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$L(3, 1)$ LABELING OF LADDER GRAPH, MILLIPEDE GRAPH, TADPOLE GRAPH AND GRID GRAPH

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Abstract

$L(3, 1)$ labeling is one of a particular model of frequency assignment problem of $L(h, k)$ labeling. A $L(3, 1)$ labeling of a graph G is a function f from the vertex set $V(G)$ to the set of positive integers such that for any two vertices u, v if $d(u, v) = 1$ then $| f(u) - f(v) | \geq 3$ and if $d(u, v) = 2$ then $| f(u) - f(v) | \geq 1$. In $L(3, 1)$ labeling, λ is the smallest positive integer which denotes the maximum label used.

In this paper, we consider $L(3, 1)$ labeling of Ladder graph, Millipede graph, Tadpole graph and Grid graph.

1. Introduction

Graph Labeling is assigning labels to vertices or edges or both under certain conditions. The applications are found in communication networks, astronomy and so on.

The wireless communication networks have the radio frequencies allotted to them, which are not adequate. The interference by unconstrained

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transmitters will disturb the communication. Hale. W took up this problem in terms of graph labeling. Griggs and Robert proposed a variation in channel assignment problem. According to him any two transmitters which are close will receive different channels so as to avoid interference. $L(2, 1)$ labeling is a result of this problem introduced by Griggs J. and Yeh R. [2]. $L(3, 1)$ labeling was introduced by Sumanto Ghosh and Anita Pal [5] whose definition is as follows.

Definition 1. ($L(3, 1)$ labeling)**.** [5]

Let the graph $G = (V, E)$, $L(3, 1)$ labeling is a function f that assigns labels for every u, v belonging to the set of positive integers, if $d(u, v) = 1$ then $|f(u) - f(v)| \geq 3$ and if $d(u, v) = 2$ then $|f(u) - f(v)| \geq 1$. $L(3, 1)$ labeling number, $\lambda(G)$ is the smallest number λ with λ as the maximum label such that G has $L(3, 1)$ labeling.

Definition 2 (Ladder Graph)**.** A Ladder graph is obtained as the Cartesian product of two path graphs, one of which has only edge i.e. $L_{(n, 1)} = P_n \times P_2$ and is denoted by L_n. The graph consists of $2n$ vertices and $3n - 2$ edges. Let $v_1, v_2, \ldots, v_n$ be the vertices along the left side of the ladder graph and $v_{n+1}, v_{n+2}, \ldots, v_{2n}$ be the vertices along the right side of the ladder graph. All the internal vertices of the ladder graph have degree 3, the vertices at the ends have degree 2 and the diameter of the graph is n.

Definition 3 (Millipede graph)**.** Millipede graph is a graph made from Ladder graph L_n by adding r number of path graphs which has length 1 at each vertex of the ladder graph and is denoted by $L_n \odot K_r$.

Definition 4 (Tadpole graph)**.** The Tadpole graph $T(m, n)$ is obtained by joining a cycle $C_m (m \geq 3)$ to the path P_n at one of the vertices of C_m.

Definition 5. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The Cartesian product of G_1 and G_2 which is denoted by $G_1 \times G_2$ is the graph with vertex set $V(G) = (v_{ij}, 1 \leq i \leq m, 1 \leq j \leq n)$. The product $P_m \times P_n$ is called planar grid. There are mn vertices and $2mn - m - n$ edges.

2. Main Results

$L(3, 1)$ labeling number of Ladder graph L_n, Millipede graph $L_n \odot K_r$, Tadpole graph $T(m.n)$ and Grid graph $P_m \times P_n$ are determined in this section.

Theorem 2.1. *$L(3, 1)$ labeling number of the ladder graph is* $\lambda(L_n) = 7$ $(n \geq 2)$.

Proof. In the ladder graph L_n, suppose v receives the label 0 where $d(v) = 3$ then $N(v) = \{v_1, v_2, v_3\}$ receives the labels 3, 6, 4, since two of the neighbours of v has common neighbour v_4 as shown in Figure 1, v_4 can receive minimum label as 7. Thus $\lambda(L_n) = 7$.

The vertices on the left consecutively receive labels 0, 3, 6, 0, 3, 6, . . . and the corresponding vertices on the right receive labels 4, 7, 1, 4, 7, 1, ... See Figure 1.

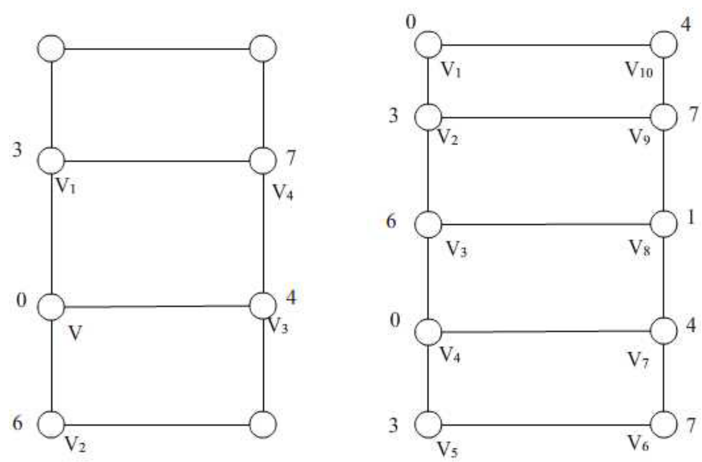

Figure 1. $L(3, 1)$ Labeling of Ladder graph.

Theorem 2.2. *$L(3, 1)$ Labeling number of $(L_2 \odot K_r)$ is equal to $6 + r$ if* $(r \geq 2)$.

Proof. The vertices of L_2 are labeled as v_1, v_2, v_3, v_4 consecutively in the anticlockwise direction and the corresponding pendant vertices attached to each of v_i's are labeled as $v_i^1, v_i^2, \ldots, v_i^r$.

Consider a vertex v_1. Without loss of generality, suppose v_1 is labelled with 0, then the adjacent vertices of v_1 namely v_2, v_4, v_1^1 receive labels 3, 4, 5. Vertex v_3 adjacent to v_2 and v_4 receives label ≥ 7. Then for $G = L_2 \odot K_r$

Define $f : V(G) \to \{0, 1, 2, \ldots\}$ as follows

$f(v_1) = 0,\ f(v_2) = 3$

$f(v_3) = 7,\ f(v_4) = 4$

$f(v_1^i) = 4 + i (1 \leq i \leq r)$ since $N(v_1) = \{v_2, v_4\}$

$f(v_2^1) = 6,\ f(v_2^i) = 6 + i (2 \leq i \leq r)$ since $N(v_2) = \{v_1, v_3\}$

$f(v_3^1) = 0,\ f(v_3^2) = 1,\ f(v_3^3) = 2,\ f(v_3^i) = 6 + i (4 \leq i \leq r)$ since

$N(v_3) = \{v_2, v_4\}$

$f(v_4^1) = 6,\ f(v_4^i) = 6 + i (2 \leq i \leq r)$ since $N(v_4) = \{v_1, v_3\}$

In view of the above, we infer that the maximum label assigned to the vertex is $6 + r$.

Thus we conclude that $\lambda(L_2 \odot K_r) = 6 + r$ if $(r \geq 2)$ See Figure 2.

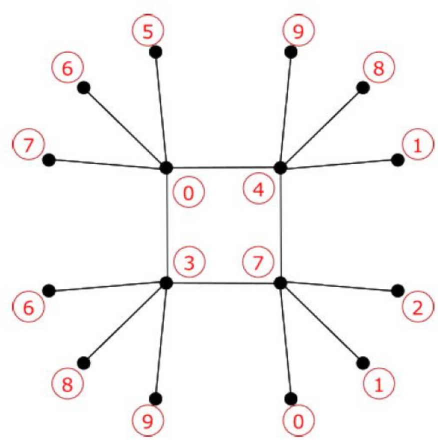

Figure 2. $L(3, 1)$ Labeling of Millipede graph.

Theorem 2.3. *$L(3, 1)$ labeling number of $(L_3 \odot K_r)$ is equal to $7 + r$ if $(r \geq 2)$.*

Proof. The vertices of L_3 are labeled as $v_1, v_2, v_3, v_4, v_5, v_6$ consecutively in the anticlockwise direction and the corresponding pendant vertices attached to each of v_i's are labeled as $v_i^1, v_i^2, \ldots, v_i^r$.

In view of ladder graph L_n, the vertices v_i of $L_3 \odot K_r$ are labelled as follows

Define $f : V(G) \to \{0, 1, 2, \ldots\}$ as follows $f(v_1) = 0$, $f(v_2) = 3$, $f(v_3) = 6$ $f(v_4) = 1$, $f(v_5) = 7$, $f(v_6) = 4$

Now

$f(v_1^i) = 4 + i(1 \leq i \leq r)$ since $N(v_1) = \{v_2, v_4\}$

$f(v_2^i) = 7 + i(1 \leq i \leq r)$ since $N(v_2) = \{v_1, v_3, v_5\}$

$f(v_3^1) = 0$, $f(v_3^2) = 1$, $f(v_3^3) = 2$

$f(v_3^i) = 5 + i(4 \leq i \leq r)$ since $N(v_3) = \{v_2, v_6\}$

$f(v_4^1) = 4$, $f(v_4^2) = 5$, $f(v_4^i) = 5 + i(3 \leq i \leq r)$ since $N(v_4) = \{v_3, v_5\}$

$f(v_5^1) = 0$, $f(v_5^2) = 2$, $f(v_5^3) = 4$,

$f(v_5^i) = 6 + i(4 \leq i \leq r)$ since $N(v_5) = \{v_2, v_4, v_6\}$

$f(v_6^1) = 1$, $f(v_6^i) = 6 + i(2 \leq i \leq r)$ since $N(v_1) = \{v_1, v_5\}$

In view of the above, we infer that the maximum value assigned to the vertex is $7 + r$. Thus we conclude $\lambda(L_3 \odot K_r) = 7 + r$ if $(r \geq 2)$. See Figure 3.

Remark 1. In view of the above theorem, we conclude that if $v_1, v_2, v_3, \ldots, v_n$ are labelled as $0, 3, 6, 0, 3, 6, \ldots$ and $v_{n+1}, v_{n+2}, v_{n+3}, \ldots, v_{2n}$ are labelled as $1, 7, 4, 1, 7, 4, \ldots$ then the $L(3, 1)$ labeling number of the Millipede graph $L_n \odot K_r$ is $\lambda(L_n \odot K_r) = 7 + r$ $(r \geq 2)$.

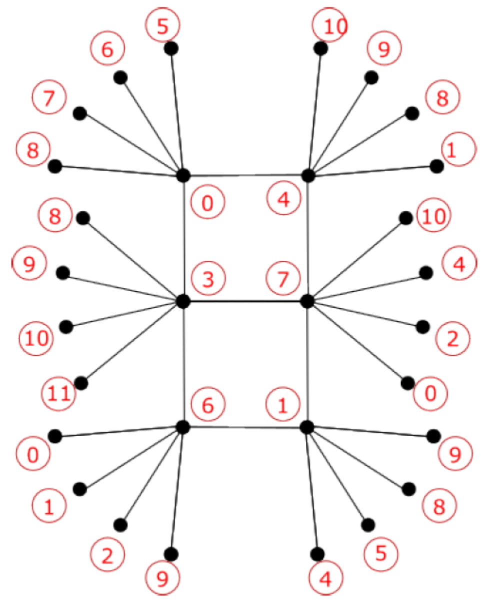

Figure 3. $L(3, 1)$ Labeling of Millipede graph.

Theorem 2.4. *$L(3, 1)$ labeling number of the Tadpole graph $T(m, n)$ is* 6 *for* $(n > 3)$.

Proof. Let the vertices of C_n be labelled as $v, v_1, v_2, \ldots, v_{n-1}$ in the anticlockwise direction. The path P_n is attached with $v = u_1$ and the consecutive vertices in P_n are labelled as $u_2, u_3, \ldots, u_n$. Suppose $v = u_1$ is assigned the value 0, then the adjacent vertices of u_1, namely v_1, v_{n-1}, u_2 receive labels 3, 4, 5 when n is odd and they receive labels 4, 5, 3 when n is even.

Case 1. n odd

$L(3, 1)$ labeling number of C_n is 6 for $n \geq 3$ as in [5]

Hence the remaining vertices of C_n namely $v_2, v_3, \ldots, v_{n-2}$ receive labels $3, 6, 0, \ldots, 1$ and the remaining vertices $u_3, u_4, \ldots, u_n$ of the path P_n receive labels $1, 4, 0, 5, 1, 4, \ldots$

Thus $\lambda(T(n, m) = 6(n > 3))$.

Case 2. n even

As discussed in Case 1, when $n > 4$, the remaining vertices of C_n namely $v_2, v_3, \ldots, v_{n-2}$ receive labels $4, 1, 5, \ldots, 2$ and v_{n-2} receives the value 1 when n is 4. The vertices $u_3, u_4, \ldots, u_n$ of the path P_n receive labels $6, 0, 3, 6, 0, \ldots$.

Thus $\lambda(T(n, m) = 6(n > 3))$. See Figure 4.

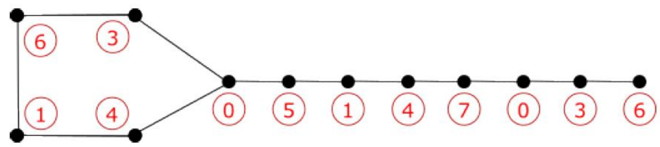

Figure 4. $L(3, 1)$ Labeling of Tadpole graph $T(5, 8)$

Theorem 2.5. *$L(3, 1)$ labeling number of the grid $P_m \times P_n$ is $\lambda(P_m \times P_n) = 8$ for all $m, n > 3$.*

Proof. Consider a vertex v whose degree is 4. Let v_1, v_2, v_3, v_4 be the vertices adjacent to v in the anticlockwise direction. Let v_{ij} be the vertices adjacent to both v_i and v_j, $(i, j = 1, 2, 3, 4)$. Without loss of generality, suppose v is labelled as 0 then v_i, $(i = 1, 2, 3, 4)$ can receive the labels 3, 4, 5, 6.

If v_1 receives the label 3, v_2 receives 4, v_3 as 5 and v_4 as 6, then v_{12} receives labels ≥ 7 and v_{14} receives labels ≥ 8.

If v_1 receives the label 3, v_2, v_3, v_4 receives labels as 6, 4, 5 respectively, then v_{12} receives labels ≥ 9 and v_{14} receives labels ≥ 8. Other cases are dealt in a similar manner. Therefore $\lambda(P_m \times P_n) \geq 8$. Now to show $\lambda(P_m \times P_n) = 8$, let the vertices of the grid graph be labelled as v_{ij}, $(1 \leq i \leq m)$ $(1 \leq j \leq n)$, where m and n denote the number of rows and columns of the grid. The vertices v_{i1}, are labelled as $0, 3, 6, 0, 3, 6, \ldots$ vertices v_{i2}, adjacent to v_{i1}, are labelled as $4, 7, 1, 4, 7, 1, \ldots$ and vertices v_{i3} are labelled as $8, 2, 5, 8, 2, 5, \ldots$. The vertices from v_{i4} follow a cyclic pattern of labeling as 3, 6, 0, 3, …, and for $i = 1, 2, \ldots, m,$ in a similar manner $v_{ij}, j = 5, 6, \ldots, n$ are labelled as 7, 1, 4, 7, …, 2, 5, 8, 2, …, 6, 0, 3, 6, … and so on. We infer from the above that $\lambda(P_m \times P_n) = 8$ for all $m, n > 3.$

See Figure 5.

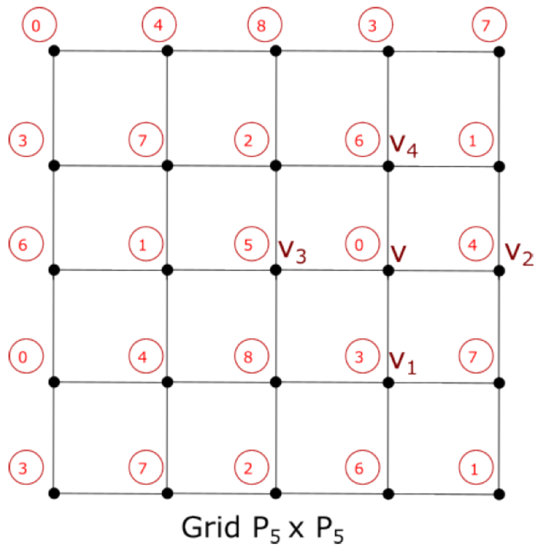

Figure 5. $L(3, 1)$ Labelling of grid $P_5 \times P_5$.

3. Conclusion

$L(3, 1)$ labeling number of Ladder graph, Millipede graph, Tadpole graph and Grid graph has been obtained. Research on more graphs is under progress.

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