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THE p_i-CAYLEY GRAPH AND THE PRIME POWER CAYLEY GRAPH FOR THE CYCLIC GROUP OF ORDER pqr AND THEIR PROPERTIES

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Abstract

A Cayley graph associated to a group is a graph that is formed depending on the specific subset of the group. Two variations od Cayley graph, namely p_i-Cayley graph and prime power Cayley graph have been defined based on the order of the group. The order of a group can be written as a product of prime powers. The p_i-Cayley graph is constructed using the subset that contains elements of the group with prime power order for each prime, while the prime power Cayley graph is constructed using subset that contain elements with prime power order for all primes. In this paper, these two variations of the Cayley graph are constructed for the cyclic group of order pqr, where p, q, and r are primes with $p < q < r$. Some properties of the graphs, which are the diameter, chromatic number, and planarity, are determined.

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Introduction

Cayley graphs were introduced by Arthur Cayley in 1878 [1]. The Cayley graph of a group G consists of vertices corresponding to the elements of the group, with two distinct vertices, x and y, being adjacent if $xy - 1 \in S$, where S is a subset of the group. In 2010, Khosravi and Mahmoudi [2] investigated the characterization of Cayley graphs on rectangular groups. In 2018, Huang et al. [3] explored perfect codes in Cayley graphs. In 2021, Saba [4] investigated on the structure of Cayley graphs of dihedral groups for different valencies.

There are several variations of Cayley graphs. Tolue introduced two variations of Cayley graphs, namely the prime order Cayley graph [5] and the composite order Cayley graph [6] in 2015 and 2019, respectively. In 2020, Farrokhi [7] introduced the relative Cayley graph. Zulkarnain et al. introduced the p_i-Cayley graph [8] and the prime power Cayley graph [9] in 2020 and 2022, respectively. In 2023, Aikawa et al. [10] introduced left and right Cayley graphs. The differences between these variations of Cayley graphs are in the subsets.

In this paper, the p_i-Cayley graph and prime power Cayley graph are constructed for a cyclic group of order pqr where p, q, and r are primes and $p < q < r.$ The construction of the p_i-Cayley graph and prime power Cayley graph involves classifying the subsets based on the orders of the elements, partitioning the vertices into several sets, and lastly, finding the adjacency and non-adjacency between the vertices. Meanwhile, for the prime power Cayley graph, the graphs are constructed by using the subgraphs of the p_i-Cayley graph.

This paper is structured into four main sections. The first section contains an introduction to the study that involves some background and literature review on variations of Cayley graphs. The second section contains some basic background on groups and graphs. The third section discusses the main results of this study. Finally, the fourth section gives an overall summary of this study.

Preliminaries

In this section, some definitions and propositions that are used in this study are presented. The p_i-Cayley graph associated to a group is defined as follows:

Definition 1 [8]. (p_i-Cayley Graph Associated to a Group) Let G be a group with $|G| = p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot P_n^{k_n}$, where p_i are primes, and $k_i \in N$ for $i = 1, 2, \ldots, n$. Define $S^{(p_i)} = \{x \in G| \, x| = p_i^{r_i}, 1 \leq r_i \leq k_i\} \subseteq G$ with $S^{(p_i)} = S^{(p_i)} := \{s^{-1} \mid s \in S^{(p_i)}\}$. The p_i-Cayley graph associated to G with respect to $S^{(p_i)}$, denoted as p_i-Cay$(G, S^{(p_i)})$, is a graph in which the vertices are the elements of G, and two distinct vertices, g and h, are adjacent if $gh^{-1} \in S^{(p_i)}$ for all $g, h \in G$.

The prime power Cayley graph associated to a group is defined as follows:

Definition 2 [9]. (Prime Power Cayley Graph Associated to a Group) Let G be a group with $|G| = p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot P_n^{k_n}$, where p_i are primes, and $k_i \in N$ for $i = 1, 2, \ldots, n$. Define $S = \{x \in G| \, x| = p_i^{r_i}, 1 \leq r_i \leq k_i, 1 \leq i \leq n\} \subseteq G$ with $S = S^{-1} := \{s^{-1} \mid s \in S\}$. The prime power Cayley graph associated to G with respect to S, denoted asg $Cay(G, S)$, is a graph in which the vertices are elements of G, and two distinct vertices, g and h, are adjacent if $gh^{-1} \in S$ for all $g, h \in G$.

Next, the definition of a subgraph is given in the following.

Definition 3 [11]. (Subgraph) A graph Γ' is called a subgraph of a graph Γ if $V(\Gamma') \subseteq V(\Gamma)$ and $E(\Gamma') \subseteq E(\Gamma)$.

The diameter of a complete graph is given as follows.

Proposition 1 [12] (Diameter of Complete Graph). *The diameter of a complete graph is* 1, *or in symbols*, $diam(K_n) = 1$, *where* K_n *denotes the complete graph with n vertices*.

Next, the definition of chromatic number of a graph is given.

Definition 4 [13] (Chromatic Number)**.** The proper coloring of a graph Γ is the coloring of the vertices and edges with minimal number of colors such that no two vertices should have the same color. The minimum number of colours is called as the chromatic number, $\chi(\Gamma)$ and the graph is called properly coloured graph.

Lastly, the theorem on planarity, which is called Kuratowski's theorem, is given as follows.

Theorem 1 [11] (Kuratowski's Theorem)**.** *A graph is planar if and only if it contains no subdivision of either* K_5 *or* $K_{3,3}$.

Results and Discussion

This section is divided into three parts. First, the construction of the p_i-Cayley graph of the cyclic group of order pqr is explained. Second, the construction of the prime power Cayley graph of the cyclic group of order pqr is demonstrated.

Lastly, some properties of the obtained graphs are determined. The construction of the p_i-Cayley graph of the cyclic group of order pqr is shown below.

The p_i-Cayley Graph of the Cyclic Group of Order pqr

In this subsection, the construction of the p_i-Cayley graph is presented, as explained in Lemma 1 through Lemma 7, and Theorem 8.

Let C_{pqr} be the cyclic group of order pqr, that is $|C_{pqr}| = pqr$, where $p < q < r$. Since the order of the group is a product of three distinct primes, p, q, and r, three different subsets can be formed as follows:

1. The subset $S^{(p)}$ contains elements with order p, which can be expressed as:

$$S^{(p)} = \{x \in C_{pqr} \mid |x| = p\} = \{x^{qr}, x^{2qr}, \ldots, x^{(p-1)qr}\}.$$

2. The subset $S^{(q)}$ contains elements with order q, and it can be represented as:

$$S^{(q)} = \{x \in C_{pqr} \mid | x | = q\} = \{x^{pr}, x^{2pr}, \ldots, x^{(q-1)pr}\}.$$

3. The subset $S^{(r)}$ contains elements with order r, and it can be represented as:

$$S^{(r)} = \{x \in C_{pqr} \mid | x | = r\} = \{x^{pq}, x^{2pq}, \ldots, x^{(r-1)pq}\}.$$

Based on these subsets, three p_i-Cayley graphs can be constructed for this group; the p-Cayley graph with respect to $S^{(p)}$, the q-Cayley graph with respect to $S^{(q)}$, and the r-Cayley graph with respect to $S^{(r)}$. The construction of the p-Cayley graph with respect to $S^{(p)}$ is presented in the following lemmas and theorem.

First, the elements of this group are divided into different sets, namely Set A_i and Set B, as stated in Lemma 1.

Lemma 1. *Let C_{pqr} be a cyclic group of order pqr generated by x. Let $S^{(p)} = \{x \in C_{pqr} \mid | x | = p\} = \{x^{qr}, x^{2qr}, \ldots, x^{(p-1)qr}\}$. Define $A_i = \{x^{lqr+i} \mid 1 \leq l \leq p-1\}$ for $i = 1, 2, \ldots, qr$ and $B = \{x^j \mid 1 \leq j \leq qr\}$ $= \{x, x^2, \ldots, x^q r\}$. Then, $B \cup A_1 \cup A_2 \cup \ldots \cup A_{qr} = G$.*

Proof. Let $C_{pqr} = \langle x \rangle = \{e, x, x^2, \ldots, x^{pqr-1}\}$. Consider $B \cup A_1 \cup A_2 \cup \ldots \cup A_{qr}$ in C_{pqr}. For $g \in C_{pqr}$, where $g = x^k$ and $1 \leq k \leq pqr$, there are two cases to examine:

1. If $1 \leq k \leq qr$, then $x \in B$.

2. If $k = lq + i$ for $1 \leq l \leq p-1$, then $x \in A_i$, where $1 \leq i \leq qr$.

Therefore, $B \cup A_1 \cup A_2 \cup \ldots \cup A_{qr}$ in C_{pqr}. ∎

Next, the relations between every two distinct vertices are observed by referring to the condition given in the definition of p_i-Cayley graph, which states that distinct vertices g and h in C_{pqr} are adjacent if $gh^{-1} \in S^{(p)}$. The adjacency between all vertices are discussed from Lemma 2 until Lemma 7. In the following lemma, the adjacency of p^{-1} vertices in each A_i, where $i = 1, 2, \ldots, qr$, are observed.

Lemma 2. *Let* C_{pqr} *be a cyclic group of order* pqr *generated by* x. *Let* $S^{(p)} = \{x \in C_{pqr} : |x| = p\} = \{x^{qr}, x^{2qr}, \ldots, x^{(p-1)qr}\}$. *Define* $A_i = \{x^{lqr+i} \mid 1 \leq l \leq p-1\}$ *for* $i = 1, 2, \ldots, qr$. *Then, all vertices in each* A_i *are adjacent with each other for* $1 \leq i \leq qr$.

Proof. Let $C_{pqr} = \langle x \rangle = \{e, x, x^2, \ldots, x^{pqr-1}\}$. Let g and h be elements in A_i where $g = x^{lqr+i}$ and $h = x^{l'qr+i}$ for $1 \leq l \neq l' \leq p-1$. Then, gh^{-1} is in $S^{(p)}$ since $gh^{-1} = x^{(l-l')qr} \in S^{(p)}$. Therefore, all elements in each A_i for $1 \leq i \leq qr$ are adjacent to each other. ■

Following from this lemma, there are qr-components of the complete graphs where in each complete graph contains $p-1$ vertices.

In the previous lemma, the vertices in each A_i turned out to be adjacent with each other. Next, the relations between the vertices in two distinct sets, labelled as A_i and A_j for $i \neq j = 1, 2, \ldots, qr$ are observed in the following lemma.

Lemma 3. *Let* C_{pqr} *be a cyclic group of order* pqr *generated by* x. *Let* $S^{(p)} = \{x \in C_{pqr} : |x| = p\} = \{x^{qr}, x^{2qr}, \ldots, x^{(p-1)qr}\}$. *Define* $A_i = \{x^{lqr+i} \mid 1 \leq l \leq p-1\}$ *and* $A_j = \{x^{lq+j} \mid 1 \leq l \leq p-1\}$ *for* $i, j = 1, 2, \ldots, qr$. *Then, the vertices in* A_i *and* A_j *for* $i \neq j$ *are not adjacent.*

Proof. Let $C_{pqr} = \langle x \rangle = \{e, x, x^2, \ldots, x^{pqr-1}\}$. Let $A_i = \{x^{lqr+i} \mid 1 \leq l \leq p-1\}$ and $A_j = \{x^{lq+j} \mid 1 \leq l \leq p-1\}$ for $i \neq j$. The product of x^{lqr+i} and the inverse of x^{lqr+j} is x^{i-j}, and this element is not in $S^{(p)}$. Therefore, x^{lqr+i} and x^{lqr+j} are not adjacent for $1 \leq l \leq p-1$, which implies that the vertices in different sets, A_i and A_j, are not adjacent for $i \neq j$. ■

Lemma 2 and Lemma 3 showed that even though the vertices in each A_i are adjacent with each other, but there is no adjacency between the vertices in different sets of A_i. Hence, each set A_i formed a component of the graph. Therefore, there are qr -components of graph with $p-1$ vertices are formed since there are qr sets of A_i. In the following lemma, the adjacency between two distinct vertices in Set B are studied.

Lemma 4. *Let C_{pqr} be a cyclic group of order pqr generated by x. Let $S^{(p)} = \{x \in C_{pqr} : |x| = p\} = \{x^{qr}, x^{2qr}, \ldots, x^{(p-1)qr}\}$. Define $B = \{x^j \mid 1 \leq j \leq qr\} = \{x, x^2, \ldots, x^q r\}$. Then, there is no adjacent vertices in set B.*

Proof. Let $C_{pqr} = \langle x \rangle = \{e, x, x^2, \ldots, x^{pqr-1}\}$. Let $x_i \in B$ and $x_j \in B$ for $1 \leq i \neq j \leq qr$ and $i > j$. Then, $(x^i)(x^j)^{-1} = x^{1-j} \notin S^{(p)}$. Therefore, there is no adjacent vertices in B. ■

This lemma showed that all vertices in set B are not adjacent to each other. Next, the adjacency between the vertices in set A_i with set B are studied as explained in Lemma 3 until Lemma 5.

Lemma 5. *Let C_{pqr} be a cyclic group of order pqr generated by x and $S^{(p)} = \{x^{qr}, x^{2qr}, \ldots, x^{(p-1)qr}\}$. Define the sets:*

$$A_i = \{x^{lqr+i} \mid 1 \leq l \leq p-1\} \text{ for } i = 1, 2, \ldots, qr,$$

$$B = \{x^j \mid 1 \leq j \leq qr\} = \{x, x^2, \ldots, x^q r\}.$$

Then, each vertex x^j in set B is adjacent to every vertex in set A_i for $1 \leq i = j \leq qr$.

Proof. Let $C_{pqr} = \langle x \rangle = \{e, x, x^2, \ldots, x^{pqr-1}\}$. For $1 \leq j \leq qr$, define $A_i = \{x^{lqr+i} \mid 1 \leq l \leq p-1\}$ and $x^j \in B$. Consider $(x^{lq+i})(x^j)^{-1}$, which is an element in $S^{(p)}$ since $(x^{lq+i})(x^j)^{-1} = x^{lq} \in S^{(p)}$. Therefore, x^j in B is adjacent to all vertices in A_i for $1 \leq i = j \leq qr$. ∎

Based on Lemma 5, the adjacency between the vertices in set A_i and B is demonstrated. Each vertex in A_i is adjacent to only one vertex in set B such that $x^j \in B$ being adjacent to all elements in A_j for $1 \leq j \leq q$. The next lemma focuses on the relationships between $x^j \in B$ and the vertices in set A_i for $1 \leq i \neq j \leq q$.

Lemma 6. *Let C_{pqr} be a cyclic group of order pqr generated by x. Let $S^{(p)} = \{x \in C_{pqr} : |x| = p\} = \{x^{qr}, x^{2qr}, \ldots, x^{(p-1)qr}\}$. Define $A_i = \{x^{lqr+i} \mid 1 \leq l \leq p-1\}$ for $i = 1, 2, \ldots, qr$ and $B = \{x^j \mid 1 \leq j \leq qr\} = \{x, x^2, \ldots, x^q r\}$. Then, for $i \neq j$, $x^{lqr+i} \in A_i$ and $x^j \in B$ are not adjacent.*

Proof. Let $C_{pqr} = \langle x \rangle = \{e, x, x^2, \ldots, x^{pqr-1}\}$. Now, define sets $A_i = \{x^{lqr+i} \mid 1 \leq l \leq p-1\}$ and let x_j belong to set B for $i \neq j$. Consider the product of x^{lqr+i} and the inverse of x_j, which is equal to $x^{lqr+i-j}$. The element $x^{lqr+i-j}$ does not belong to set $S^{(p)}$. Therefore, x^{lqr+i} and x_j from sets A_i and B, respectively, are not adjacent for $i \neq j$. ∎

From the previous lemma, there is no adjacency between $x^{lqr+i} \in A_i$ and $x_j \in B$. Therefore, all vertices in A_i and B form a complete graph, as explained in the next lemma.

Lemma 7. *Let C_{pqr} be a cyclic group of order pqr generated by x. Let $S^{(p)} = \{x \in C_{pqr} : |x| = p\} = \{x^{qr}, x^{2qr}, \ldots, x^{(p-1)qr}\}$. Define $A_i = \{x^{lqr+i} \mid 1 \leq l \leq p-1\}$ for $i = 1, 2, \ldots, qr$. Then, $\{x^j\} \cup A_j$ is a complete graph, K_p.*

Proof. Let $C_{pqr} = \langle x \rangle = \{e, x, x^2, \ldots, x^{pqr-1}\}$. Based on Lemma 2, all vertices in set A_i are adjacent to each other. According to Lemma 3, $\{x^j\}$ is adjacent to the vertices in A_j. Thus, there are p vertices in $\{x^j\} \cup A_j$, and each vertex is adjacent to every other vertex. Therefore, $\{x^j\} \cup A_j$ forms a complete graph, K_p. ∎

This lemma demonstrates that a complete graph with p vertices is constructed from $x^{lqr+i} \in A_i$ and $x_j \in B$ for $i \neq j$.

Following from this lemma, the next theorem explains how q complete graphs are formed.

Theorem 2. *Let C_{pqr} be a cyclic group of order pqr, and let $S^{(p)} = \{x \in C_{pqr} : |x| = p\}$ with $S^{(p)} = S^{(p)-1}$. Then, the p-Cayley graph of C_{pqr} is the union of qr copies of complete graphs of order p, p-Cay $(C_{pqr}, S^{(p)}) = qrK_p$.*

Proof. Let $C_{pqr} = \langle x \rangle = \{e, x, x^2, \ldots, x^{pqr-1}\}$. Let $S^{(p)} = \{x \in C_{pqr} : |x| = p\} = \{x^{qr}, x^{2qr}, \ldots, x^{(p-1)qr}\}$. Define $A_i = \{x^{lqr+i} \mid 1 \leq l \leq p-1\}$ for $i = 1, 2, \ldots, qr$ and $B = \{x^j \mid 1 \leq j \leq qr\}$ $= \{x, x^2, \ldots, x^q r\}$.

All elements of Set A_i and Set B are in C_{pqr}, as proved in Lemma 1. Then, the vertices in each A_i are adjacent to each other, as proved in Lemma 2. Furthermore, there is no adjacency between the vertices in two distinct sets of A. Each set in A is an independent set because the vertices of set A_i are not adjacent to the vertices in set A_j for $i \neq j$, as proved in Lemma 3.

Besides, the vertices in set B are not adjacent to each other as showed in Lemma 4. Then, the vertices in set B is adjacent with the vertices in a set A such that $\{x^j\} \in B$ is adjacent with $x^{lqr+i} \in A_i$ for $i = j$. Hence, the adjacency of these elements, $\{x^j\} \in B$ and $x^{lqr+i} \in A_i$ for $i = j$ formed a component of graph. This component contains p vertices since there are $p-1$ vertices from Set A_i and a vertex from Set B. Following from this, a graph with qr-components is formed where each component contains p-vertices. This is because there are q vertices in Set B which each vertex is connected with all vertices of exactly one set from Set A. Therefore, based on Definition 1, the p-Cayley graph of the cyclic group of order pqr for $p < q < r$ is the union of qr copies of complete graphs with p vertices, qrK_p. ■

Since the value of p is not equal to q and r, and both are primes, the q-Cayley graph and r-Cayley graph has the same graph pattern as the p-Cayley graph, presented in the following two theorems. The proof is similar to that of Theorem 2.

Theorem 3. *Let C_{pqr} be a cyclic group of order pqr, and let $S^{(p)} = \{x \in C_{pqr} : |x| = p\}$. Then, q-Cay $(C_{pqr}, S^{(p)}) = qrK_p$.*

Theorem 4. *Let C_{pqr} be a cyclic group of order pqr, and let $S^{(p)} = \{x \in C_{pqr} : |x| = p\}$. Then, r-Cay$(C_{pqr}, S^{(p)}) = qrK_p$.*

In the next subsection, the construction of the prime power Cayley graph for the cyclic group of order pqr is explained.

The Prime Power Cayley Graph of Cyclic Group of Order pqr

In this subsection, the construction of the prime power Cayley graph for a cyclic group of order pqr is presented in the following theorems. The prime power Cayley graph of C_{pqr} with respect to S is labeled as $Cay(C_{pqr}, S)$.

Theorem 5. *Let C_{pqr} be a cyclic group of order pqr, and let $S^{(p)} = \{x \in C_{pqr} : |x| = p, q \text{ or } r\}$. Then, $Cay(C_{pqr}, S)$ contains three subgraphs, qrK_p, prK_q and pqK_r.*

Proof. Let C_{pqr} be a cyclic group of order pqr, and let $S^{(p)} = \{x \in C_{pqr} : |x| = p, q \text{ or } r\}$. From Theorem 2, p-$Cay(C_{pqr}, S^{(p)})$ with respect to $S^{(p)} \subseteq S$ is qrK_p, from Theorem 3, q-$Cay(C_{pqr}, S^{(q)})$ with respect to $S^{(q)} \subseteq S$ is prK_q, from Theorem 4, r-$Cay(C_{pqr}, S^{(r)})$ with respect to $S^{(r)} \subseteq S$ is prK_r. By Definition 2, $V(gCay(C_{pqr}, S)) = C_{pqr}$ and two different vertices, g and h are adjacent if $gh^{-1} \in S$. Since all vertices in S are also vertices in $S^{(p)}, S^{(q)}$ and $S^{(r)}$ of $p\text{-}Cay(C_{pqr}, S^{(p)})$, $q\text{-}Cay(C_{pqr}, S^{(q)})$, and $r\text{-}Cay(C_{pqr}, S^{(r)})$, respectively, then the edges of $Cay(C_{pqr}, S)$ is the combination of edges in $p\text{-}Cay(C_{pqr}, S^{(p)})$, $q\text{-}Cay(C_{pqr}, S^{(q)})$, and $r\text{-}Cay(C_{pqr}, S^{(r)})$. Therefore, based on Definition 3, $Cay(C_{pqr}, S)$ contains three subgraphs, qrK_p, prK_q and pqK_r. ∎

In Theorem 6, the structure of $Cay(C_{pqr}, S)$ is explored.

Theorem 6. *Let C_{pqr} be a cyclic group of order pqr, and let $S^{(p)} = \{x \in C_{pqr} : |x| = p, q \text{ or } r\}$ such that $S = S^{-1}$. Then, $Cay(C_{pqr}, S)$ is a $(p + q + r - 3)$ -regular graph.*

Proof. Let C_{pqr} be a cyclic group of order pqr, and let $S^{(p)} = \{x \in C_{pqr} : |x| = p, q \text{ or } r\}$. Since all vertices in S are also vertices of $S^{(p)}, S^{(q)}$ and $S^{(r)}$ in $p\text{-}Cay(C_{pqr}, S^{(p)})$, $q\text{-}Cay(C_{pqr}, S^{(q)})$ and $r\text{-}Cay(C_{pqr}, S^{(r)})$, respectively, then the edges of $Cay(C_{pqr}, S)$ is the combination of edges in $p\text{-}Cay(C_{pqr}, S^{(p)})$, $q\text{-}Cay(C_{pqr}, S^{(q)})$ and $r\text{-}Cay(C_{pqr}, S^{(r)})$. From Theorem 2, $p\text{-}Cay(C_{pqr}, S) = qrK_p$. Hence, each vertex in the $p\text{-}Cay(C_{pqr}, S)$ has degree $p - 1$. Additionally, according to Theorem 3, $q\text{-}Cay(C_{pqr}, S^{(q)}) = prK_q$. Hence, each vertex in the $q\text{-}Cay(C_{pqr}, S^{(q)})$ has degree $q - 1$. Then, according to Theorem 4, $r\text{-}Cay(C_{pqr}, S^{(r)}) = pqK_r$. Hence, each vertex in the $r\text{-}Cay(C_{pqr}, S^{(r)})$ has degree $r - 1$. Therefore, each vertex in $Cay(C_{pqr}, S)$ has degree $(p - 1) + (q - 1) + (r - 1) = p + q + r - 3$. Consequently, $Cay(C_{pqr}, S)$ forms a $(p + q + r - 3)$ -regular graph. ∎

In the next subsection, some properties of the graphs obtained are determined.

Some Properties of the p_i-Cayley Graph and the Prime Power Cayley Graph for the Cyclic Group of Order pqr

In this section, some properties of the p_i-Cayley graph of a cyclic group of order pqr are studied. The properties covered in this section are the diameter, chromatic number, and planarity of the graph.

The $p\text{-}Cay(C_{pqr}, S^{(p)})$ is explored in Proposition 2 until Proposition 4.

Proposition 2. *Let C_{pqr} be a cyclic group of order pqr, and let $S^{(p)} = \{x \in C_{pqr} : |x| = p\}$ with $S^{(p)} = S^{(p)-1}$. Then, $diam(p\text{–}Cay(C_{pqr}, S^{(p)})) = \infty$.*

Proof. From Theorem 2, $p\text{-}Cay(C_{pqr}, S^{(p)}) = qrK_p$. By Proposition 1, $diam(p\text{–}Cay(C_{pqr}, S^{(p)})) = \infty$. ■

Next, the computation for the chromatic number of $p\text{-}Cay(C_{pqr}, S^{(p)})$ is provided in the following.

Proposition 3. *Let C_{pqr} be a cyclic group of order pqr, and let $S^{(p)} = \{x \in C_{pqr} : |x| = p\}$ with $S^{(p)} = S^{(p)-1}$. Then, $\chi(p\text{–}Cay(C_{pqr}, S^{(p)})) = p$.*

Proof. From Theorem 2, $p\text{-}Cay(C_{pqr}, S^{(p)}) = qrK_p$. Consider any two distinct vertices in $p\text{-}Cay(C_{pqr}, S^{(p)})$, denoted as x_i and x_j for $0 \le i \ne j \le pqr - 1$. The colouring between x_i and x_j uses different colours, as these two vertices are adjacent to each other for all $0 \le i \ne j \le p - 1$. Hence, there are p different colours of the vertices in $p\text{-}Cay(C_{pqr}, S^{(p)})$.

Therefore, based on Definition 4, $\chi(p\text{–}Cay(C_{pqr}, S^{(p)})) = p$. ■

The proposition below demonstrates the planarity of $p\text{-}Cay(C_{pqr}, S^{(p)})$.

Proposition 4. *Let* C_{pqr} *be a cyclic group of order* pqr, *and let* $S^{(p)} = \{x \in C_{pqr} : |x| = p\}$ *with* $S^{(p)} = S^{(p)-1}$. *Then*, $p\text{-}Cay(C_{pqr}, S^{(p)})$ *is a non-planar graph*.

Proof. From Theorem 2, $p\text{-}Cay(C_{pqr}, S^{(p)}) = qrK_p$. The smallest possible p, q, r are $p = 2, q = 3, r = 5$, then $pqr = 30$.

There exists a subdivision of K_5 or $K_{3,3}$ in $p\text{-}Cay(C_{pqr}, S^{(p)})$. Therefore, according to Theorem 1, $p\text{-}Cay(C_{pqr}, S^{(p)})$ is a non-planar graph, due to the presence of a K_5 or $K_{3,3}$ subdivision. ∎

By using the same method, the $q\text{-}Cay(C_{pqr}, S^{(q)})$ is explored in Proposition 5.

Proposition 5. *Let* C_{pqr} *be a cyclic group of order* pqr, *and let* $S^{(q)} = \{x \in C_{pqr} : |x| = q\}$ *with* $S^{(q)} = S^{(q)-1}$. *Then*,

1. $diam\,(q\text{–}Cay(C_{pqr}, S^{(q)})) = \infty$.

2. $\chi(q\text{–}Cay(C_{pqr}, S^{(q)})) = p$.

3. $q\text{-}Cay(C_{pqr}, S^{(q)})$ *is a non-planar graph*.

By using the same method, the $r\text{-}Cay(C_{pqr}, S^{(r)})$ is explored in Proposition 6.

Proposition 6. *Let* C_{pqr} *be a cyclic group of order* pqr, *and let* $S^{(r)} = \{x \in C_{pqr} : |x| = r\}$ *with* $S^{(r)} = S^{(r)-1}$. *Then*,

1. $diam\,(r\text{–}Cay(C_{pqr}, S^{(r)})) = \infty$.

2. $\chi(r\text{–}Cay(C_{pqr}, S^{(r)})) = r$.

3. $r\text{-}Cay(C_{pqr}, S^{(r)})$ *is a non-planar graph*.

Conclusion

In this paper, the p_i-Cayley graph and prime power Cayley graph are constructed for the cyclic group of order pqr, p, q, r primes and $p < q < r$. The graphs formed for the p_i-Cayley graph of the cyclic group of order pqr are qrK_p, pqK_q, and pqK_r. The prime power Cayley graph for the cyclic group of order pqr is a $(p + q + r - 3)$ -regular graph.

In addition, some properties of the p_i-Cayley graph and the prime power Cayley graph for the cyclic group of order pqr are obtained, which are diameter, chromatic number and planarity.

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