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NON DIFFERENTIABLE INEXACT NEWTON-SECANT-TYPE METHOD FOR GENERALIZED EQUATIONS

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Abstract

A multitude of applications spanning various fields can be distilled down to the resolution of generalized equations that incorporate Banach space-valued operators. The predominant approach to solving these equations involves iterative methods, where a sequence is systematically generated to approximate a solution, contingent upon specific conditions being met for the initial point and the operators employed in the process. Specialized Secant-type methods have been devised, with their adaptations converging to familiar techniques like Newton's method, modified Newton's method, Secant method, Kurchatov method, and Steffensen's method, among others. In this paper, we are concerned with the problem of approximating a solution of the generalized equation $f(x) + f_1(x) + G(x) \ni 0$, where B_1, B_2 are Banach spaces, $f : B_1 \to B_2$ is a differentiable operator in the sense of Fréchet, $f_1 : B_1 \to B_2$ is a continuous operator whose differentiability is not assumed and $G : B_1 \rightrightarrows B_2$ is a set valued operator. We demonstrate that the above problem can be resolved using Newton-Secant-type method. The theory is complemented by numerical applications.

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1. Introduction

We are concerned with the problem of approximating a solution of the generalized equation

$$f(x) + f_1(x) + G(x) \ni 0, \tag{1.1}$$

where B_1, B_2 are Banach spaces, $f : B_1 \to B_2$ is a differentiable operator in the sense of Fréchet, $f_1 : B_1 \to B_2$ is a continuous operator whose differentiability is not assumed and $G : B_1 \rightrightarrows B_2$ is a set valued operator. The usual case is when $f_1 = 0$ [29].

Newton-Secant-type method for generating a sequence approximating a solution $x_* \in B_1$ of the equation (1.1) is

$$x_{-1}, x_0 \in B_1, \quad f(x_n) + f_1(x_n) + A(x_{n+1} - x_n)(x_{n+1}, x_n) + G(x_{n+1}) \ni 0,$$

$$n = 0, 1, 2, \ldots \tag{1.2}$$

where $A : B_1 \times B_1 \to \mathcal{L}(B_1, B_1)$ is usually a conscious approximation to f'.

If $G = \{0\}$ and $A(x, y) = f'(x)$, the method (1.2) reduces to the iteration introduced by Zincenko [43] for solving the equation

$$f(x) + f_1(x) = 0. \tag{1.3}$$

The operator $G : B_1 \rightrightarrows B_2$ is assumed to be closed, non-empty, convex cone in the Banach space B_2 which is denoted by C for simplicity. The semi-local convergence analysis of the method (1.2) is developed using generalized continuity conditions and majorizing sequences. The idea is taken from [29], where we used such conditions to solve the equation (1.3). We also show that even specializations of our results provide finer error analysis than existing ones [29].

2. Mathematical Background

The definition of a convex process was inaugurated by Rockafellar [37] and studied by Robinson [32], Dontchev [18] and others [1-43]. In order to make the study as self contained as possible, we present the definition of a convex process. More properties can be found in [35] and the references within.

Definition 2.1. An operator P from a linear space $B_1 \times B_1$ into the linear space B_2 is a convex process if

$$P(x, \tilde{x}) + P(y, \tilde{y}) \subset P(x + y, \tilde{x} + \tilde{y}) \text{ for each } x, y, \tilde{x}, \tilde{y} \in B_1,$$

$$P(a(x, \tilde{x})) = aP(x, \tilde{x}) \text{ for each } a > 0 \text{ and } x, \tilde{x} \in B_1$$

$$\text{and } P(0, 0) = 0.$$

Then, the domain denoted by dom, the range by rge and the inverse by P^{-1} are defined as: $domP$ is the set of elements from B_1 with $P(x) \neq 0$, rge of P is $U\{P(x); x \in dom(P)\}$ and $P^{-1}(y) = \{x; y \in P(x)\}$.

The sets $domP$ and $rgeP$ are convex set and the inverse is a convex process.

If the norm of P, $\| P \|$ is finite, the convex process is normed.

The following Banach-type perturbation Lemma is needed [27].

Lemma 2.2. *Let* P, P_1 *be convex processes from* B_1 *into* B_2. *Set* $dom(P) = D_1$ *and* $rge(P) = D_2$. *Suppose* P, P^{-1} *and* P_1 *are normed, and that* $\| P^{-1} \| \cdot \| P_1 \| < 1; C \subset dom(P_1), P_1(P) \subset P_1, P_1$ *is closed and* $(P - P_1)(x)$ *is closed for each* $x \in D_1$. *Then,* $(P - P_1)^{-1}$ *satisfies*

$$rge(P) \subset rge(P - D_1),$$

$$(P - P_1)_{D_2}^{-1} \text{ is a normed convex process};$$

$$\text{and } \| P^{-1}(P - P_1)_{D_2}^{-1} \| \leq \frac{1}{1 - \| P_1 \|}.$$

We assume familiarity with majorizing sequences and their importance to the study of the convergence of iterative methods [4-7].

3. Algorithm

Let $\Omega \subset B_1$. For each fixed $x, y \in \Omega$, define

$$P(x, y) = z = A(x, y)z - C, \quad z \in B_1.$$

Clearly, P is a normed convex process from B_1 into B_2, with inverse

$$P^{-1}(x, y)(z) = \{\bar{x} \in B_1, A(x, y)(z) \in z + C\},$$

which is also a convex process.

For a starting element $x_{-1}, x_0 \in \Omega, P^{-1} = P^{-1}(x_{-1}, x_0)$ such that $P^{-1}(-f(x_0) - f_1(x_0)) \neq \phi$, consider x_1 as the sum of x_0 and a projection of the origin in B_1 on $P^{-1}(-f(x_0) - f_1(x_0))$. This process is repeated with x_1 replacing x_0. At the n-th step, we get x_n and define x_{n+1} as the sum of x_n and a projection of the origin in B_1 on $P^{-1}(-f(x_n) - f_1(x_n))$. Equivalently, the algorithm is, if x_{n-1}, x_n are computed, the iterate x_{n+1} is any solution of the minimization problem:

$$\text{minimize } \{\| x - x_n \| : f(x_n) + f_1(x_n) + A(x_{n-1}, x_n)(x - x_n) \in C\}. \tag{3.1}$$

Thus, we arrived at:

Algorithm: Newton-Secant-type-Cone $(f, f_1, C, x_{-1}, x_0, \epsilon)$.

Step 1. If $P^{-1}(-f(x_0) - f_1(x_0)) = \phi$, terminate with failure.

Step 2. If $\epsilon_1 < \epsilon$ do:

- Pick a solution of the problem

$$\text{minimize } \{\| x - x_0 \| : f(x_0) + f_1(x_0) + A(x_{-1}, x_0)(x - x_0) \in C\}.$$

- $e_1 = \| x - x_0 \|; x_0 = x$.

Step 3. Return x.

This algorithm reduces to the one in [29] if $A = f'$. If $G = 0$, the algorithm reduces to the one considered by us. Moreover, if $C = \{0\}$, then it reduces to the Zincenko iteration [43].

Remark 3.1. The continuity of $A(x_n)$ and since C is closed and convex, imply that the feasible set of (3.1) is a closed convex set for each $n = 1, 2, \ldots$. Then, the existence of a feasible element $\bar{x}$ implies that any solution of (3.1) is in the intersection of the feasible set of (3.1) with the closed ball of center x_n and radius $\| \bar{x} - x_n \|$. But, B_1 is reflexive and the function $\| x - x_n \|$ is weakly lower semi-continuous, a solution exists (see [29]). Thus, if (3.1) is feasible, then it is solvable and by convexity, this implies that any local solution is global. This important remark is used in Section 4.

4. Convergence

The following conditions are needed in the semi-local convergence of the method (1.2).

Assume:

(H_1) There exist elements $x_{-1}, x_0 \in \Omega$ so that $P = P(x_{-1}, x_0)$ maps B_1 onto B_2.

(H_2) Let $s_0 \geq 0$. There exists a continuous and non-decreasing function $v_0 : [0, +\infty) \times [0, +\infty) \to (-\infty, +\infty)$ so that for each $x \in \Omega$,

$$\| P^{-1}(A(x, y) - A(x_{-1}, x_0)) \| \leq v_0(\| x - x_{-1} \|, \| y - x_0 \|).$$

(H_3) The equation $v_0(s_0 + t, t) - 1 = 0$ has a smallest positive solution denoted by ρ, satisfying $\rho \leq s_0$.

(H_4) There exists a continuous and non-decreasing function $v : [0, \rho - s_0) \times [0, \rho) \to (-\infty, +\infty)$ so that for each $x, y, z \in U(x_0, \rho - s_0)$

$$\| P^{-1}(f(x) - f(y) + f_1(x) - f_1(y) + A(x, z)(y - x)) \| \leq v(\| y - x \|, \| z - x \|).$$

(H_5) $\| x_1 - x_0 \| \leq s_1 \leq \rho - s_0$ where x_1 exists by the Remark 3.1 and is obtained by the Algorithm. Define the scalar sequence $\{s_n\} s_{-1} = 0, s_0 = \| x_0 - x_{-1} \|, s_1 \geq s_0$, where s_1 is given by

$$s_{n+1} = s_n + \frac{v(s_{n-1} - s_{n-2}, s_n - s_{n-1})}{1 - v_0(s_{n-1} + s_0, s_n)}, \quad n = 0, 1, 2, \ldots \tag{4.1}$$

The sequence $\{s_n\}$ is shown to be majorizing in Theorem 4.1 for the method (1.2). However, first a convergence condition is needed for it.

(H_6) There exists $\rho_0 \in [s_1, \rho - s_0]$ so that for each $n = 0, 1, 2, \ldots$

$$v_0(s_{n-1} + s_0, s_n) < 1 \text{ and } s_n \leq \rho$$

and the existence of the $\lim_{n \to +\infty} s_n$ denoted by s_*. This limit is the unique least upper bound of the sequence $\{s_n\}$ and

(H_7) $U[x_0, s_* - s_0] \subset D.$

Next, the main semi-local convergence result is developed under the conditions (H_1)-(H_7) and the preceding notation.

Theorem 4.1. *Assume that the conditions* (H_1)-(H_7) *are valid. Then, the sequence* $\{x_n\}$ *generated by the Algorithm* (3.1) *is well defined in* $U(x_0, s_* - s_0)$, *remains in* $U(x_0, s_* - s_0)$ *for each* $n = 0, 1, 2, \ldots$ *and is convergent to some* $x_* \in U[x_0, s_* - s_0]$ *solving the equation* (1.1).

Moreover, the following error estimates are valid.

$$\| x_* - x_n \| \leq s_* - s_n, \quad n = 0, 1, 2, \ldots. \tag{4.2}$$

Proof. Mathematical induction is employed to show the assertion

$$\| x_{n+1} - x_n \| \leq s_{n+1} - s_n, \quad n = 0, 1, 2, \ldots. \tag{4.3}$$

The definition of the iterate s_1, s_0 in (H_5) and (4.1) imply the existence of x_1 and that the assertion (4.3) is valid if $n = 0$.

Assume $x_1, \ldots, x_m, s_{-1}, s_0, \ldots, s_m$ exists and

$$\| x_m - x_{m-1} \| \leq s_m - s_{m-1} \text{ for each } m = 0, 1, 2, \ldots, n \tag{4.4}$$

It follows that

$$\| x_m - x_0 \| \leq \| x_m - x_{m-1} \| + \ldots + \| x_1 - x_0 \|$$

$$\leq s_m - s_{m-1} + \ldots + s_1 - s_0 = s_m - s_0.$$

Thus, the iterates $x_m \in U(x_0, s_* - s_0)$. By (H_1), (H_2), (H_3) and (H_7), it follows that

$$\| P^{-1}(A(x_{m-1}, x_m) - A(x_{-1}, x_0)) \| \leq v_0(\| x_{m-1} - x_{-1} \|, \| x_m - x_0 \|)$$

$$\leq v_0(s_{m-1} + s_0, s_m) \leq v_0(s_* + s_0, s_*) < 1, \tag{4.5}$$

by the definition of s_* in (H_6). Moreover, we have

$$P(x_{m-1}, x_m)(x) = A(x_{m-1}, x_m)(x) - C$$

$$= (A(x_{-1}, x_0) + (A(x_{m-1}, x_m) - A(x_{-1}, x_0)))(x) - C$$

$$= (P(x_{-1}, x_0) - Q_m),$$

$$Q_m = A(x_{-1}, x_0) - A(x_{m-1}, x_m).$$

By Lemma 2.2 and (4.5), $P(x_m)$ maps B_1 into B_2, $P^{-1}(x_m)$ is normed and

$$\| P^{-1}(x_{m-1}, x_m)P \| \leq \frac{1}{1 - \| P^{-1}(A(x_{m-1}, x_m) - A(x_{-1}, x_0)) \|}$$

$$\leq \frac{1}{1 - v_0(s_0 + s_{m-1}, s_m)}. \tag{4.6}$$

Moreover, (3.1) is solvable for $m = n$, since $P(x_m)$ maps B_1 into B_2, which establishes the existence of x_{m+1} solving (3.1). Then, consider the problem to find x solving

$$f(x_m) + f_1(x_m) + A(x_{m-1}, x_m)(x - x_m) \in f(x_{m-1}) + f_1(x_{m-1})$$

$$+A(x_{m-2}, x_{m-1})(x_m - x_{m-1}) + C. \tag{4.7}$$

Notice that the right hand side of (4.7) is in the cone C, so x_m solves (3.1). Thus, any x satisfying (4.7) is also feasible for (3.1). We can write (4.7) as

$$x - x_m \in P^{-1}(x_{m-1}, x_m)(- f(x_m) - f_1(x_m) + f(x_{m-1}) + f_1(x_{m-1})$$

$$+A(x_{m-2}, x_{m-1})(x_m - x_{m-1})). \tag{4.8}$$

But the right hand side of (4.8) has an element of least norm. Hence, there exists $\bar{x}$ satisfying (4.8), (4.7) and

$$\begin{aligned}\| \bar{x} - x_m \| &\leq P^{-1}(x_{m-1}, x_m)(-f(x_m) + f(x_{m-1}) + A(x_{m-1}, x_m)(x_m - x_{m-1}) \\ &\quad + (f_1(x_m) - f_1(x_{m-1}))) \\ &\leq \frac{v(\| x_{m-1} - x_{-1} \|, \| x_m - x_0 \|)}{1 - v_0(s_0 + s_{m-1}, s_m)} \\ &\leq \frac{v(s_0 + s_{m-1}, s_m)}{1 - v_0(s_0 + s_{m-1}, s_m)} \\ &= s_{m+1} - s_m, \end{aligned} \tag{4.9}$$

where we also used (H_4), (4.3) (for $n = m$) and (4.6). Then, by (4.9), it follows

$$\| x_{m+1} - x_m \| \leq \| \bar{x} - x_m \| \leq s_{m+1} - s_m$$

$$\begin{aligned}\text{and } \| x_{m+1} - x_0 \| &\leq \| x_{m+1} - x_m \| + \| x_m - x_0 \| \\ &\leq s_{m+1} - s_m + s_m - s_0 \\ &= s_{m+1} - s_0 < s_* - s_0.\end{aligned}$$

Thus, the induction for (4.3) is completed and the iterate

$$x_{m+1} \in U(x_0, s_* - s_0). \tag{4.10}$$

So, the sequence $\{x_n\}$ is complete in a Banach space B_1 and as such it is convergent to some $x_* \in U[x_0, s_* - s_0]$ (since $U[x_0, s_* - s_0]$ is a closed set). We can write

$$\begin{aligned}E_m &= (f(x_{m+1}) + f_1(x_{m+1}) - f(x_*) - f_1(x_*)) - (f(x_{m+1}) - f(x_m) \\ &\quad - A(x_{m-1}, x_m)(x_{m+1} - x_m)) + (f_1(x_{m+1})).\end{aligned}$$

Then, we get

$$E_m \in C - f(x_*) - f_1(x_*).$$

By the continuity of functions f and f_1

$$E_m \to 0 \text{ as } m \to +\infty.$$

But, $C - f(x_*) - f_1(x_*)$ is closed. Thus, we conclude $f(x_*) + g(x_*) \in C$, which implies that x_* solves the equation (1.1).

5. Specializations and Applications

Application 5.1. Let us compare our results to the ones in [29]. That is we take $A(x_{n-1}, x_n) = f'(x)$. The conditions in [29] are non-affine invariant form

(T_1) $\| P^{-1}(x_0) \| \leq b, \| f'(x) - f'(y) \| \leq L_1 \| x - y \|$.

(T_2) $\| f_1(x) - f_1(y) \| \leq L_2 \| x - y \|$.

and the majorizing sequence $\{t_n\}$ is defined by

$$t_0 = 0, t_1 \geq 0, \tag{5.1}$$

$$t_{n+1} = t_n + \frac{b}{1 - bL_1 t_n}\left(\frac{L_1}{2}(t_n - t_{n-1}) + L_2\right)(t_n - t_{n-1}). \tag{5.2}$$

(T_3) $\| x_1 - x_0 \| \leq t_1 \leq \min\left(\frac{1}{bL_1}, \frac{(1 - bL_2)^2}{2bL_1}\right)$.

(T_4) $U[x_0, t_*] \subset D$, where $t_* = \frac{1}{bL_1}(1 - bL_1 - \sqrt{(1 - bL_2)^2 - 4bL_1 t_1})$.

In our case, specialize the functions

$$v_0(s, t) = v_0(t) = l_2 t$$

$$\text{and} \quad v(s, t) = v(t) = \frac{l_1}{2}t^2 + l_2 t.$$

then, the iteration $\{s_n\}$ becomes

$$s_{n+1} = s_n + \frac{1}{l_2 s_n}\left(\frac{l_1}{2}(s_n - s_{n-1}) + l_2\right)(s_n - s_{n-1}). \tag{5.3}$$

Notice that

$$l_1 \leq bL_1 \tag{5.4}$$

$$\text{and} \quad l_2 \leq bL_2. \tag{5.5}$$

Our results are given in affine invariant form. The advantages of affine over non-affine invariant form are well known [29]. It follows by (5.1)-(5.5) and a simple inductive argument that

$$0 \leq s_n \leq t_n \tag{5.6}$$

$$0 \leq s_{n+1} - s_n \leq t_{n+1} - t_n \tag{5.7}$$

$$\text{and} \quad 0 \leq s_* \leq t_*. \tag{5.8}$$

Notice also that our sufficient semi-local convergence conditions (H_5) and (H_6) are weaker than (T_3). That is the conditions (T_1)-(T_4) imply the conditions (H_1)-(H_7) but not necessarily vice versa.

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