)$, where $x \in \mathbb{R}$, has a derivative $\varphi'(x)$ which is also a continuous function, formula (57) holds. Also, since the integral of a continuous function $\varphi(x)$, where $x \in \mathbb{R}$, is also a continuous function, formula (58) applies. Finally, relation (59) holds.

$$\frac{d\phi(\xi)}{d\xi} = \phi'(\xi) = \phi'\left(\lim_{x \to +\infty} x\right) = \lim_{x \to +\infty} \phi'(x) \tag{57}$$

$$\int \phi(\xi) d\xi = \int \phi\left(\lim_{x \to +\infty} x\right) dx = \lim_{x \to +\infty} \int \phi(x) dx \tag{58}$$

$$d\xi = d\left(\lim_{x \to +\infty} (x)\right) = \lim_{x \to +\infty} (dx) \qquad \square(59)$$

Thus, the derivative/integral of an infinite number function, $\varphi(\xi)$, is easily calculated from the derivative/integral of the function $\varphi(x)$, where $x \in \mathbb{R}$, if, in the position of x, we consider ξ. For example,

$$\frac{d}{d\xi}(\ln \xi) = \frac{1}{\xi}$$

$$\int 5 d\xi = 5\xi + C \quad \text{where } C \in \mathbb{R}.$$

Remark. Based on the above, and if $F(x)$ is the integral of the function $f(x)$, where $x \in \mathbb{R}$, the improper integral of $f(x)$, using infinite numbers, is as follows:

$$\lim_{t \to +\infty} \int_{x=0}^{t} f(x) dx = \int_{x=0}^{\xi} f(x) dx = F(x)\Big|_{x=0}^{\xi} = F(\xi) - F(0).$$

Definition 4. By definition, a series of infinite terms, A_n, is called ordinary if this series presents $n \in \mathbb{N}$ in its (nth) term and not in all its terms.

For example, for $n \in \mathbb{N}$, the following series $A_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} + \ldots$ is an ordinary series, whereas the series

$A'_n = \ln\frac{1}{n} + \ln\frac{2}{n} + \ldots + \ln\frac{n}{n} + \ldots$ is not an ordinary series since it presents n in all the terms of the series.

When taking Definition 4 into account, the following theorem complements and formulates the calculations of the derivatives of infinite number functions, as briefly mentioned in [38].

Theorem 2. *The derivative of an ordinary series of numbers, A_n, written as an infinite number function, $A(\xi)$, is the last term (or, equivalently, the average of the last terms) of this infinite number function if the criterion of equality between them is satisfied.*

Proof. Based on eqs. (57), (58), and (59), the following also applies:

$$\frac{d}{d\xi}\phi(\xi) = \lim_{x\to+\infty}\frac{d}{dx}\phi(x) = \lim_{x\to+\infty}\frac{\phi(x) - \phi(x - dx)}{dx}$$

$$= \frac{\lim_{x\to+\infty}\phi(x) - \lim_{x\to+\infty}\phi(x - dx)}{\lim_{x\to+\infty} dx} = \frac{\phi(\xi) - \phi(\xi - d\xi)}{d\xi}$$

$$\text{or} \quad \frac{d}{d\xi}\phi(\xi) = \frac{\phi(\xi) - \phi(\xi - d\xi)}{d\xi}. \tag{60}$$

With the use of infinite numbers, the ordinary series, A_n, below can be scripted as an infinite number function, $A(\xi)$.

$$A_n = a_1 + a_2 + \ldots + a_n + \ldots \quad \text{where } n \in N$$

$$A(\xi) = a_1 + a_2 + \ldots + a_\xi \tag{61}$$

As an example, the series $A_n = 1^2 + 2^2 + 3^2 + \ldots + n^2 + \ldots$ can be transcribed as the infinite number function $A(\xi) = 1^2 + 2^2 + 3^2 + \ldots + \xi^2$.

Then, according to eq. (60), the derivative of the infinite function, $A(\xi)$, is given by (62).

$$\frac{d}{d\xi}A(\xi) = \frac{A(\xi) - A(\xi - d\xi)}{d\xi}. \tag{62}$$

The infinitesimal dx is an infinitely small quantity of x (where $x \in \mathbb{R}$), and hence it is true that $(dx/x) = 0$ (for $x \neq 0$). Additionally, based on eqs. (1) and (59), it holds that

$$\frac{d\xi}{\xi} = \frac{\lim_{x \to +\infty}(dx)}{\lim_{x \to +\infty}(x)} = \lim_{x \to +\infty}\left(\frac{dx}{x}\right) = \lim_{x \to +\infty}(0) = 0$$

Thus, $\frac{d\xi}{\xi} = 0$.

Therefore, considering that $(d\xi/\xi) = 0$, we can assume $d\xi = 1$, given that $1/\xi = 0$. Thus, eq. (62) is transformed into eq. (63) if, instead of $d\xi$, we assume 1. Moreover, instead of $d\xi = 1$, we can also assume $d\xi = 2$ or $d\xi = 3 \ldots$ or $d\xi = \alpha$, when $\alpha \in \mathbb{R}$ since $\alpha/\infty = 0$ $\forall a \in \mathbb{R}$.

$$\frac{d}{d\xi}A(\xi) = \frac{A(\xi) - A(\xi - 1)}{1} = A(\xi) - A(\xi - 1) \tag{63}$$

Hence,

$$\frac{d}{d\xi}A(\xi) = (a_1 + a_2 + \ldots + a_\xi) - (a_1 + a_2 + \ldots + a_{\xi-1}) = a_\xi \tag{64}$$

Consequently, the derivative of the infinite series function $A(\xi)$ is equal to its last infinite term (a_ξ).

Nevertheless, if $d\xi = 2$ is assumed instead of $d\xi = 1$, then eq. (62) is transformed into eq. (65) and finally into eq. (66).

$$\frac{d}{d\xi}A(\xi) = \frac{A(\xi) - A(\xi - 2)}{2} \tag{65}$$

which implies

$$\frac{d}{d\xi}A(\xi) = \frac{(a_1 + a_2 + \ldots + a_\xi) - (a_1 + a_2 + \ldots + a_{\xi-2})}{2} = \frac{a_{\xi-1} + \alpha_\xi}{2}$$

or

$$\frac{d}{d\xi}A(\xi) = \frac{a_{\xi-1} + \alpha_\xi}{2} \tag{66}$$

Hence, when taking the above into account, formulas (64) and (66) as well as all the corresponding formulas that arise for $d\xi = 1, 2, 3, 4, \ldots$ should be equal to each other, given that the derivative of $A(\xi)$ should be the same in all these cases. This means that it is necessary to apply the following condition (criterion):

$$\alpha_\xi = \frac{\alpha_{\xi-1} + \alpha_\xi}{2} = \frac{\alpha_{\xi-2} + \alpha_{\xi-1} + \alpha_\xi}{3} = \ldots$$

Therefore, this criterion should always be taken into consideration before using just the last term as the derivative of an ordinary series. □

Let us now see the following examples that show when the above criterion is satisfied and the derivative can be determined:

Consider the following ordinary series B_n, which is written in infinite numbers as $B(\xi)$ (below).

$$B_n = 1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 + n^2 + \ldots$$

$$B(\xi) = 1^2 + 2^2 + 3^2 + \ldots + (\xi - 1)^2 + \xi^2.$$

According to the above, its derivative is expressed either by its last term or, equivalently, by the average of the last terms. So, the following criterion must therefore apply:

$$\xi^2 = \frac{(\xi-1)^2 + \xi^2}{2} = \frac{(\xi-2)^2 + (\xi-1)^2 + \xi^2}{3} = \ldots$$

Since $\xi - 1$ can be replaced by ξ (similarly for $\xi - 2$ etc), we find that the above criterion is satisfied; thus, the derivative does not change value if instead of the last term (ξ^2), we use the average of more last terms.

However, let us now look at the case of the following ordinary series of numbers, L_n, which is written as $L(\xi)$. This is a very fast-growing series.

$$L_n = 2^1 + 2^2 + 2^3 + \ldots + 2^{n-1} + 2^n + \ldots$$

$$L(\xi) = 2^1 + 2^2 + 2^3 + \ldots + 2^{\xi-1} + 2^\xi$$

The above criterion imposes that

$$2^{\xi} = \frac{2^{\xi-1} + 2^{\xi}}{2} = \frac{2^{\xi-2} + 2^{\xi-1} + 2^{\xi}}{3} = \dots$$

But this is not true since

$$\frac{2^{\xi-1} + 2^{\xi}}{2} = \frac{2^{\xi}/2 + 2^{\xi}}{2} = \frac{3}{4}2^{\xi}$$

Additionally,

$$\frac{2^{\xi-2} + 2^{\xi-1} + 2^{\xi}}{3} = \frac{2^{\xi}/4 + 2^{\xi}/2 + 2^{\xi}}{3} = \frac{7}{12}2^{\xi}.$$

Therefore, in this case, the derivative of $L(\xi)$ cannot be calculated by the above method since the respective criterion does not apply. This usually happens in a rapidly increasing series, L_n, where n is in the exponent of the nth term l_n. For example, $l_n = 2^n$ or $l_n = 3^n$ or $l_n = n^n$, etc.

In addition, it should be noted that there are cases of series of numbers where the nth term is more complicated than in the above examples, and it can be a whole expression.

For example, by considering the following ordinary series in eq. (67), it is apparent that this series is not produced only from the fraction $\left(-\frac{1}{3n}\right)$; the whole expression $\left(\frac{1}{3n-2} - \frac{1}{3n}\right)$ constitutes its nth term (containing $n \in \mathbb{N}$).

$$A_n = \frac{1}{1} - \frac{1}{3} + \frac{1}{4} - \frac{1}{6} + \dots + \frac{1}{3n-2} - \frac{1}{3n} + \dots \qquad n \in \mathbb{N} \tag{67}$$

By using infinite numbers, this series is written as eq. (68):

$$A(\xi) = \frac{1}{1} - \frac{1}{3} + \frac{1}{4} - \frac{1}{6} + \dots + \frac{1}{3\xi-2} - \frac{1}{3\xi} \tag{68}$$

where it is obvious that its derivative is the whole expression of the right-hand member of eq. (69), given that the aforementioned criterion is also satisfied.

$$\frac{d}{d\xi}A(\xi) = \frac{1}{3\xi - 2} - \frac{1}{3\xi} \tag{69}$$

Indeed, since $\xi - 1$ can be replaced by ξ, the respective criterion is satisfied:

$$\frac{1}{3\xi - 2} - \frac{1}{3\xi} = \frac{\left(\frac{1}{3\xi - 2} - \frac{1}{3\xi}\right) + \left(\frac{1}{3(\xi - 1) - 2} - \frac{1}{3(\xi - 1)}\right)}{2} = \ldots$$

Theorem 3. *An ordinary infinite series* (*a series with infinitely many numbers*) *converges if it can be written as a simple infinite number*, $A(\xi)$, *that represents a finite value number and not infinity* (*series convergence criterion*).

Proof. Using the infinity unit, (ξ), we can write any infinite series of numbers as $A(\xi)$, and by considering Theorem 2, we can easily find the derivative of this series (if the respective criterion is also fulfilled). In addition, by using integration, we can calculate, again, the initial series $A(\xi)$ as a simple infinite number. If this final infinite number represents infinity, then it is obvious that the series diverges; otherwise, it converges (finite value number). This is a simple series convergence criterion that is easy to apply and has a wide range of applications. □

Examples. Let us consider the previous series from eq. (67), where we saw that its derivative is given by eq. (69). By using integration, we obtain relation (70):

$$A(\xi) = \int \frac{d}{d\xi}A(\xi)d\xi = \int\left(\frac{1}{3\xi - 2} - \frac{1}{3\xi}\right)d\xi$$

$$= \int\left(\frac{1}{3\xi - 2}\right)d\xi - \int\left(\frac{1}{3\xi}\right)d\xi = \frac{1}{3}\ln(3\xi - 2) - \frac{1}{3}\ln(\xi) + C$$

$$= \frac{1}{3}\ln\left(\frac{3\xi - 2}{\xi}\right) + C \approx \frac{1}{3}\ln(3) + C \quad \text{where } C \in \mathbb{R}. \tag{70}$$

So, according to eq. (70), the series is equal to a finite number, and, therefore, it converges.

On the other hand, if we consider the same series of eq. (67) but with all of its individual fractions being positive, as shown in eq. (71), then the respective integral is calculated as in eq. (72).

$$A(\xi) = \frac{1}{1} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{3\xi - 2} + \frac{1}{3\xi} \tag{71}$$

$$A(\xi) = \int \left(\frac{1}{3\xi - 2} + \frac{1}{3\xi} \right) d\xi = \frac{1}{3} \ln(3\xi - 2) + \frac{1}{3} \ln(\xi) + C$$

$$= \frac{1}{3} \ln((3\xi - 2) \cdot \xi) + C = \frac{1}{3} \ln(3\xi^2 - 2\xi) + C$$

$$\approx \frac{1}{3} \ln(3\xi^2) + C \quad \text{where } C \in \mathbb{R} \tag{72}$$

So, based on Theorem 3, the series is calculated as being equal to infinity, and therefore it diverges.

Lemma 10 completes and formulates some of the calculations mentioned in [38].

Lemma 10. *The harmonic series that diverts is an ordinary series, and its value in infinite numbers is given by eq.* (73) *below, and the limit of equation* (74) *also applies.*

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{\xi} = \ln \xi + C \quad \textit{where } C \in \mathbb{R} \tag{73}$$

$$\lim_{n \to \infty} \frac{\ln n}{1 + \frac{1}{2} + \dots + \frac{1}{n}} = 1 \quad n \in \mathbb{N} \tag{74}$$

Proof. When taking into account the fact that the harmonic series that is depicted in eq. (75) presents $n \in \mathbb{N}$ in its (nth) term and not in all its terms, it is, therefore, an ordinary series according to Definition 4. Thus, Theorem 2 applies. By using infinite numbers, the series in eq. (75) is written as formula (76), and according to Theorem 2, its derivative is equal to its last term $1/\xi$ (eq. (77)), given that the respective criterion is also fulfilled.

$$A_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots \quad n \in \mathbb{N} \tag{75}$$

$$A(\xi) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{\xi} \tag{76}$$

$$\frac{d}{d\xi} A(\xi) = \frac{1}{\xi} \tag{77}$$

By integrating in eq. (77), the following is obtained:

$$A(\xi) = \int \frac{d}{d\xi} A(\xi) d\xi = \int \frac{1}{\xi} d\xi$$

$$\Rightarrow A(\xi) = \ln \xi + C \quad \text{where } C \in \mathbb{R} \text{ or}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{\xi} = \ln \xi + C \tag{78}$$

Therefore, according to Theorem 3, the harmonic series deviates and tends to infinity and, of course, relation (78) calculates its value in infinite numbers, where, as is known, C is the Euler-Mascheroni constant $(C = \gamma = 0.57721 \ldots)$. The left-hand side of eq. (78) gives the full form (FF) while the $(\ln \xi)$ part of the right-hand side is the simplified form (SF), and finally, the real constant C is the advancement (AD). It is interesting to note that the derivative of SF is $1/\xi$, which is also the derivative of FF (so, both are of the same degree, as expected).

Furthermore, according to Section III and (eqs. (17) and (18)), from eq. (78), the following is obtained:

$$\frac{\ln \xi}{1 + \frac{1}{2} + \ldots + \frac{1}{\xi}} = 1 \tag{79}$$

or equivalently

$$\lim_{n \to \infty} \frac{\ln n}{1 + \frac{1}{2} + \ldots + \frac{1}{n}} = 1 \tag{80}$$

Hence, the limit of eq. (80) is proved. It should be noted that if we take the average of the last two terms of eq. (76) as the derivative of the infinite series $A(\xi)$ (instead of only the last term), we come again to the same result of eq. (80), as is shown below. The same happens if we obtain more of the last terms of this series:

$$\frac{d}{d\xi} A(\xi) = \frac{1}{2}\left(\frac{1}{\xi - 1} + \frac{1}{\xi}\right) \Rightarrow$$

$$A(\xi) = \int \frac{d}{d\xi} A(\xi) d\xi = \int \frac{1}{2}\left(\frac{1}{\xi - 1} + \frac{1}{\xi}\right) d\xi$$

$$= \frac{1}{2}(\ln(\xi - 1) + \ln(\xi)) + C = \frac{1}{2}(\ln(\xi) + \ln(\xi)) + C$$
$$= \ln \xi + C \qquad \text{where } C \in \mathbb{R} \qquad \square$$

Consequently, according to the above, an ordinary series of infinite terms can be transformed in an equivalent manner as a simple infinite number function if one takes the derivative of the series, which is expressed by its last (infinite) term and proceeds to its integration. Of course, a condition for doing this is that the relevant criterion applies. However, what if ξ (or a function of it), in addition to the last term, also appears in all the terms of the series (not ordinary series), as is characteristically shown in the following series (81):

$$A(\xi) = \frac{1}{\xi^2} + \frac{2}{\xi^2} + \frac{3}{\xi^2} + \dots + \frac{\xi - 1}{\xi^2} + \frac{\xi}{\xi^2}. \tag{81}$$

Certainly, $A(\xi - 1)$ is then given by relation (82):

$$A(\xi - 1) = \frac{1}{(\xi - 1)^2} + \frac{2}{(\xi - 1)^2} + \frac{3}{(\xi - 1)^2} + \dots + \frac{\xi - 1}{(\xi - 1)^2}. \tag{82}$$

However, in this case, the difference, $A(\xi) - A(\xi - 1)$, does not result in the last term, ξ/ξ^2, of the series $A(\xi)$ since the remaining $(\xi - 1)$ terms do not cancel each other during the subtraction; therefore, the last term of the series, $A(\xi)$, no longer expresses the derivative of the series.

Of course, the same is true for the following series, where ξ appears within the $\ln(\cdot)$ function in all terms of the series:

$$A(\xi, \xi) = \frac{1}{\xi}\ln\frac{1}{\xi} + \frac{1}{\xi}\ln\frac{2}{\xi} + \dots + \frac{1}{\xi}\ln\frac{\xi}{\xi}.$$

Theorem 4. *If a series, A_n, of infinite terms (which, when using infinite numbers, is written as $A(\xi)$, where ξ^{th} is its last infinite term) includes ξ (or a function of ξ) in all the terms of the series, then can ξ appearing in all terms be replaced by the natural number, n, so that the series is converted into a new ordinary series (only having ξ in its last term). In this new series, the last term is its derivative, where by integration, the equivalent of the new series is obtained, and, furthermore, by setting $n = \xi$, the equivalent of the original series, A_n, can be calculated.*

Proof. Consider the following series (83):

$$A(\xi, \xi) = a_{1,\xi} + a_{2,\xi} + \dots + a_{\xi,\xi}. \tag{83}$$

If ξ, which appears in all the terms of the series, $A(\xi, \xi)$, is replaced by the natural number, n, then we obtain the following new series: $A(\xi, n)$ (see eq. (84)), which is an ordinary series where ξ only appears in its last term.

$$A(\xi, n) = a_{1,n} + a_{2,n} + \dots + a_{\xi,n}. \tag{84}$$

Moreover, relation (85) holds that

$$A(\xi, \xi) = \lim_{n\to\xi} A(\xi, n). \tag{85}$$

In this new series, $A(\xi, n)$, the last term $\alpha_{\xi,n}$ expresses its derivative since eq. (86) applies:

$$A(\xi, n) - A(\xi - 1, n) = (a_{1,n} + a_{2,n} + \dots + a_{\xi-1,n} + a_{\xi,n})$$

$$-(a_{1,n} + a_{2,n} + \dots + a_{\xi-1,n}) = a_{\xi,n}. \tag{86}$$

Consequently, when integrating the last term, $\alpha_{\xi,n}$ (derivative), we obtain the equivalent of the new series: $A(\xi, n)$. Finally, according to eq. (85), when taking the limit of $A(\xi, n)$ of n tending to infinity (ξ), the equivalent of the original series, $A(\xi, \xi)$, is calculated. □

Examples. Let us now look at non-ordinary series (81). If we replace ξ appearing in all the terms of the series with n, we obtain relation (87), where the last term (ξ/n^2) of this new series $A(\xi, n)$ now expresses its derivative.

$$A(\xi, n) = \frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \dots + \frac{\xi - 1}{n^2} + \frac{\xi}{n^2} \quad n \in \mathbb{N} \tag{87}$$

Therefore, by integrating the last term: derivative (ξ/n^2), we obtain

$$A(\xi, n) = \int \frac{\xi}{n^2} d\xi = \frac{1}{n^2}\int \xi d\xi = \frac{1}{n^2}\left(\frac{\xi^2}{2} + C(\xi)\right). \tag{88}$$

It is noted that in this case, the constant due to the integration of the infinite number function is not a real number, C, but a function, $C(\xi)$, one degree lower than the calculated function, $\xi^2/2$, and, more specifically, $C(\xi) = C_1\xi + C_2$, where $C_1, C_2 \in \mathbb{R}$. In other words, $C(\xi)$ is the advancement of the calculated integral $\xi^2/2$. That is, it holds that

$$A(\xi, n) = \int \frac{\xi}{n^2} d\xi = \frac{1}{n^2}\int \xi d\xi = \frac{1}{n^2}\left(\frac{\xi^2}{2} + C_1 \cdot \xi + C_2\right).$$

The reason is that if we take the derivative of the above integration result (see relation (89)), we obtain $(\xi + C_1)$, which is the infinite number, ξ, in its generality, i.e., with an advancement, C_1.

$$\frac{d}{d\xi}\left(\frac{\xi^2}{2} + C_1\xi + C_2\right) = \xi + C_1 \tag{89}$$

Furthermore, when taking the limit of $A(\xi, n)$ of n tending to infinity, we have

$$A(\xi, \xi) = \lim_{n \to \xi} A(\xi, n) = \frac{1}{\xi^2}\left(\frac{\xi^2}{2} + C_1 \cdot \xi + C_2\right) = \frac{1}{2} \tag{90}$$

Alternatively, without using the infinite numbers, series (81) is calculated as shown in eq. (91).

$$\begin{aligned} A_n &= \lim_{n \to +\infty}\left(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \dots + \frac{n}{n^2}\right) \\ &= \lim_{n \to +\infty} \frac{1}{n^2}(1 + 2 + 3 + \dots + n) = \lim_{n \to +\infty} \frac{1}{n^2}\frac{n(n+1)}{2} \\ &= \lim_{n \to +\infty}\left(\frac{1}{2} + \frac{1}{2n}\right) = \frac{1}{2}. \end{aligned} \tag{91}$$

Hence, the result of eq. (90) is the same as that of eq. (91), and thus, it is confirmed to be the correct one.

Let us now take another non-ordinary series, that of eq. (92):

$$A(\xi, \xi) = \frac{1^3}{\xi^4} + \frac{2^3}{\xi^4} + \frac{3^3}{\xi^4} + \dots + \frac{\xi^3}{\xi^4}. \tag{92}$$

We form the series $A(\xi, n)$, i.e., ξ appearing in all terms is replaced by $n \in \mathbb{N}$.

$$A(\xi, n) = \frac{1^3}{n^4} + \frac{2^3}{n^4} + \frac{3^3}{n^4} + \ldots + \frac{\xi^3}{n^4}.$$

Consequently, by integrating its last term, (ξ^3/n^4), we obtain

$$A(\xi, n) = \frac{1}{n^4}\int \xi^3 d\xi = \frac{1}{n^4}\left(\frac{\xi^4}{4} + C_1\xi^3 + C_2\xi^2 + C_3\xi + C_4\right)$$

where $C_1, C_2, C_3, C_4 \in \mathbb{R}$

Also, here, the advancement $C(\xi)$ is a polynomial that is one degree lower than the infinite number function $\xi^4/4$. Furthermore, when taking the limit of $A(\xi, n)$ of n tending to infinity, we have

$$A(\xi, \xi) = \lim_{n\to\xi} A(n, \xi) = \frac{1}{\xi^4}\left(\frac{\xi^4}{4} + C_1\xi^3 + C_2\xi^2 + C_3\xi + C_4\right)$$

$$= \frac{1}{4} + \frac{C_1}{\xi} + \frac{C_2}{\xi^2} + \frac{C_3}{\xi^3} + \frac{C_4}{\xi^4} = \frac{1}{4} \tag{93}$$

Alternatively, without using the infinite numbers, series (92) is calculated as shown in eq. (94):

$$A_n = \lim_{n\to+\infty}\left(\frac{1^3}{n^n} + \frac{2^3}{n^4} + \frac{3^3}{n^4} + \ldots + \frac{\xi^3}{n^4}\right)$$

$$= \lim_{n\to+\infty} \frac{1}{n^4}(1^3 + 2^3 + 3^3 + \ldots + n^3) = \lim_{n\to+\infty} \frac{1}{n^4}\frac{n^2(n+1)^2}{4}$$

$$= \lim_{n\to+\infty} \frac{(n+1)^2}{4n^2} = \lim_{n\to+\infty} \frac{n^2 + 2n + 1}{4n^2}$$

$$= \lim_{n\to+\infty}\left(\frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}\right) = \frac{1}{4} \tag{94}$$

Therefore, the result of eq. (93) is the same as that of eq. (94), and thus, it is confirmed to be the correct one.

Let us now take a more complex case and calculate the limit, *LIM*, of expression (95):

$$LIM = \lim_{n \to +\infty} \frac{\frac{1 \ln 1}{n^2} + \frac{2 \ln 2}{n^2} + \frac{3 \ln 3}{n^2} + \ldots + \frac{n \ln n}{n^2}}{\ln n} \tag{95}$$

By using the infinite numbers, the series of the numerator of expression (95) is written as per eq. (96).

$$A(\xi, \xi) = \frac{1 \ln 1}{\xi^2} + \frac{2 \ln 2}{\xi^2} + \frac{3 \ln 3}{\xi^2} + \ldots + \frac{\xi \ln \xi}{\xi^2} \tag{96}$$

We form the series $A(\xi, n)$.

$$A(\xi, n) = \frac{1 \ln 1}{n^2} + \frac{2 \ln 2}{n^2} + \frac{3 \ln 3}{n^2} + \ldots + \frac{\xi \ln \xi}{n^2} \quad n \in \mathbb{N} \tag{97}$$

Consequently, by integrating the last term: $\frac{\xi \ln \xi}{n^2}$, we obtain

$$A(\xi, n) = \frac{1}{n^2} \int \xi \ln \xi d\xi = \frac{1}{n^2}\left(\frac{\xi^2}{2} \ln \xi - \frac{\xi^2}{4} + C_1 \xi + C_2\right)$$

where $C_1, C_2 \in \mathbb{R}$.

Furthermore, by taking the limit of $A(\xi, n)$ of n tending to infinity, we have

$$A(\xi, \xi) = \lim_{n \to \xi} A(n, \xi) = \frac{1}{\xi^2}\left(\frac{\xi^2}{2} \ln \xi - \frac{\xi^2}{4} + C_1 \xi + C_2\right)$$

$$= \frac{\ln \xi}{2} - \frac{1}{4} + \frac{C_1}{\xi} + \frac{C_2}{\xi^2} = \frac{\ln \xi}{2} - \frac{1}{4}. \tag{98}$$

Therefore, eq. (95) finally transforms into eq. (99), and thus, $LIM = 1/2$.

$$LIM = \frac{\frac{\ln \xi}{2} - \frac{1}{4}}{\ln \xi} = \frac{1}{2} \tag{99}$$

Of course, in all the above cases, the relative criterion of the series applies.

It is noted that in some cases (as mentioned above), these non-ordinary series can be solved more simply if ξ (or a function of ξ) can be extracted as a common factor of the series. In this case, the series turns into an ordinary series, where its derivative as well as its integral can be easily calculated. For example,

$$A(\xi) = \frac{1}{\xi^2} + \frac{2}{\xi^2} + \frac{3}{\xi^2} + \dots + \frac{\xi - 1}{\xi^2} + \frac{\xi}{\xi^2}$$

$$= \frac{1}{\xi^2}(1 + 2 + 3 + \dots + \xi)$$

Given that

$$(1+2+3+\dots+\xi) = \int \xi d\xi = \frac{\xi^2}{2} + C_1\xi + C_2 \quad \text{where } C_1, C_2 \text{ real numbers}$$

it applies that

$$A(\xi) = \frac{1}{\xi^2}\left(\frac{\xi^2}{2} + C_1\xi + C_2\right) = \frac{1}{2} + \frac{C_1}{\xi} + \frac{C_2}{\xi^2} = \frac{1}{2}$$

However, this (i.e., extracting a common factor in a non-ordinary series) is not always possible, as, e.g., in the following series, where n (or ξ respectively) appears inside the $\tan(\cdot)$ function, and where we must use Theorem 4.

$$A_n = \lim_{n \to +\infty}\left(\frac{1}{n}\tan\frac{1}{n} + \frac{1}{n}\tan\frac{2}{n} + \dots + \frac{1}{n}\tan\frac{n}{n}\right) \tag{100}$$

$$\Leftrightarrow A(\xi, \xi) = \frac{1}{\xi}\tan\frac{1}{\xi} + \frac{1}{\xi}\tan\frac{2}{\xi} + \dots + \frac{1}{\xi}\tan\frac{\xi}{\xi}$$

We form the series $A(\xi, n)$

$$A(\xi, n) = \frac{1}{n}\tan\frac{1}{n} + \frac{1}{n}\tan\frac{2}{n} + \dots + \frac{1}{n}\tan\frac{\xi}{n} \quad n \in \mathbb{N}$$

Consequently, by integrating the last term, $\frac{1}{n}\tan\frac{\xi}{n}$, we obtain

$$A(\xi, n) = \frac{1}{n}\int \tan\frac{\xi}{n}d\xi = \frac{1}{n}\left(-n\ln\left|\cos\frac{\xi}{n}\right| + C\right) = -\ln\left|\cos\frac{\xi}{n}\right| + \frac{C}{n},$$

where C is a real number.

Furthermore, by taking the limit of $A(\xi, n)$ of n tending to infinity, we have

$$A(\xi, \xi) = \lim_{n\to+\infty} A(\xi, n) = -\ln\left|\cos\frac{\xi}{\xi}\right| + \frac{C}{\xi}$$

$$= -\ln\cos(1) = 0.615626\ldots. \tag{101}$$

Therefore, it is interesting to notice that the sum of the infinite terms (limits), which are written below and are all zero (tending to zero), nicely results in the number: $-\ln\cos(1)$.

$$\lim_{n\to+\infty}\left(\frac{1}{n}\tan\frac{1}{n}\right), \lim_{n\to+\infty}\left(\frac{1}{n}\tan\frac{2}{n}\right), \ldots, \lim_{n\to+\infty}\left(\frac{1}{n}\tan\frac{n}{n}\right).$$

This result is confirmed numerically, since the sum of the first thousand terms of the series equals 0.616405.... Let us also look at the following non-ordinary series, where, again, it is not possible to extract n (contained within the logarithm in all terms of the series) as a common factor:

$$A_n = \lim_{n\to+\infty}\left(\frac{1}{n}\ln\frac{1}{n} + \frac{1}{n}\ln\frac{2}{n} + \ldots + \frac{1}{n}\ln\frac{n}{n}\right) \tag{102}$$

$$\Leftrightarrow A(\xi, \xi) = \frac{1}{\xi}\ln\frac{1}{\xi} + \frac{1}{\xi}\ln\frac{2}{\xi} + \ldots + \frac{1}{\xi}\ln\frac{\xi}{\xi}$$

However, we form the series $A(\xi, n)$

$$A(\xi, n) = \frac{1}{n}\ln\frac{1}{n} + \frac{1}{n}\ln\frac{2}{n} + \ldots + \frac{1}{n}\ln\frac{\xi}{n} \quad n \in \mathbb{N}$$

Consequently, by integrating the last term, $\frac{1}{n}\ln\frac{\xi}{n}$ we obtain

$$A(\xi, n) = \frac{1}{n}\int \ln\frac{\xi}{n}d\xi = \frac{1}{n}\left(\int \ln\xi d\xi - \int \ln n d\xi\right)$$

$$= \frac{1}{n}(\xi\ln\xi - \xi - \xi\ln n + C) = \frac{1}{n}\left(\xi\ln\frac{\xi}{n} - \xi + C\right).$$

Furthermore, when taking the limit of $A(\xi, n)$ of n tending to infinity, we have

$$A(\xi, \xi) = \lim_{n \to +\infty} A(n, \xi) = \frac{1}{\xi}\left(\xi \ln \frac{\xi}{\xi} - \xi + C\right)$$

$$= \ln \frac{\xi}{\xi} - 1 + \frac{C}{\xi} = -1 \quad \text{where } C \in \mathbb{R}. \tag{103}$$

In conclusion, complicated limits and series can be easily calculated using this general method with infinite numbers, which would be difficult to calculate otherwise.

As we have seen in the previous relations ((70), (73), (88), etc.), the calculation of the integral of the infinite number function $A(\xi)$ also includes the constant C or, more generally, the advancement $C(\xi)$, which was not necessary to calculate in the aforementioned examples. However, a question remains whether it is possible to calculate it, as is the case for the integrals of real functions $f(x)$, $x \in \mathbb{R}$ (where the corresponding constant is calculated from the initial conditions, i.e., for $x = 0$.

Theorem 5. *If the series* $a_1 + a_2 + \ldots + a_\xi$ *can* (*by integrating its derivative*) *be written as a simple infinite number* $A(\xi) + C(\xi)$, *where* $C(\xi)$ *is the advancement*, *then by setting*, *sequentially*, *where* ξ *the natural numbers* $1, 2, 3, \ldots,$ *we can calculate* (*approximately or exactly*) *the advancement* $C(\xi)$.

Proof. We have

$$A(\xi) + C(\xi) = a_1 + a_2 + \ldots + a_\xi \tag{104}$$

Given that $\xi = \lim_{n \to +\infty} n$, where $n \in \mathbb{N}$, eq. (104) converts (equivalently) to

$$A\left(\lim_{n \to +\infty} n\right) + C\left(\lim_{n \to +\infty} n\right) = \lim_{n \to +\infty} (a_1 + a_2 + \ldots + a_n)$$

and further

$$\lim_{n \to +\infty} (A(n) + C(n)) = \lim_{n \to +\infty} (a_1 + a_2 + \ldots + a_n) \tag{105}$$

Therefore, the two functions of n in the above relation, namely the function $(A(n) + C(n))$ and the function $(a_1 + a_2 + \ldots + \alpha_n)$, have the same limit when n tends to infinity. This means that either *i*) the two functions take the same value for each value of n (equal) or ii) the two functions do not take the same value for each value of n (unequal); however, as n tends to infinity, the values that these receive progressively approach each other.

In the first case where the functions $(A(n) + C(n))$ and $(\alpha_1 + \alpha_2 + \ldots + \alpha_n)$ are equal, for n, if we use the values $n = 1, 2, 3, \ldots,$ we obtain the corresponding equations:

$$A(1) + C(1) = \alpha_1 \qquad (105\_1)$$

$$A(2) + C(2) = \alpha_1 + \alpha_2 \qquad (105\_2)$$

$$A(3) + C(3) = \alpha_1 + \alpha_2 + \alpha_3 \qquad (105\_3)$$

……

So, if the advancement C is a simple real number, then each of the above equations will calculate the value of that real number C. If, on the other hand, the advancement $C = C(\xi)$ is an infinite number that includes more real numbers $C_1, C_2, \ldots,$ then a corresponding number of equations needs to be obtained from eqs. (105_1), (105_2), (105_3) ..., to calculate these real numbers.

In the second case, where the functions are unequal, again, by setting $n = 1, 2, 3, \ldots,$ we obtain equations (105_1), (105_2), (105_3)..., where these equations are now approximate, and each higher value of n gives a better approximation. □

Examples. Let us calculate the following series:

$$A(\xi) = 1 + 2 + 3 + \ldots + \xi$$

By integrating the last term, ξ, we obtain relation (106), where $C_1\xi + C_2$ is the advancement of the infinite number $\xi^2/2$ (one degree lower than $\xi^2/2$).

$$A(\xi) = \int \xi d\xi = \frac{\xi^2}{2} + C_1\xi + C_2 \quad \text{where } C_1, C_2 \in \mathbb{R} \qquad (106)$$

Based on eqs. (105_1) and (105_2), we have

$$\frac{1^2}{2} + C_1 \cdot 1 + C_2 = 1 \Rightarrow C_1 + C_2 = \frac{1}{2}$$

$$\frac{2^2}{2} + C_1 \cdot 2 + C_2 = 1 + 2 \Rightarrow 2C_1 + C_2 = 1$$

Thus, $C_1 = \frac{1}{2} \wedge C_2 = 0$

Therefore, $C(\xi) = C_1\xi + C_2 = \xi/2$, and eq. (106) becomes (107):

$$A(\xi) = \frac{\xi^2}{2} + \frac{\xi}{2} \tag{107}$$

Of course, this result is the same as the one we obtain if we take the limit $\lim_{n\to+\infty} \frac{n(n+1)}{2}$, so the calculation $C(\xi) = \xi/2$ is confirmed to be correct.

If instead of equations (105_1) and (105_2), we use eqs. (105_3) and (105_4), we, again, obtain $C(\xi) = \xi/2$, as we see below. The same, of course, applies if we use any other pair of the infinite number equations: (105_1), (105_2), (105_3) …

$$\frac{3^2}{2} + C_1 \cdot 3 + C_2 = 1 + 2 + 3 \Rightarrow 3C_1 + C_2 = 1.5$$

$$\frac{4^2}{2} + C_1 \cdot 4 + C_2 = 1 + 2 + 3 + 4 \Rightarrow 4C_1 + C_2 = 2$$

Therefore, $C_1 = \frac{1}{2} \wedge C_2 = 0 \Rightarrow C(\xi) = \frac{\xi}{2}$.

Finally, if instead of the first-degree advancement above, we consider a higher degree (a fact that cannot happen), for example, a second-degree advancement, i.e., $C(\xi) = C_1\xi^2 + C_2\xi + C_3$, where $C_1, C_2, C_3 \in \mathbb{R}$, by using eqs. (105_1), (105_2), and (105_3), we come, again, to the same result (first-degree advancement), as can be seen below:

$$\frac{1^2}{2} + C_1 \cdot 1^2 + C_2 \cdot 1 + C_3 = 1 \Rightarrow C_1 + C_2 + C_3 = \frac{1}{2}$$

$$\frac{2^2}{2} + C_1 \cdot 2^2 + C_2 \cdot 2 + C_3 = 1 + 2 \Rightarrow 4C_1 + 2C_2 + C_3 = 1$$

$$\frac{3^2}{2} + C_1 \cdot 3^2 + C_2 \cdot 3 + C_3 = 1 + 2 + 3 \Rightarrow 9C_1 + 3C_2 + C_3 = 1.5$$

Hence, $C_1 = 0 \wedge C_2 = \frac{1}{2} \wedge C_3 = 0.$

Therefore, $C(\xi) = \xi/2$ is calculated, again, exactly as before, and, of course, the advancement $C(\xi)$ is of a first-degree.

Let us now consider the famous series of relation (108). This is the well-known Basel problem posed by Pietro Mengoli in 1650, which was solved by Leonhard Euler in 1734, where this sum is equal to $\pi^2/6 = 1.644934\ldots$

$$A(\xi) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{\xi^2}. \tag{108}$$

By integrating the last term $1/\xi^2$ we obtain the relation below, where C is the advancement of the considered infinite number.

$$A(\xi) = \int \frac{1}{\xi^2} d\xi = -\frac{1}{\xi} + C$$

Certainly, here, the advancement C cannot be a polynomial function of ξ since the sum of the series is not infinite but is finite. By sequentially applying eqs. (105_1), (105_2), (105_3) …, we obtain

$$(105\_1) \Rightarrow -\frac{1}{1} + C = \frac{1}{1^2} \Rightarrow C = 2$$

$$(105\_2) \Rightarrow -\frac{1}{2} + C = \frac{1}{1^2} + \frac{1}{2^2} \Rightarrow C = \frac{7}{4} = 1.75$$

$$(105\_3) \Rightarrow -\frac{1}{3} + C = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \Rightarrow C = \frac{61}{36} = 1.6944\ldots.$$

Therefore, we observe that, as the value of n increases, the approximation of the real number C becomes better, i.e., closer to the correct value: $C = 1.644934\ldots.$

Let us now take $n = 50$ and calculate the corresponding sum of the first 50 terms of the series:

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{50^2} = 1.625133\ldots.$$

This value of the sum of the first 50 terms of the series implies a calculation error equal to 1.2038%.

Now, by using the corresponding formula (105_50), as shown in eq. (109), we obtain a value of $C = 1.645133\ldots,$ which implies a much smaller error, equal to 0.0121%, i.e., 99.67 times smaller.

$$-\frac{1}{50} + C = 1.625133\ldots \Rightarrow C = 1.645133\ldots. \tag{109}$$

In order to achieve the same accuracy result (as before), by directly calculating the sum of the terms, we need to obtain 5030 terms from the series (instead of the 50 terms before), which means much more laborious calculations.

Consequently, the developed numerical calculation method (using the infinite numbers) is important in the cases where the sum of a series is not known in an analytical way and the calculation is necessarily carried out numerically. For example, for the series of relation (110), the sum of which is not analytically known, we have

$$A(\xi) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \ldots + \frac{1}{\xi^3}. \tag{110}$$

$$A(\xi) = \int \frac{1}{\xi^3}\, d\xi = -\frac{1}{2\xi^2} + C \quad \text{where } C \in \mathbb{R}$$

For $n = 50,$ the corresponding sum of the first 50 terms of the series is

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \ldots + \frac{1}{50^3} = 1.2018609\ldots.$$

Now, using the corresponding formula (105_50), as seen in relation (111), we obtain the value $C = 1.202061\ldots.$

$$-\frac{1}{2 \cdot 50^2} + C = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \ldots + \frac{1}{50^3} \tag{111}$$

This numerical method incredibly shortens calculations, increases the accuracy of the result, and saves time.

Theorem 6. *The ratio of two infinite number real functions that represent infinity is equal to the ratio of their derivatives so long as the ratio of the derivatives is finite*, $+\infty$, *or* $-\infty$ (*L'Hopital's rule in infinite functions*).

Proof. Let us consider the real functions $\phi_1(x)$ and $\phi_2(x)$, where $x \in \mathbb{R}$. These functions are continuous, differentiable, and non-oscillating. Suppose, also, that $\lim\limits_{x \to +\infty} \phi_1(x) = \infty$ and $\lim\limits_{x \to +\infty} \phi_2(x) = \infty$. Then, according to L'Hopital's rule, relation (112) applies:

$$\lim_{x \to +\infty} \frac{\phi_1(x)}{\phi_2(x)} = \lim_{x \to +\infty} \frac{\phi_1'(x)}{\phi_2'(x)} \tag{112}$$

so long as the limit is finite, $+\infty$, or $-\infty$.

Relation (112) is transformed equivalently:

$$\frac{\lim\limits_{x \to +\infty} \phi_1(x)}{\lim\limits_{x \to +\infty} \phi_2(x)} = \frac{\lim\limits_{x \to +\infty} \phi_1'(x)}{\lim\limits_{x \to +\infty} \phi_2'(x)}$$

$$\Leftrightarrow \frac{\phi_1\left(\lim\limits_{x \to +\infty} x\right)}{\phi_2\left(\lim\limits_{x \to +\infty} x\right)} = \frac{\phi_1'\left(\lim\limits_{x \to +\infty} x\right)}{\phi_2'\left(\lim\limits_{x \to +\infty} x\right)}$$

$$\Leftrightarrow \frac{\phi_1(\xi)}{\phi_2(\xi)} = \frac{\phi_1'(\xi)}{\phi_2'(\xi)}.$$ □

Example. When making use of Theorem 6 above (L'Hopital rule in infinite functions), relation (74), which is repeated below, is proved much easier.

$$\frac{\ln \xi}{1 + \frac{1}{2} + \ldots + \frac{1}{\xi}} = 1.$$

Indeed, according to Theorem 2, the derivative of the infinite function of the denominator is equal to its last term $(1/\xi)$, whereas the derivative of the numerator is also $(1/\xi)$. Therefore, the left member of the above relation is transformed as follows:

$$\frac{\ln \xi}{1 + \frac{1}{2} + \ldots + \frac{1}{\xi}} = \frac{\frac{1}{\xi}}{\frac{1}{\xi}} = 1.$$

VI. Abstract Structure and Properties of the Infinite Numbers Set

In light of the above, it is important to investigate what kind of abstract structure the new set of infinite numbers has and what the similarities and differences are compared to other related mathematical concepts. As is known, an abstract structure includes precise rules of behavior that can be used to determine whether a candidate implementation actually matches the abstract structure in question, and this must be free from contradictions. Certainly, this also happens with infinite numbers, as precise rules of behavior have been established, and there is no contradiction.

In mathematics, an algebraic structure consists of a non-empty set, *S* (called the underlying set, carrier set, or domain), a collection of operations on S of finite arity (typically binary operations) and a finite set of identities, known as axioms, that these operations must satisfy. Based on what was exposed in the previous Sections and the above-mentioned definitions, the set of infinite numbers, **A**, is indeed an algebraic structure.

Moreover, in mathematics, a field is a set on which addition, subtraction, multiplication, and division are defined and behave as the corresponding operations on rational and real numbers do. The fields that are most familiar are the field of rational numbers, the field of real numbers, and the field of complex numbers. Many other fields, such as the fields of rational functions, algebraic function fields, algebraic number fields, and p-adic fields, are regularly used and studied in mathematics, especially in number theory. Based on what was stated in the previous Sections and the four arithmetic operations defined, it is obvious that the set of infinite numbers is a field. Furthermore, the set of infinite numbers is an algebraically closed field since every operation on infinite numbers yields another infinite number.

In mathematics, an ordered field is a field together with a total ordering of its elements that is compatible with the field operations. The basic example of an ordered field is the field of real numbers. However, in the set of infinite numbers, **A**, ordering is not applied, as infinite numbers with different angles cannot be compared, as was already mentioned above.

As is known, a Hardy field (*H*) is a field consisting of the germs of real-valued functions at infinity that is closed under differentiation [24, 31, 32], where a germ of an object (in general) in a topological space is an equivalence class of that object and others of the same kind that captures their shared local properties. Loosely speaking, Hardy fields are the natural domain of asymptotic analysis, where all rules hold without qualifying conditions. It is interesting to note that we can place an ordering on H by saying $f < g$ if $(g - f)$ is eventually strictly positive, meaning that there is a real number, U, such that $[g(x) - f(x)] > 0$ for all $x \geq U$. However, in contrast to Hardy fields, in the set of infinite numbers, **A**, ordering is not applied, as infinite numbers with different angles cannot be compared [25, 26].

In addition to these, rings in mathematics are algebraic structures that generalize fields; multiplication does not need to be commutative, and multiplicative inverses do not need to exist. In other words, a ring is a set that is equipped with two binary operations that satisfy properties analogous to those of the addition and multiplication of integers. Ring elements can be numbers, such as integers or complex numbers, but they can also be non-numerical objects, such as polynomials, square matrices, functions, and power series. Of course, the set of infinite numbers that constitutes a field is also a ring, which is a broader concept than a field.

Whether a ring is commutative (that is, whether the order in which two elements are multiplied might change the result) has profound implications on its behavior. Commutative algebra, the theory of commutative rings, is a major branch of ring theory, highly influenced by problems and ideas of algebraic number theory and algebraic geometry. The infinite numbers form a commutative algebra over the real numbers since the commutative law for multiplication applies, as is explained in Section III.

Additionally, it should be mentioned that in the set of infinite numbers, **A**, the "Archimedean property" is not satisfied for any pair of elements, as is true, for example, for numbers ξ and ξ^2. The reason is that there is no integer, n, so that $n\xi > \xi^2$, given that $\frac{n\xi}{\xi^2} = \frac{n}{\xi} = 0$ for every natural number n, which means that the number ξ is infinitesimal with respect to the number ξ^2. Therefore, the infinite numbers, **A**, constitute an algebraic structure that is "non-Archimedean", as there are pairs of numbers that do not satisfy the

Archimedean property, while the sets of the integers, the rational numbers and the real numbers, together with the operation of addition and the usual ordering (≤), are Archimedean groups [27, 28, 33, 34].

In model theory, a "transfer principle" states that all statements of some language that are true for one structure are true for another structure. For hyperreal numbers, the transfer principle is concerned with the logical relation between the properties of the real numbers, $\mathbb{R}$, and the properties of a larger field, denoted ${}^{*}\mathbb{R}$, called the hyperreal numbers, which include, in particular, infinitesimal numbers.

However, it is noted that the hyperreal numbers form an ordered field, whereas the set of infinite numbers is not an ordered field, as infinite numbers cannot be compared if they have different angles. On the other hand, as mentioned above, the real numbers form an Archimedean field. Thus, a similar transfer principle cannot be applied to infinite numbers, in contrast to hyperreal numbers [29].

In summary, infinite numbers, which are a superset of the complex numbers, form a rich structure that is simultaneously a field, a non-ordered ring, a "non-Archimedean" algebraic structure, an algebraically closed field, and a commutative algebra over the real numbers.

Finally, it should be noted that although the real technological systems are finite, infinity may appear during their operation in various ways, and, therefore, respective analysis is required. For instance, a system can lead to instability, meaning that some physical quantity tends to infinity and, thus, during analysis/synthesis, it needs to be fully investigated and properly dimensioned. Moreover, a system with a very large number of elements (a distributed system) can be simulated as a system of an infinite number of elements (see applications in infinite electrical networks below).

Therefore, the investigation of infinity is very useful (not only in theory but also in practice) and finds important applications in real technological systems.

VII. Indicative Mathematical and Technological System Applications Based on the Above Approach

A. *Applications for calculating limits of the form* $0 \cdot \infty$ *or* ∞/∞ *with series of numbers, where L'Hopital's rule cannot be applied.*

A1. *Find the following limit* (113):

$$\lim_{n\to\infty}\left(\frac{1}{n^2 - \ln(n^2+1)}\right)\cdot\left(\frac{1^3}{1^2+1}+\frac{2^3}{2^2+1}+\ldots+\frac{n^3}{n^2+1}\right) \tag{113}$$

1st Proof. By using infinite numbers, the ordinary series of eq. (114) below is written as formula (115), and, according to Theorem 2, its derivative is equal to its last term (eq. (116)), given that the respective criterion is also satisfied.

$$A_n = \frac{1^3}{1^2+1}+\frac{2^3}{2^2+1}+\ldots+\frac{n^3}{n^2+1}+\ldots \quad n \in \mathbb{N} \tag{114}$$

$$A(\xi) = \frac{1^3}{1^2+1}+\frac{2^3}{2^2+1}+\ldots+\frac{\xi^3}{\xi^2+1} \tag{115}$$

$$\frac{d}{d\xi}A(\xi) = \frac{\xi^3}{\xi^2+1}. \tag{116}$$

By taking the integrals in eq. (116), the following is obtained:

$$A(\xi) = \int \frac{d}{d\xi}A(\xi)d\xi = \int \frac{\xi^3}{\xi^2+1}d\xi$$

$$\Rightarrow A(\xi) = \frac{1}{2}(\xi^2 - \ln(\xi^2+1)) + C(\xi) \tag{117}$$

where $C(\xi)$ is the advancement, which is one degree lower than the function $A(\xi)$.

When combining (115) and (117), we obtain

$$A(\xi) = \frac{1^3}{1^2+1}+\frac{2^3}{2^2+1}+\ldots+\frac{\xi^3}{\xi^2+1} = \frac{1}{2}(\xi^2 - \ln(\xi^2+1)) + C(\xi)$$

Furthermore, according to Section III (eq. (17)), the following is obtained:

$$\frac{\frac{1^3}{1^2+1}+\frac{2^3}{2^2+1}+\ldots+\frac{\xi^3}{\xi^2+1}}{\frac{1}{2}(\xi^2-\ln(\xi^2+1))}=1$$

$$\Rightarrow\left(\frac{1}{\xi^2-\ln(\xi^2+1)}\right)\left(\frac{1^3}{1^2+1}+\frac{2^3}{2^2+1}+\ldots+\frac{\xi^3}{\xi^2+1}\right)=\frac{1}{2}$$

$$\Rightarrow\lim_{n\to\infty}\left(\frac{1}{n^2-\ln(n^2+1)}\right)\left(\frac{1^3}{1^2+1}+\frac{2^3}{2^2+1}+\ldots+\frac{n^3}{n^2+1}\right)=\frac{1}{2}$$ □

2nd Proof. Using infinite numbers, the last infinite term of an ordinary series depicts the first derivative, and, therefore, by applying Theorem 6 (L'Hopital's rule in infinite functions), we obtain the following:

First, relation (113) is written in infinite numbers:

$$L=\lim_{n\to\infty}\left(\frac{1}{n^2-\ln(n^2+1)}\right)\left(\frac{1^3}{1^2+1}+\frac{2^3}{2^2+1}+\ldots+\frac{n^3}{n^2+1}\right)$$

$$=\left(\frac{1}{\xi^2-\ln(\xi^2+1)}\right)\left(\frac{1^3}{1^2+1}+\frac{2^3}{2^2+1}+\ldots+\frac{\xi^3}{\xi^2+1}\right)$$

$$=\frac{\frac{1^3}{1^2+1}+\frac{2^3}{2^2+1}+\ldots+\frac{\xi^3}{\xi^2+1}}{\xi^2-\ln(\xi^2+1)}$$

Now, by taking the first derivative of the numerator and denominator, we have

$$L=\frac{\frac{\xi^3}{\xi^2+1}}{2\xi-\frac{2\xi}{\xi^2+1}}=\frac{\frac{\xi^3}{\xi^2}}{2\xi}=\frac{\xi}{2\xi}=\frac{1}{2}$$

It is really a very simple way to calculate the limit of expression (113). Additionally, it is a method that can be applied in all corresponding cases. □

In order to examine the correctness of the above result (and the proposed method), the limit of expression (113) is calculated again below, with an additional proof based on conventional algebra, which is certainly more difficult:

3rd Proof.

$$\lim_{n\to\infty}\left(\frac{1}{n^2-\ln(n^2+1)}\right)\left(\frac{1^3}{1^2+1}+\frac{2^3}{2^2+1}+\ldots+\frac{n^3}{n^2+1}\right)$$

$$=\lim_{n\to\infty}\left(\frac{1}{1-\frac{\ln(n^2+1)}{n^2}}\right)\cdot\frac{1}{n^2}\left(\frac{1^3}{1^2+1}+\frac{2^3}{2^2+1}+\ldots+\frac{n^3}{n^2+1}\right)$$

Given the fact that $\lim_{n\to\infty}\left(\frac{1}{1-\frac{\ln(n^2+1)}{n^2}}\right)=1$

we have to prove that

$$\lim_{n\to\infty}\frac{1}{n^2}\left(\frac{1^3}{1^2+1}+\frac{2^3}{2^2+1}+\ldots+\frac{n^3}{n^2+1}\right)=\frac{1}{2}$$

Since it is valid that $k-1<\frac{k^3}{k^2+1}<k\ \forall\, k=1,2,\ldots,n$

adding from $k=1$ to $k=n$ and taking into account that $1+2+\ldots+n=\frac{n(n+1)}{2}$, we have

$$\frac{(n-1)n}{2n^2}<\frac{1}{n^2}\left(\frac{1^3}{1^2+1}+\frac{2^3}{2^2+1}+\ldots+\frac{n^3}{n^2+1}\right)<\frac{n(n+1)}{2n^2}.$$

If we take the limits of all the members of inequality, we conclude that our expression is equal to $\frac{1}{2}$, as it is also proved by using infinite numbers. □

In the same way, the limits of complicated expressions with a series of numbers, such as in eqs. (118)-(120), where $n\in\mathbb{N}$, are easily determined. On the contrary, proving them without the use of infinite numbers is not an easy task.

$$\lim_{n\to\infty} \frac{4(n+e)^{5/2}}{(1+e)^{3/2} + (2+e)^{3/2} + \ldots + (n+e)^{3/2}} = 10 \tag{118}$$

$$\lim_{n\to\infty} \frac{n - \pi \cdot \ln(n+\pi)}{\frac{1}{1+\pi} + \frac{2}{2+\pi} + \ldots + \frac{n}{n+\pi}} = 1 \tag{119}$$

$$\lim_{n\to\infty} \frac{(n-2e)\cdot\sqrt{n+e}}{\frac{1}{\sqrt{1+e}} + \frac{2}{\sqrt{2+e}} + \ldots + \frac{n}{\sqrt{n+e}}} = \frac{3}{2} \tag{120}$$

We notice that the above limits (relations (118)-(120)) are of the form ∞/∞ since both the numerator and the denominator tend to infinity. However, L'Hopital's rule cannot be used according to the classical method (without infinite numbers) since we have not functions but series involved, where, of course, there are no derivatives.

Let us now calculate the limit of the ratio of the series:

A2. *Find the limit of the following expression*, *with ratio of infinite series*:

$$\lim_{n\to\infty} \frac{A_n}{B_n} = \lim_{n\to\infty} \frac{\frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \ldots + \frac{1}{3n-1}}{\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \ldots + \frac{1}{4n-1}} \quad \text{where } n \in \mathbb{N} \tag{121}$$

Proof 1. By using infinite numbers, the above expression is written as the following ratio, $A(\xi)/B(\xi)$, of the corresponding infinite number functions:

$$\frac{A(\xi)}{B(\xi)} = \frac{\frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \ldots + \frac{1}{3\xi-1}}{\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \ldots + \frac{1}{4\xi-1}}$$

According to Theorem 2, the derivative of the infinite number function of the numerator $A(\xi)$ is $dA(\xi)/d\xi = 1/(3\xi - 1)$, which also takes into account the fact that the respective criterion applies. Therefore, the following integral is calculated:

$$A(\xi) = \int \frac{d}{d\xi} A(\xi) d\xi = \int \frac{1}{3\xi - 1} d\xi = \frac{1}{3} \int \frac{1}{3\xi-1} d(3\xi - 1) = \frac{1}{3} \ln(3\xi - 1) + C_1$$

$$= \frac{1}{3}\ln(3\xi) + C_1 = \frac{1}{3}\ln(3) + \frac{1}{3}\ln(\xi) + C_1 \qquad \text{where } C_1 \in \mathbb{R}$$

Hence: $A(\xi) = \frac{1}{3}\ln(3) + \frac{1}{3}\ln(\xi) + C_1$

Similarly, for the denominator $B(\xi)$, we obtain

$$B(\xi) = \frac{1}{4}\ln(4) + \frac{1}{4}\ln(\xi) + C_2 \qquad \text{where } C_2 \in \mathbb{R}$$

Therefore, the following applies:

$$\frac{A(\xi)}{B(\xi)} = \frac{\frac{1}{3}\ln(3) + \frac{1}{3}\ln(\xi) + C_1}{\frac{1}{4}\ln(4) + \frac{1}{4}\ln(\xi) + C_2} = \frac{\frac{1}{3}\ln(\xi)}{\frac{1}{4}\ln(\xi)} = \frac{4}{3} \qquad \square$$

Proof 2. By applying L'Hopital's rule to the ratio of infinite functions, $A(\xi)/B(\xi)$, we obtain

$$\frac{\frac{dA(\xi)}{d\xi}}{\frac{dB(\xi)}{d\xi}} = \frac{\frac{1}{3\xi - 1}}{\frac{1}{4\xi - 1}} = \frac{4\xi - 1}{3\xi - 1} = \frac{4\xi}{3\xi} = \frac{4}{3} \qquad \square$$

A3. *Find the limit of the following expression*:

$$\lim_{n\to\infty} \frac{A_n}{B_n} = \lim_{n\to\infty} \frac{(a + b) + (2a + b) + (3a + b) + \ldots + (na + b)}{(c + d) + (2c + d) + (3c + d) + \ldots + (nc + d)} \qquad (122)$$

where $n \in \mathbb{N}$ *and* $a, b, c, d \in \mathbb{R}$.

Proof. According to Theorem 2, the derivative of the infinite number function of the numerator $A(\xi)$ is $dA(\xi)/d\xi = (\alpha\xi + b)$, which also takes into account the fact that the respective criterion applies. Therefore, the following integral is calculated, where $C_1(\xi)$ is the advancement:

$$A(\xi) = \int (\alpha\xi + b)d\xi = \frac{\alpha\xi^2}{2} + b\xi + C_1(\xi)$$

$$C_1(\xi) = C_{11}\xi + C_{12} \qquad \text{where } C_{11}, C_{12} \in \mathbb{R}$$

Similarly, for the denominator $B(\xi)$, we obtain

$$B(\xi) = \int (c\xi + d)d\xi = \frac{c\xi^2}{2} + d\xi + C_2(\xi)$$

$$C_2(\xi) = C_{21}\xi + C_{22} \quad \text{where } C_{21}, C_{22} \in \mathbb{R}$$

Therefore, the following applies:

$$\frac{A(\xi)}{B(\xi)} = \frac{\frac{a\xi^2}{2} + b\xi + C_{11}\xi + C_{12}}{\frac{c\xi^2}{2} + d\xi + C_{21}\xi + C_{22}} = \frac{\frac{a\xi^2}{2}}{\frac{c\xi^2}{2}} = \frac{a}{c}$$

For example, for $\alpha = 1.5$, $b = 1$, $c = 4.2$, $d = 2$, we obtain

$$\lim_{n\to\infty} \frac{A_n}{B_n} = \lim_{n\to\infty} \frac{2.5 + 4 + 5.5 + 7 + \ldots + (1.5n + 1)}{6.2 + 10.4 + 14.6 + 18.8 + \ldots + (4.2n + 2)} = \frac{1.5}{4.2} = 0.357\ldots$$

In particular, for $b = 0$ and $d = 0$, without using infinite numbers, we have

$$\lim_{n\to\infty} \frac{A_n}{B_n} = \lim_{n\to\infty} \frac{a + 2a + 3a + \ldots + na}{c + 2c + 3c + \ldots + nc} = \lim_{n\to\infty} \frac{a(1 + 2 + 3 + \ldots + n)}{c(1 + 2 + 3 + \ldots + n)} = \frac{a}{c}$$

where it is confirmed that the result of our method is true, that is, a/c. □

A4. *Calculate the following unusual expression involving a series of numbers and improper integrals*:

$$\frac{\pi \cdot \lim_{t\to\infty} \int_{x=\pi}^{t} \frac{1}{x} dx - e \cdot \lim_{t\to\infty} \int_{x=e}^{t} \frac{dx}{\sqrt{x^2 + e^2}}}{\lim_{n\to\infty} \left(\frac{1}{1} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{3n-2} + \frac{1}{3n} \right)} \tag{123}$$

where $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

Proof. This expression is transformed as:

$$\frac{\pi \cdot \ln x|_{x=\pi}^{\infty} - e \cdot \ln(x + \sqrt{x^2 + e^2})|_{x=e}^{\infty}}{\lim_{n\to\infty} \left(\frac{1}{1} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{3n-2} + \frac{1}{3n} \right)}$$

Taking also into account eqs. (71) and (72), we notice that the above expression is of the form $\frac{\infty - \infty}{\infty}$ which is certainly undefined, moreover, normal L'Hopital's rule cannot be applied, and consequently cannot be calculated. If, however, we use the infinite numbers, this expression is transformed as below where $C \in \mathbb{R}$, and is easily calculated.

$$\frac{\pi \cdot (\ln \xi - \ln \pi) - e \cdot (\ln(\xi + \sqrt{\xi^2 + e^2}) - \ln(e + e\sqrt{2}))}{\frac{1}{3}\ln(3\xi^2) + C}$$

$$= \frac{(\pi - e)\ln \xi}{\frac{2}{3}\ln(\sqrt{3}\xi)} = \frac{(\pi - e)\ln \xi}{\frac{2}{3}\ln \xi + \frac{2}{3}\ln(\sqrt{3})} = \frac{(\pi - e)\ln \xi}{\frac{2}{3}\ln \xi} = \frac{3(\pi - e)}{2}.$$

It is an elegant result, achieved by a clear, smart, elegant and general method.

B. *Application of the extended* (*in the infinite numbers*) *bilateral Laplace transform to solve specific* 2nd *order differential equations*, *defined piecewise over the entire domain of real numbers* $(-\infty, +\infty)$.

B1. *Find the solution of the differential equation below*:

$$y''(t) + 2y'(t) = f(t) \tag{124}$$

where the following limits apply:

$$\lim_{t \to -\infty} y(t) = \pi \quad \text{and} \quad \lim_{t \to -\infty} y'(t) = 1 \tag{125}$$

and where

$$f(t) = e^{5t} \quad \text{for } t \in (-\infty, -3)$$

$$f(t) = 9 \quad \text{for } t \in [-3, +4)$$

$$f(t) = \sin t \quad \text{for } t \in [+4, +\infty). \tag{126}$$

Proof. By using the infinite numbers/functions, we can certainly obtain the bilateral Laplace transform of both sides of eq. (124). Given that, for the bilateral Laplace transform, the first derivative property is given by eq. (127), the second derivative property is calculated by using eq. (128).

$$L(y'(t)) = sY(s) + y(t)e^{-st} \Big|_{-\infty}^{+\infty} \tag{127}$$

$$L(y''(t)) = L((y'(t))')$$

$$= s^2 Y(s) + sy(t)e^{-st}\Big|_{-\infty}^{+\infty} + y'(t)e^{-st}\Big|_{-\infty}^{+\infty} \tag{128}$$

where $L(\cdot)$ is the Laplace transformation symbol.

Moreover, by using infinite numbers, the limit, $\lim_{t\to-\infty} y(t) = \pi$, is equivalently written as $y(-\xi) = \pi$, and the limit, $\lim_{t\to-\infty} y'(t) = 1$, is equivalently written as $y'(-\xi) = 1$.

Based on eqs. (127) and (128), eq. (124) transforms

$$s^2 Y(s) + sy(\xi)e^{-s\xi} - sy(-\xi)e^{s\xi} + y'(\xi)e^{-s\xi} - y'(-\xi)e^{s\xi}$$

$$+ 2sY(s) + 2y(\xi)e^{-s\xi} - 2y(-\xi)e^{s\xi}$$

$$= \int_{t=-\xi}^{t=\xi} f(t)e^{-st}dt$$

where, for $s > 0$, we have

$$s^2 Y(s) + 2sY(s) - (s+2)y(-\xi)e^{s\xi} - y'(-\xi)e^{s\xi} = \int_{t=-\xi}^{t=\xi} f(t)e^{-st}dt$$

and given that $y(-\xi) = \pi$ and $y'(-\xi) = 1$, we have

$$s^2 Y(s) + 2sY(s) - \pi(s+2)e^{s\xi} - e^{s\xi} = \int_{t=-\xi}^{t=\xi} f(t)e^{-st}dt$$

$$= \int_{t=-\xi}^{t=-3} e^{5t}e^{-st}dt + \int_{t=-3}^{t=4} 9e^{-st}dt + \int_{t=4}^{t=\xi} \sin t \cdot e^{-st}dt$$

$$= -\frac{1}{s-5}e^{-(s-5)t}\Big|_{t=-\xi}^{t=-3} - \frac{9}{s}e^{-st}\Big|_{t=-3}^{t=4} - \frac{se^{-st}\sin t}{1+s^2}\Big|_{t=4}^{t=\xi} - \frac{e^{-st}\cos t}{1+s^2}\Big|_{t=4}^{t=\xi}$$

Therefore,

$$(s^2+2s)Y(s) = -\frac{1}{s-5}e^{-(s-5)t}\Big|_{t=-\xi}^{t=-3} - \frac{9}{s}e^{-st}\Big|_{t=-3}^{t=4}$$

$$-\frac{se^{-st}\sin t}{1+s^2}\Big|_{t=4}^{t=\xi} - \frac{e^{-st}\cos t}{1+s^2}\Big|_{t=4}^{t=\xi} + \pi(s+2)\cdot e^{s\xi} + e^{s\xi}.$$

Thus,

$$Y(s) = -\frac{1}{s(s+2)(s-5)}e^{-(s-5)t}\Big|_{t=-\xi}^{t=-3} - \frac{9e^{-st}}{s^2(s+2)}\Big|_{t=-3}^{t=4}$$

$$-\frac{se^{-st}\sin t}{s(s+2)(1+s^2)}\Big|_{t=4}^{t=\xi} - \frac{e^{-st}\cos t}{s(s+2)(1+s^2)}\Big|_{t=4}^{t=\xi} + \frac{\pi\cdot e^{s\xi}}{s} + \frac{e^{s\xi}}{s(s+2)}.$$

Hence,

$$Y(s) = \frac{e^{(s-5)\xi} - e^{(s-5)3}}{s(s+2)(s-5)} - 9\frac{e^{-4s} - e^{3s}}{s^2(s+2)}$$

$$+\frac{(e^{-s4}\sin 4 - e^{-s\xi}\sin\xi)}{(s+2)(1+s^2)} + \frac{e^{-s4}\cos 4 - e^{-s\xi}\cos\xi}{s(s+2)(1+s^2)}$$

$$+\frac{\pi\cdot e^{s\xi}}{s} + \frac{e^{s\xi}}{s(s+2)}. \tag{129}$$

Furthermore,

$$Y(s) = -\frac{1}{10}\frac{e^{(s-5)\xi}}{s} + \frac{1}{14}\frac{e^{(s-5)\xi}}{s+2} + \frac{1}{35}\frac{e^{(s-5)\xi}}{s-5}$$

$$+\frac{1}{10}\frac{e^{(s-5)3}}{s} - \frac{1}{14}\frac{e^{(s-5)3}}{s+2} - \frac{1}{35}\frac{e^{(s-5)3}}{s-5} + \frac{9}{4}\frac{e^{-4s}}{s} - \frac{9}{2}\frac{e^{-4s}}{s^2}$$

$$-\frac{9}{4}\frac{e^{-4s}}{s+2} - \frac{9}{4}\frac{e^{3s}}{s} + \frac{9}{2}\frac{e^{3s}}{s^2} + \frac{9}{4}\frac{e^{3s}}{s+2}$$

$$+\frac{1}{5}\frac{1}{(s+2)}e^{-s4}\sin 4 - \frac{1}{5}\frac{s-2}{s^2+1}e^{-s4}\sin 4$$

$$-\frac{1}{5}\frac{1}{(s+2)}e^{-s\xi}\sin\xi+\frac{1}{5}\frac{s-2}{s^2+1}e^{-s\xi}\sin\xi$$

$$+\frac{1}{2}\frac{1}{s}e^{-s4}\cos 4-\frac{1}{10}\frac{1}{s+2}e^{-s4}\cos 4-\frac{1}{5}\frac{2s+1}{1+s^2}e^{-s4}\cos 4$$

$$-\frac{1}{2}\frac{1}{s}e^{-s\xi}\cos\xi+\frac{1}{10}\frac{1}{s+2}e^{-s\xi}\cos\xi+\frac{1}{5}\frac{2s+1}{1+s^2}e^{-s\xi}\cos\xi$$

$$+\frac{\pi\cdot e^{s\xi}}{s}+\frac{1}{2}\frac{e^{s\xi}}{s}-\frac{1}{2}\frac{e^{s\xi}}{s+2}.$$

Furthermore,

$$Y(s)=-\frac{1}{10}e^{-5\xi}\frac{e^{\xi\cdot s}}{s}+\frac{1}{14}e^{-7\xi}\frac{e^{\xi(s+2)}}{s+2}+\frac{1}{35}\frac{e^{\xi(s-5)}}{s-5}$$

$$+\frac{1}{10}e^{-15}\frac{e^{3s}}{s}-\frac{1}{14}e^{-21}\frac{e^{3(s+2)}}{s+2}-\frac{1}{35}\frac{e^{3(s-5)}}{s-5}+\frac{9}{4}\frac{e^{-4s}}{s}-\frac{9}{2}\frac{e^{-4s}}{s^2}$$

$$-\frac{9}{4}\frac{e^{-4s}}{s+2}-\frac{9}{4}\frac{e^{3s}}{s}+\frac{9}{2}\frac{e^{3s}}{s^2}+\frac{9}{4}\frac{e^{3s}}{s+2}$$

$$+\frac{1}{5}e^{8}\frac{e^{-4(s+2)}}{(s+2)}\sin 4-\frac{1}{5}e^{-4s}\frac{s}{s^2+1}\sin 4+\frac{1}{5}e^{-4s}\frac{2}{s^2+1}\sin 4$$

$$-\frac{1}{5}e^{2\xi}\frac{e^{-\xi(s+2)}}{(s+2)}\sin\xi+\frac{1}{5}e^{-\xi s}\frac{s}{s^2+1}\sin\xi-\frac{1}{5}e^{-\xi s}\frac{2}{s^2+1}\sin\xi$$

$$+\frac{\cos 4}{2}\cdot\frac{e^{-4s}}{s}-\frac{\cos 4}{10}\cdot e^{8}\frac{e^{-4(s+2)}}{s+2}-\frac{\cos 4}{5}\cdot\frac{2s\cdot e^{-4s}}{1+s^2}-\frac{\cos 4}{5}\cdot\frac{e^{-4s}}{1+s^2}$$

$$-\frac{\cos\xi}{2}\cdot\frac{e^{-\xi s}}{s}+\frac{\cos\xi}{10}\cdot e^{2\xi}\frac{e^{-\xi(s+2)}}{s+2}+\frac{\cos\xi}{5}\cdot\frac{2s\cdot e^{-\xi s}}{1+s^2}+\frac{\cos\xi}{5}\cdot\frac{e^{-\xi s}}{1+s^2}$$

$$+\pi\frac{e^{\xi s}}{s}+\frac{1}{2}\frac{e^{\xi s}}{s}-\frac{1}{2}e^{-2\xi}\frac{e^{\xi(s+2)}}{s+2}.$$

It is noted that the inverse bilateral Laplace transform of the expression $e^{-\alpha s}\frac{s}{s^2+1}$ for any $\alpha>0$ is $L^{-1}\left(e^{-as}\frac{s}{s^2+1}\right)=\cos(t-\alpha)u(t-\alpha)$, that is, the same as the one-side Laplace transform given that for $t<\alpha$, and, consequently, for $t<0$, the function $\cos(t-\alpha)u(t-\alpha)$ is zero.

Therefore, taking the inverse Laplace transform of $Y(s)$, we have

$$y(t) = -\frac{1}{10}e^{-5\xi}u(t-(-\xi)) + \frac{1}{14}e^{-7\xi}e^{-2t}u(t-(-\xi)) + \frac{1}{35}e^{5t}u(t-(-\xi))$$

$$+\frac{1}{10}e^{-15}u(t-(-3)) - \frac{1}{14}e^{-21}e^{-2t}u(t-(-3)) - \frac{1}{35}e^{5t}u(t-(-3))$$

$$+\frac{9}{4}u(t-4) - \frac{9}{2}(t-4)u(t-4) - \frac{9}{4}e^{-2t}u(t-4)$$

$$-\frac{9}{4}u(t-(-3)) + \frac{9}{2}(t+3)u(t-(-3)) + \frac{9}{4}e^{-2t}u(t-(-3))$$

$$+\frac{1}{5}e^{8}\sin 4\cdot e^{-2t}u(t-4) - \frac{1}{5}\sin 4\cdot\cos(t-4)u(t-4)$$

$$+\frac{2}{5}\sin 4\cdot\sin(t-4)u(t-4)$$

$$-\frac{1}{5}\sin\xi\cdot e^{2\xi}e^{-2t}u(t-\xi) + \frac{1}{5}\sin\xi\cdot\cos(t-\xi)u(t-\xi)$$

$$-\frac{2}{5}\sin\xi\cdot\sin(t-\xi)u(t-\xi)$$

$$+\frac{\cos 4}{2}u(t-4) - \frac{\cos 4}{10}\cdot e^{8}e^{-2t}u(t-4) - \frac{2\cos 4}{5}\cos(t-4)u(t-4)$$

$$-\frac{\cos 4}{5}\cdot\sin(t-4)u(t-4)$$

$$-\frac{\cos\xi}{2}u(t-\xi) + \frac{\cos\xi}{10}\cdot e^{2\xi}e^{-2t}u(t-\xi) + \frac{2\cos\xi}{5}\cos(t-\xi)u(t-\xi)$$

$$+\frac{\cos\xi}{5}\cdot\sin(t-\xi)u(t-\xi)$$

$$+\pi\cdot u(t-(-\xi)) + \frac{1}{2}u(t-(-\xi)) - \frac{1}{2}e^{-2\xi}e^{-2t}u(t-(-\xi)).$$

Since $u(t-(-\xi)) = 1$ and $u(t-\xi) = 0$, we obtain the solution:

$$y(t) = -\frac{1}{10}e^{-5\xi} + \frac{1}{14}e^{-7\xi}e^{-2t} + \frac{1}{35}e^{5t}$$

$$+\frac{1}{10}e^{-15}u(t-(-3)) - \frac{1}{14}e^{-21}e^{-2t}u(t-(-3)) - \frac{1}{35}e^{5t}u(t-(-3))$$

$$+\frac{9}{4}u(t-4)-\frac{9}{2}(t-4)u(t-4)-\frac{9}{4}e^{-2t}u(t-4)$$

$$-\frac{9}{4}u(t-(-3))+\frac{9}{2}(t+3)u(t-(-3))+\frac{9}{4}e^{-2t}u(t-(-3))$$

$$+\frac{1}{5}e^{8}\sin 4\cdot e^{-2t}u(t-4)-\frac{1}{5}\sin 4\cdot\cos(t-4)u(t-4)$$

$$+\frac{2}{5}\sin 4\cdot\sin(t-4)u(t-4)$$

$$+\frac{\cos 4}{2}u(t-4)-\frac{\cos 4}{10}\cdot e^{8}e^{-2t}u(t-4)-\frac{2\cos 4}{5}\cdot e^{-4s}\cos(t-4)u(t-4)$$

$$-\frac{\cos 4}{5}\cdot\sin(t-4)u(t-4)$$

$$+\pi+\frac{1}{2}-\frac{1}{2}e^{-2\xi}e^{-2t}. \tag{130}$$

The above relation can be written equivalently:

For $t<-3$:

$$y(t)=-\frac{1}{10}e^{-5\xi}+\frac{1}{14}e^{-7\xi}e^{-2t}+\frac{1}{35}e^{5t}+\pi+\frac{1}{2}-\frac{1}{2}e^{-2\xi}e^{-2t} \tag{131}$$

For $-3<=t<+4$:

$$y(t)=-\frac{1}{10}e^{-5\xi}+\frac{1}{14}e^{-7\xi}e^{-2t}+\frac{1}{35}e^{5t}+\frac{1}{10}e^{-15}-\frac{1}{14}e^{-21}e^{-2t}-\frac{1}{35}e^{5t}$$

$$-\frac{9}{4}+\frac{9}{2}(t+3)+\frac{9}{4}e^{-2t}+\pi+\frac{1}{2}-\frac{1}{2}e^{-2\xi}e^{-2t} \tag{132}$$

For $t>=+4$:

$$y(t)=-\frac{1}{10}e^{-5\xi}+\frac{1}{14}e^{-7\xi}e^{-2t}+\frac{1}{35}e^{5t}+\frac{1}{10}e^{-15}-\frac{1}{14}e^{-21}e^{-2t}-\frac{1}{35}e^{5t}$$

$$+\frac{9}{4}-\frac{9}{2}(t-4)-\frac{9}{4}e^{-2t}-\frac{9}{4}+\frac{9}{2}(t+3)+\frac{9}{4}e^{-2t}$$

$$+\frac{1}{5}e^{8}\sin 4\cdot e^{-2t}-\frac{1}{5}\sin 4\cdot\cos(t-4)+\frac{2}{5}\sin 4\cdot\sin(t-4)$$

$$+\frac{\cos 4}{2}-\frac{\cos 4}{10}\cdot e^{8}e^{-2t}-\frac{2\cos 4}{5}\cdot\cos(t-4)-\frac{\cos 4}{5}\cdot\sin(t-4)$$

$$+\pi+\frac{1}{2}-\frac{1}{2}e^{-2\xi}e^{-2t}. \tag{133}$$

Let us now proceed to verify the above solution, which is fully described by eq. (130). First, we will examine whether eq. (130) and, consequently, eq. (131) satisfy the specific initial conditions, i.e., if $y(-\xi)=\pi$ and $y'(-\xi)=1$ holds. For $t=-\xi$, equation (131) is transformed into eq. (134):

$$y(-\xi)=-\frac{1}{10}e^{-5\xi}+\frac{1}{14}e^{-7\xi}e^{-2(-\xi)}+\frac{1}{35}e^{5(-\xi)}+\pi+\frac{1}{2}-\frac{1}{2}e^{-2\xi}e^{-2(-\xi)}.$$

Therefore,

$$y(-\xi)=-\frac{1}{10}e^{-5\xi}+\frac{1}{14}e^{-5\xi}+\frac{1}{35}e^{-5\xi}+\pi+\frac{1}{2}-\frac{1}{2}e^{0}=\pi \tag{134}$$

Thus, $y(-\xi)=\pi$

From equation (131), for the $t<-3$ results:

$$y'(t)=(-2)\frac{1}{14}e^{-7\xi}e^{-2t}+5\frac{1}{35}e^{5t}-(-2)\frac{1}{2}e^{-2\xi}e^{-2t}$$

$$\Rightarrow y'(-\xi)=(-2)\frac{1}{14}e^{-7\xi}e^{-2(-\xi)}+\frac{5}{35}e^{5(-\xi)}-(-2)\frac{1}{2}e^{-2\xi}e^{-2(-\xi)}$$

$$=-\frac{1}{7}e^{-5\xi}+\frac{1}{7}e^{-5\xi}+e^{0}=1.$$

Thus, $y'(-\xi)=1$

Therefore, both the specific initial conditions are satisfied.

Let us now see if the differential equation (124) is also verified by the solution in question. To this end, let us form the derivatives $y'(t)$ and $y''(t)$.

For $t<-3$:

$$y'(t)=-\frac{1}{7}e^{-7\xi}e^{-2t}+\frac{1}{7}e^{5t}+e^{-2\xi}e^{-2t}$$

$$y''(t)=\frac{2}{7}e^{-7\xi}e^{-2t}+\frac{5}{7}e^{5t}-2e^{-2\xi}e^{-2t}.$$

So, it holds that

$$y''(t) + 2y'(t) = e^{5t}$$

For $-3 <= t < 4$:

$$y'(t) = -\frac{1}{7}e^{-7\xi}e^{-2t} + \frac{1}{7}e^{5t} + \frac{2}{14}e^{-21}e^{-2t} - \frac{5}{35}e^{5t}$$

$$+ \frac{9}{2} - \frac{9}{2}e^{-2t} + \frac{2}{2}e^{-2\xi}e^{-2t}$$

$$y''(t) = \frac{2}{7}e^{-7\xi}e^{-2t} + \frac{5}{7}e^{5t} - \frac{2}{7}e^{-21}e^{-2t} - \frac{25}{35}e^{5t} + \frac{9}{1}e^{-2t} - 2e^{-2\xi}e^{-2t}.$$

So, it holds that

$$y''(t) + 2y'(t) = e^{5t} - e^{5t} + 9 = 9$$

For $t >= 4$:

$$y'(t) = -\frac{1}{7}e^{-7\xi}e^{-2t} + \frac{1}{7}e^{5t} + \frac{2}{14}e^{-21}e^{-2t} - \frac{5}{35}e^{5t} - \frac{9}{2} + \frac{9}{2}e^{-2t} + \frac{9}{2} - \frac{9}{2}e^{-2t}$$

$$-\frac{2}{5}e^{8}\sin 4 \cdot e^{-2t} + \frac{1}{5}\sin 4 \cdot \sin(t-4) + \frac{2}{5}\sin 4 \cdot \cos(t-4)$$

$$+ \frac{\cos 4}{5} \cdot e^{8}e^{-2t} + \frac{2\cos 4}{5} \cdot \sin(t-4) - \frac{\cos 4}{5} \cdot \cos(t-4) + \frac{2}{2}e^{-2\xi}e^{-2t}$$

$$y''(t) = \frac{2}{7}e^{-7\xi}e^{-2t} + \frac{5}{7}e^{5t} - \frac{2}{7}e^{-21}e^{-2t} - \frac{25}{35}e^{5t} - \frac{9}{1}e^{-2t} + \frac{9}{1}e^{-2t}$$

$$+ \frac{4}{5}e^{8}\sin 4 \cdot e^{-2t} + \frac{1}{5}\sin 4 \cdot \cos(t-4) - \frac{2}{5}\sin 4 \cdot \sin(t-4)$$

$$-2\frac{\cos 4}{5} \cdot e^{8}e^{-2t} + \frac{2\cos 4}{5} \cdot \cos(t-4) + \frac{\cos 4}{5} \cdot \sin s(t-4)$$

$$-\frac{2}{1}e^{-2\xi}e^{-2t}.$$

So, it holds that

$$y''(t) + 2y'(t) = \sin 4\cos(t-4) + \cos 4\sin(t-4)$$

$$= \sin(t-4+4) = \sin t.$$

Consequently, it is confirmed that eq. (130) (which is written in infinite numbers), is, indeed, the solution to the problem, in the broader set of infinite number functions. Certainly, the infinite number terms, as seen in eq. (130) are zero and can be omitted. Consequently, our solution is finally described by the following function (135), which is a real function.

$$y(t) = \frac{1}{35}e^{5t} + \frac{1}{10}e^{-15}u(t-(-3)) - \frac{1}{14}e^{-21}e^{-2t}u(t-(-3))$$
$$-\frac{1}{35}e^{5t}u(t-(-3)) + \frac{9}{4}u(t-4) - \frac{9}{2}(t-4)u(t-4) - \frac{9}{4}e^{-2t}u(t-4)$$
$$-\frac{9}{4}u(t-(-3)) + \frac{9}{2}(t+3)u(t-(-3)) + \frac{9}{4}e^{-2t}u(t-(-3))$$
$$+\frac{1}{5}e^{8}\sin 4\cdot e^{-2t}u(t-4) - \frac{1}{5}\sin 4\cdot\cos(t-4)u(t-4)$$
$$+\frac{2}{5}\sin 4\cdot\sin(t-4)u(t-4)\frac{\cos 4}{2}u(t-4) - \frac{\cos 4}{10}\cdot e^{8}e^{-2t}u(t-4)$$
$$-\frac{2\cos 4}{5}\cdot e^{-4s}\cos(t-4)u(t-4) - \frac{\cos 4}{5}\cdot\sin(t-4)u(t-4)$$
$$+\pi+\frac{1}{2}. \qquad \square \; (135)$$

C. *Applications in analyzing complicated infinite electrical networks*

In [38], the infinite electrical network of Figure 1 was considered, and current I was calculated in the simplified case where all impedances, Z, are equal to a net resistance $Z = R = 1\Omega$.

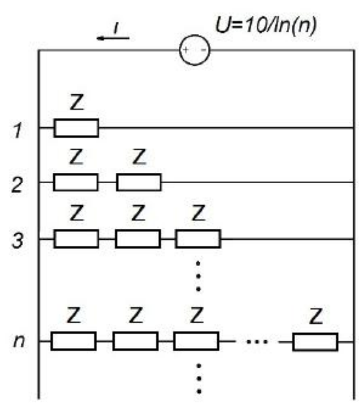

Figure 1. Infinite electrical network.

C1. *In the present study* (*for the electric circuit of Figure* 1, where n tends to infinity), *in place of the resistors* $(R = 1\Omega)$, *we consider the general case of impedances* (*e.g.*, $Z = 0.8 + i0.6\Omega$). *So*, *in this network, each subsequent branch includes an additional impedance*, Z. *This means that the nth branch contains n impedances Z in a series*, *and all the branches are connected in parallel with each other as well as with an electrical voltage source. Solve the circuit*, *calculate the current* (I), *find the current lag with respect to voltage* (U), *and*, *finally*, *determine the ratio of the total* (*circuit equivalent*) *reactance to the total resistance*.

Proof. According to Kirchhoff's two laws, the following equation applies:

$$\frac{1}{Z_{total}} = \frac{1}{0.8+i0.6} + \frac{1}{2\cdot(0.8+i0.6)} + \ldots + \frac{1}{n\cdot(0.8+i0.6)} + \ldots$$

$$= \frac{1}{0.8+i0.6}\left(1+\frac{1}{2}+\ldots+\frac{1}{\xi}\right)$$

where Z_{total} is the total (circuit equivalent) impedance.

Therefore,

$$I = \frac{U}{Z_{total}} = U\cdot\frac{1}{Z_{total}} = \frac{10}{\ln\xi}\cdot\frac{1}{0.8+i0.6}\left(1+\frac{1}{2}+\ldots+\frac{1}{\xi}\right)$$

$$\Rightarrow I = \frac{1}{0.8+i0.6}\cdot\frac{\frac{10}{\ln\xi}}{1+\frac{1}{2}+\ldots+\frac{1}{\xi}}. \tag{136}$$

So, based on relation (79), it follows that

$$I = \frac{10}{0.8+i0.6} = 10\cdot(0.8-i0.6) = 8-i6A \text{ or}$$

$$I = 10\angle -36.87^{\circ}A$$

Thus, the current (I) is 10 A, and its lag with respect to voltage (U) is 36.87°.

Also:

$$Z_{total} = R_{total} + iX_{total} = \frac{0.8 + i0.6}{\left(1 + \frac{1}{2} + \ldots + \frac{1}{\xi}\right)} = \frac{0.8 + i0.6}{\ln \xi}$$

$$= (0.8/\ln \xi) + i(0.6/\ln \xi)$$

where X_{total} and R_{total} are the circuit's total reactance and total resistance, respectively.

Therefore, $\frac{X_{total}}{R_{total}} = \frac{0.6/\ln \xi}{0.8/\ln \xi} = \frac{0.6}{0.8} = 0.75.$ □(137)

C2. *In the electric circuit of Figure* 2, *let us consider the more complex case, where the values of the cascaded parallel resistors* $r_a, r_b, r_c, r_d, \ldots$ *are not equal to each other, as is the case in* [30] *and* [38] *but have the following values, respectively*: $1\Omega, 3\Omega, 4\Omega, 6\Omega, \ldots (3n-2)\Omega, 3n\Omega, \ldots$ *Solve the circuit.*

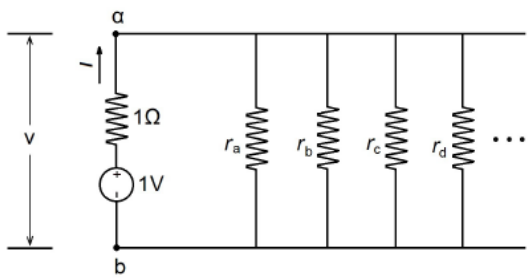

Figure 2. Infinite ladder electrical network.

Proof. The considered infinite resistive ladder network (Figure 2), which is also described in [30] but for $r_a = r_b = r_c = r_d = \ldots = r = 1\Omega$, is a typical case of an infinite network with local connectedness (locally finite connected) and finite power dissipation, and according to the "existence and uniqueness theorem", it must have only one (specific) solution in terms of voltages and currents, and, of course, Kirchhoff's two laws must apply [14, 15, 21].

In terms of infinite numbers, the last two infinite parallel branches have resistors equal to $(3\xi - 2)\Omega$ and $3\xi\Omega$. The total resistance, r_{total}, of the infinite number of parallel pure resistive branches, when also taking into account eqs. (71) and (72), is given by the following formulas:

$$\frac{1}{r_{total}} = \frac{1}{1} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{3\xi - 2} + \frac{1}{3\xi} = \frac{1}{3}\ln(3\xi^2) + C \tag{138}$$

$$\Rightarrow r_{total} = \frac{1}{\frac{1}{3}\ln 3\xi^2 + C} \quad \text{where } C \in \mathbb{R} \tag{139}$$

$$\Rightarrow r_{total} = 0$$

Therefore, there is a short circuit between nodes α and b $(v = 0)$. Certainly, the resistance r_{total}, can be calculated without the use of infinite numbers, and the result, again, is $r_{total} = 0$, given that the respective series diverges.

Hence, current I (through the voltage source) should be equal to 1 A, whereas the current through each of the purely resistive branches $(r_j \neq 0)$ is $v/(r_j\Omega) = 0/r_j = 0$. At the same time, an infinite series of zeros sums to zero. Thus, we conclude that 1 A flows toward node α, whereas 0 A flows away from it (a violation of Kirchhoff's current law). So, by using standard calculus, we must conclude that in this network as well, Kirchhoff's current law fails at node a and at node b as well, as also happens in [30]. Therefore, we reach the same unexpected conclusion as A. H. Zemanian first pointed out for these particular infinite electrical networks [30].

On the other hand, when using the proposed infinite numbers, the following is true:

Given that $I = 1V/(1\Omega + r_{total}) = 1$ A and $v = I \cdot r_{total} = r_{total}$ and, therefore (when also taking into account eq. (139)), the current of the jth parallel branch is

$$I_j = \frac{v}{r_j} = \frac{1}{\left(\frac{1}{3}\ln 3\xi^2 + C\right)} \cdot \frac{1}{r_j}$$

where $r_j = 1\Omega,\ 3\Omega,\ 4\Omega,\ 6\Omega,\ \ldots,\ 3n - 2\Omega,\ 3n\Omega,\ \ldots$.

Thus, the sum of the currents of all parallel branches I_{sum} is given by the following formula:

$$I_{sum} = \frac{1}{1 \cdot \left(\frac{\ln 3\xi^2}{3} + C\right)} + \frac{1}{3 \cdot \left(\frac{\ln 3\xi^2}{3} + C\right)} + \ldots + \frac{1}{(3n-2) \cdot \left(\frac{\ln 3\xi^2}{3} + C\right)}$$

$$+ \frac{1}{3n \cdot \left(\frac{\ln 3\xi^2}{3} + C\right)} + \ldots$$

$$= \frac{1}{\frac{1}{3}\ln 3\xi^2 + C} \cdot \left(\frac{1}{1} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{3\xi - 2} + \frac{1}{3\xi}\right)$$

which is equal to 1A, according to eq. (138). □

Consequently, also in this case, by using infinite numbers, the circuit is easily solved, and the incoming current I at node α is equal to the outgoing current I_{sum} (1A), so there is no violation of Kirchhoff's current law, as would happen if standard calculus was used.

This method is much simpler, shorter and cleaner than the one proposed by A. H. Zemanian and the result is exactly the same.

C3. *What is the power consumption across the resistance* $(r = 1)$ *of the branch* (BC) *of the electrical network of Figure* 3 *when n tends to infinity and given that all resistances are also equal to* 1?

Proof. According to A. H. Zemanian, H. Flanders, C. Thomassen, etc. [14, 22], infinite electrical networks can be much more complex than those of the previous examples. They may extend infinitely in both two or three dimensions of space. They may also not have only an infinite number of resistances but also an infinite number of sources, etc. Such a complex network, with double infinity sources (along the horizontal and vertical axis) and double infinity resistances (along the horizontal and vertical axis), is illustrated in Figure 3, where n tends to infinity.

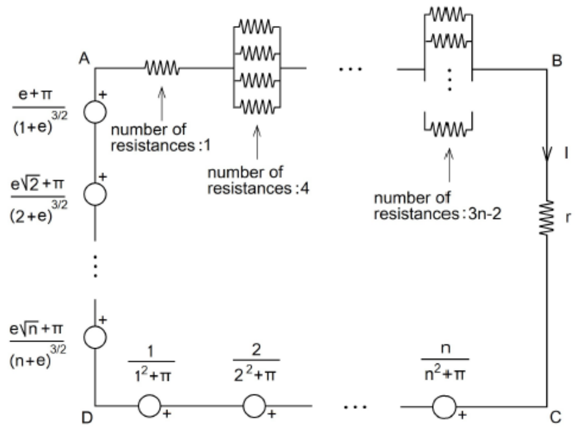

Figure 3. Infinite electrical network.

Considering that n tends to infinity (ξ), the resistance $R(\xi)$ of the horizontal branch (AB) as well as the total resistance $R_{tot}(\xi)$ of the loop are

$$R(\xi) = \frac{1}{1} + \frac{1}{4} + \frac{1}{7} + \ldots + \frac{1}{3\xi - 2} \qquad R_{tot}(\xi) = R(\xi) + 1$$

Moreover, the total loop voltage, as the current (I) points (clockwise), is $V_{tot}(\xi)$

$$V_{tot}(\xi) = \left(\frac{e + \pi}{(1 + e)^{3/2}} + \frac{e\sqrt{2} + \pi}{(2 + e)^{3/2}} + \frac{e\sqrt{3} + \pi}{(3 + e)^{3/2}} + \ldots + \frac{e\sqrt{\xi} + \pi}{(\xi + e)^{3/2}}\right)$$

$$- \left(\frac{1}{1^2 + \pi^2} + \frac{2}{2^2 + \pi^2} + \frac{3}{3^2 + \pi^2} + \ldots + \frac{\xi}{\xi^2 + \pi^2}\right)$$

Therefore, current I is

$$I = \frac{V_{tot}(\xi)}{R_{tot}(\xi)} = \frac{\left(\frac{e + \pi}{(1 + e)^{3/2}} + \frac{e\sqrt{2} + \pi}{(2 + e)^{3/2}} + \frac{e\sqrt{3} + \pi}{(3 + e)^{3/2}} + \ldots + \frac{e\sqrt{\xi} + \pi}{(\xi + e)^{3/2}}\right)}{\left(\frac{1}{1} + \frac{1}{4} + \frac{1}{7} + \ldots + \frac{1}{3\xi - 2}\right) + 1}$$

$$- \frac{\left(\frac{1}{1^2 + \pi^2} + \frac{2}{2^2 + \pi^2} + \frac{3}{3^2 + \pi^2} + \ldots + \frac{\xi}{\xi^2 + \pi^2}\right)}{\left(\frac{1}{1} + \frac{1}{4} + \frac{1}{7} + \ldots + \frac{1}{3\xi - 2}\right) + 1}. \tag{140}$$

In relation (140), we apply Theorem 6 (L'Hopital) for each of the two fractions of this relation, and we also use Theorem 2, taking into account the fact that the relevant criterion is valid for both ordinary series. So, we obtain

$$I = \frac{\frac{e\sqrt{\xi} + \pi}{(\xi + e)^{3/2}}}{\frac{1}{3\xi - 2} + 0} - \frac{\frac{\xi}{\xi^2 + \pi^2}}{\frac{1}{3\xi - 2} + 0} = \frac{\frac{e\xi^{1/2}}{\xi^{3/2}}}{\frac{1}{3\xi}} - \frac{\frac{1}{\xi}}{\frac{1}{3\xi}} = 3e - 3 = 3(e - 1)$$

Therefore, the power consumption (P) across the resistance $(r = 1)$ of the branch (BC) is

$$P = I^2 \cdot r = 9(e - 1)^2. \qquad \square$$

Solving this problem without using the infinite numbers and functions is certainly a very difficult task.

D. *Application in kinematics*

D1. *In an orthonormal* xOy *axis system* (*Figure* 4), *two material points*, *B and C*, *move on the positive half-axes* Ox *and* Oy, *respectively*. *At time* $t = 0$, *both material points are at the beginning* (*O*) *of the axes*. *At time* $t = 3$, *the material point B on the Ox half-axis is at a distance equal to ln3 from the beginning of the axis* (*position* B_3); *at time* $t = 5$, *it is at a distance equal to ln5 from the beginning of the axis* (*position* B_5), *and at time* $t = (2n - 1)$, *it is at a distance equal to* $\ln(2n - 1)$ *from the beginning of the axis* (*position* B_{2n-1}). *On the other hand*, *at time* $t = 3$, *the material point C on the* Oy *half-axis has traveled a distance of* $OC_3 = 1/3$, *at time* $t = 5$, *it has traveled an additional distance of* $C_3C_5 = 1/5$, *and at time* $t = (2n - 1)$, *it has traveled an additional distance of* $C_{2n-3}C_{2n-1} = 1/(2n - 1)$. *At each time instant*, *t*, *a right-angled triangle* $OB_3C_3, OB_5C_5, \ldots, OB_{2n-1}C_{2n-1}$ *is formed, the angle of which*, $\angle OB_{2n-1}C_{2n-1}$, *is called* φ_{2n-1} (*see Figure* 4). *What is the value of the angle* φ_{2n-1} *when t tends to infinity*? *What is the ratio* γ_B/γ_C *where* γ_B *and* γ_C *are the decelerations of the material points B and C*, *respectively*, *when t tends to infinity*? *What is the ratio of the vertical side* (OB_{2n-1}) *to the hypotenuse* $(B_{2n-1}C_{2n-1})$ *of the right-angled triangle* $OB_{2n-1}C_{2n-1}$ *when t tends to infinity*?

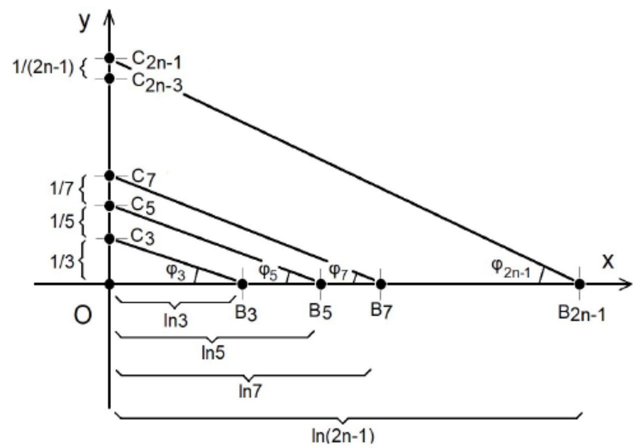

Figure 4. Ortho-normal xOy axis system.

Proof. In the right-angled triangle OB_3C_3 of Figure 4, it holds that $\tan\varphi_3 = (OC_3)/(OB_3) = (1/3)/\ln 3 = 0.30341$, and thus, $\varphi_3 = 16.87°$. Similarly, it can be found that $\varphi_5 = 18.33°$ and $\varphi_7 = 19.16°$.

If we consider the following ordinary series, $A(\xi)$, written as an infinite number function, its derivative is equal to $1/(2\xi - 1)$, and, hence, this series is recalculated from the integral of eq. (141).

Therefore, relation (142) applies.

$$A(\xi) = \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{2\xi - 1}$$

$$A(\xi) = \int \frac{d}{d\xi} A(\xi) = \int \frac{1}{2\xi - 1}\, d\xi = \frac{1}{2}\ln(2\xi - 1) + C \tag{141}$$

where $C \in \mathbb{R}$.

Thus,

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{2\xi - 1} = \frac{1}{2}\ln(2\xi - 1) + C \tag{142}$$

which is infinity.

At time $t = 2n - 1$, the total distance S_B that is traveled by material point B of Figure 4 is equal to $\ln(2n-1)$, and when n tends to infinity (ξ), the total distance $S_B(\xi)$ is expressed by eq. (143).

$$S_B(\xi) = \ln(2\xi - 1) \tag{143}$$

On the other hand, at time $t = 2n-1$, the total distance, S_C, that is traveled by material point C, is given by formula (144), and when n tends to infinity (ξ), the total distance, $S_C(\xi)$, is expressed by eq. (145).

$$S_C = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{2n-1} \tag{144}$$

$$S_C(\xi) = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{2\xi-1} \tag{145}$$

Therefore, relation (142) can be written as (146) by also taking into account eqs. (143)-(145).

$$1 + S_C(\xi) = \frac{1}{2}\ln(2\xi - 1) + C = \frac{1}{2} S_B(\xi) + C \qquad \text{or}$$

$$1 + S_C(\xi) = \frac{1}{2} S_B(\xi) + C \tag{146}$$

$$\Rightarrow \frac{S_C(\xi)}{S_B(\xi)} = \frac{1}{2}. \tag{147}$$

When $n \to \infty$, the two vertical sides, (OB_{2n-1}) and (OC_{2n-1}), of the infinite right-angled triangle $OB_{2n-1}C_{2n-1}$, are the above distances, $S_B(\xi)$ and $S_C(\xi)$, respectively. Therefore, the ratio of eq. (147) is the tangent of angle φ_{2n-1} when t tends to infinity (ξ), and thus, $\tan\varphi_\xi = 0.5$, meaning that $\varphi_\xi = 26.15°$. In conclusion, as shown above, $\varphi_3 = 16.87°$, $\varphi_5 = 18.33°$, and $\varphi_7 = 19.16°$, and when $n \to \infty$, the angle φ receives its highest value: $\varphi_\xi = 26.15°$.

According to the general definition of linear velocity from kinematics, the velocity, υ_B, of the moving material point B at time $t = 2n-1$ is calculated using eq. (148), and therefore, when n tends to infinity (ξ), velocity, υ_B, is given by (149), which, of course, is zero. However, with infinitely small infinite numbers, one can work further, and the derivative of v_B gives the acceleration /deceleration (γ_B) of material point B (150).

$$v_{B(2n-1)} = \frac{\Delta S_B}{\Delta t_B} = \frac{\ln(2n-1) - \ln(2n-3)}{(2n-1) - (2n-3)} = \frac{1}{2}\ln\left(\frac{2n-1}{2n-3}\right) \tag{148}$$

$$v_{B(2\xi-1)} = \frac{1}{2}\ln\left(\frac{2\xi-1}{2\xi-3}\right) \tag{149}$$

$$\gamma_{B(2\xi-1)} = \left[\frac{1}{2}\ln\left(\frac{2\xi-1}{2\xi-3}\right)\right]' = \frac{1}{2}\cdot\frac{2\xi-3}{2\xi-1}\cdot\frac{2(2\xi-3)-2(2\xi-1)}{(2\xi-3)^2}$$

$$= \frac{1}{2\xi-1}\cdot\frac{(-2)}{(2\xi-3)} = -\frac{2}{(2\xi-1)(2\xi-3)}. \tag{150}$$

Similarly, the velocity, v_C, of the other material point, C, at time $t = 2n-1$ when n tends to infinity is also calculated, and it is given by eq. (151). Moreover, the deceleration (γ_C) of material point B is calculated (152). By combining (150) and (152), we find that eq. (153) gives the ratio γ_B/γ_C, which is equal to the number 2. Therefore, the deceleration of material point B is twice as high as the deceleration of material point C, although both of them tend to zero (they are zero infinite numbers).

$$v_{C(2n-1)} = \frac{\Delta S_C}{\Delta t_C} = \frac{\frac{1}{2n-1}}{(2n-1)-(2n-3)} = \frac{1}{2}\cdot\frac{1}{2n-1}$$

$$v_{C(2\xi-1)} = \frac{1}{2}\cdot\frac{1}{2\xi-1} \tag{151}$$

$$\gamma_{C(2\xi-1)} = \frac{1}{2}\cdot\frac{(-1)}{(2\xi-1)^2}\cdot(2) = -\frac{1}{(2\xi-1)^2} \tag{152}$$

$$\frac{\gamma_{B(2\xi-1)}}{\gamma_{C(2\xi-1)}} = \frac{-\frac{2}{(2\xi-1)(2\xi-3)}}{-\frac{1}{(2\xi-1)^2}} = 2\cdot\frac{(2\xi-1)}{(2\xi-3)} = 2 \tag{153}$$

Based on the above, relation (147) can be written as (154). Moreover, by applying the Pythagorean theorem to the right-angled triangle $OB_{2\xi-1}C_{2\xi-1}$, the relation (155) holds. Finally, when using (154), (155) is transformed into (156), and thus, the ratio of the vertical side $(OB_{2\xi-1})$ to the hypotenuse $(B_{2\xi-1}C_{2\xi-1})$ is calculated.

$$\frac{(OC_{2\xi-1})}{(OB_{2\xi-1})} = \frac{1}{2} \tag{154}$$

$$(B_{2\xi-1}C_{2\xi-1})^2 = (OB_{2\xi-1})^2 + (OC_{2\xi-1})^2 \tag{155}$$

$$\frac{(B_{2\xi-1}C_{2\xi-1})^2}{(OB_{2\xi-1})^2} = \frac{(OB_{2\xi-1})^2}{(OB_{2\xi-1})^2} + \frac{(OC_{2\xi-1})^2}{(OB_{2\xi-1})^2} = 1 + \frac{1}{4} = \frac{5}{4}$$

$$\Rightarrow \frac{(OB_{2\xi-1})}{(B_{2\xi-1}C_{2\xi-1})} = \sqrt{\frac{4}{5}} = \frac{2\sqrt{5}}{5}. \tag{156}$$

This complex kinematics problem shows the usefulness of infinite numbers in another very different field as well. □

VIII. Conclusions

In this study, the new infinite numbers and functions were introduced, strictly formulated and proved. In particular, the fundamental definitions, lemmas, theorems, properties, and illustrative mathematical and engineering applications were presented and proved. By applying the well-founded theory of the limits of functions, infinite numbers and functions were defined and developed, which retain the important properties of real-complex numbers (arithmetic operations, powers, roots etc.). This is a very important characteristic of the proposed new numbers. These new infinite numbers can quantify infinity and they allow arithmetic operations and calculus in mathematical expressions where infinity occurs. The set of infinite numbers is a superset of the complex numbers set. They offer the possibility to extend the Laplace transform F(s) (which is widely used in the sciences) in cases where F(s) does not converge in the field of complex numbers. The extended (in the infinite numbers) bilateral Laplace transform, also proposed here, made it possible to solve specific differential equations defined piecewise over the entire domain of real numbers $(-\infty, +\infty)$. Their solutions (verified to be true), in general belong to the set of infinite number functions. However, they also include the solutions belonging to the well-known real-complex functions set. Solving these problems is not possible using the normal Laplace transform, since it is only defined for positive real values. By using the infinite numbers, interestingly, long series of infinite terms were beautifully transformed into simple, elegant infinite functions whose computation is an easy task. In this way a simple, efficient criterion for series convergence was also developed. Moreover, lemmas and theorems about the infinite functions and their derivatives/integrals were proved, and complicated, unusual limits of series of numbers, as well as ratios of the form ∞/∞ (involving series and improper integrals), were calculated in cases where L'Hopital's rule cannot be applied. In this way, solving problems that are complex or difficult becomes possible in an easy way. Additional theorems on these infinite numbers and functions were proved, and a simple-to-apply numerical method was developed for the easier and more accurate calculation of a series of numbers where the sum is not known analytically. Furthermore, it was shown that these new infinite numbers constitute an algebraic structure that is non-Archimedean, and, moreover, in contrast to Hardy fields the set of infinite numbers is not an ordered field. In general, these infinite numbers and functions with their properties (similar to those of real-complex numbers and functions) are a useful tool for solving problems where infinity occurs. As demonstrated in the indicative examples, the presented infinite numbers, functions, and their properties have important applications in terms of analyzing and solving technological and engineering systems i.e., complex infinite electrical networks and specific kinematic problems in which infinity appears. Finally, it must be noted that when the approach proposed here was applied to model and solve the above infinite electrical network examples (first published and solved by A. H. Zemanian) [30], it gave the same final results much more easily. In conclusion, as has been shown, infinite numbers easily solve problems that are quite difficult or impossible to solve by conventional methods (e.g. problems described by eqs. (100), (102), (110), (113), (118), (119), (120), (123), ((124)-(126)), also infinite electrical networks of Figures 1-3 as well as the problem of Figure 4).

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